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Maximal commutators in Orlicz spaces for the Dunkl operator on the real line

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Abstract. *On the real line, the Dunkl operators*

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \; x \in \mathbb{R}, \; \nu \geq -1/2$$

are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In the paper, in the setting $\mathbb{R}$ we study the maximal commutators $M_{b,\nu}$ in the Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$. We give necessary and sufficient conditions for the boundedness of the operators $M_{b,\nu}$ on Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$ when b belongs to $BMO(\mathbb{R}, dm_\nu)$ spaces.

Keywords. Maximal operator; Orlicz space; Dunkl operator; commutator; BMO

Mathematics Subject Classification (2010): 2010 Mathematics Subject Classification: 42B20, 42B25, 42B35

1 Introduction

On the real line, the Dunkl operators Λ_ν are differential-difference operators introduced in 1989 by Dunkl [8]. For a real parameter $\nu \geq -1/2$, we consider the *Dunkl operator*, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \;\; x \in \mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

Let $\nu > -1/2$ be a fixed number and m_ν be the *weighted Lebesgue measure* on $\mathbb{R}$, given by

$$dm_\nu(x) := \left(2^{\nu+1}\Gamma(\nu+1)\right)^{-1} |x|^{2\nu+1}\, dx, \quad x \in \mathbb{R}.$$

For any $x \in \mathbb{R}$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R} : |y| \in]\max\{0, |x|-r\}, |x|+r[\;\}$. Then $B(0,r) =]-r, r[$ and $m_\nu B(0,r) = c_\nu\, r^{2\nu+2}$, where $c_\nu := \left[2^{\nu+1}(\nu+1)\,\Gamma(\nu+1)\right]^{-1}$.

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The *maximal operator* M_ν associated by Dunkl operator on the real line is given by

$$M_\nu f(x) := \sup_{r>0} (m_\nu(B(x,r)))^{-1} \int_{B(x,r)} |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R}.$$

The *maximal commutator* $M_{b,\nu}$ associated with Dunkl operator on the real line and with a locally integrable function $b \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$ is defined by

$$M_{b,\nu} f(x) := \sup_{r>0} (m_\nu(B(x,r)))^{-1} \int_{B(x,r)} |b(x) - b(y)|\, |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R}.$$

It is well known that maximal and fractional maximal operators play an important role in harmonic analysis (see [7,24]). Also the fractional maximal function and the fractional integral, associated with D_ν differential-difference Dunkl operators play an important role in Dunkl harmonic analysis, differentiation theory and PDE's. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [4,5,18]. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space. The Dunkl theory provides a useful framework for the study of multivariable analytic structures and has gained considerable interest in various fields of mathematics and in physical applications (see, for example, [9]). The maximal function, the fractional integral and related topics associated with the Dunkl differential-difference operator have been research areas for many mathematicians such as C. Abdelkefi and M. Sifi [1], V.S. Guliyev and Y.Y. Mammadov [4–6], Y.Y. Mammadov [16], L. Kamoun [12], M.A. Mourou [19], F. Soltani [22,23], K. Trimeche [25] and others. Moreover, the results on $L_\Phi(\mathbb{R}, dm_\nu)$-boundedness of fractional maximal operator and its commutators associated with D_ν were obtained in [6,17].

Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting $\mathbb{R}$, we study the boundedness of the maximal commutator $M_{b,\nu}$ on Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$, when b belongs to the space $BMO(\mathbb{R}, dm_\nu)$, by which some new characterizations of the space $BMO(\mathbb{R}, dm_\nu)$ are given.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries in the Dunkl setting on $\mathbb{R}$

To introduce the notion of Orlicz spaces in the Dunkl setting on $\mathbb{R}$, we first recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

is denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) := \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for any } r \ge 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) := \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-*condition*, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\,\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-*condition*, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$. In what follows, for any subset E of $\mathbb{R}$, we use χ_E to denote its *characteristic function*.

Definition 2.2 (Orlicz Space). For a Young function Φ, the set

$$L_\Phi(\mathbb{R}, dm_\nu) := \left\{ f \in L_1^{\mathrm{loc}}(\mathbb{R}, dm_\nu) : \int_{\mathbb{R}} \Phi(k|f(x)|)\, dm_\nu(x) < \infty \text{ for some } k > 0 \right\}$$

is called the *Orlicz space*. If $\Phi(r) := r^p$ for all $r \in [0,\infty)$, $1 \le p < \infty$, then $L_\Phi(\mathbb{R}, dm_\nu) = L_p(\mathbb{R}, dm_\nu)$. If $\Phi(r) := 0$ for all $r \in [0,1]$ and $\Phi(r) := \infty$ for all $r \in (1,\infty)$, then $L_\Phi(\mathbb{R}, dm_\nu) = L_\infty(\mathbb{R}, dm_\nu)$. The *space* $L_\Phi^{\mathrm{loc}}(\mathbb{R}, dm_\nu)$ is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}, dm_\nu)$ for all balls $B \subset \mathbb{R}$.

$L_\Phi(\mathbb{R}, dm_\nu)$ is a Banach space with respect to the norm

$$\|f\|_{L_{\Phi,\nu}} := \inf\left\{ \lambda > 0 : \int_{\mathbb{R}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, dm_\nu(x) \le 1 \right\}.$$

For a measurable function f on $\mathbb{R}$ and $t > 0$, let

$$m(f,t)_\nu := m_\nu\{x \in \mathbb{R} : |f(x)| > t\}.$$

Definition 2.3 The *weak Orlicz space*

$$WL_\Phi(\mathbb{R}, dm_\nu) := \{ f \in L_{1,\nu}^{\mathrm{loc}}(\mathbb{R}) : \|f\|_{WL_{\Phi,\nu}} < \infty \}$$

is defined by the norm

$$\|f\|_{WL_{\Phi,\nu}} := \inf\Big\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big)_\nu \le 1 \Big\}.$$

The following analogue of the Hölder inequality is well known (see, for example, [21]).

Lemma 2.1 *Let the functions f and g be measurable on $\mathbb{R}$. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_{\mathbb{R}} |f(x)g(x)|\, dm_\nu(x) \le 2\|f\|_{L_{\Phi,\nu}} \|g\|_{L_{\widetilde{\Phi},\nu}}.$$

3 Maximal commutators $M_{b,\alpha,\nu}$ in Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$

In this section we investigate the boundedness of the maximal commutator $M_{b,\nu}$ in Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$.

The following result completely characterizes the boundedness of M_ν on Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$.

Theorem 3.1 *[3] Let Φ be a Young function.*
(i)The operator M_ν is bounded from $L_\Phi(\mathbb{R}, dm_\nu)$ to $WL_\Phi(\mathbb{R}, dm_\nu)$, and the inequality

$$\|M_\nu f\|_{WL_{\Phi,\nu}} \le C_0 \|f\|_{L_{\Phi,\nu}} \tag{3.1}$$

holds with constant C_0 independent of f.
(ii) The operator M_ν is bounded on $L_\Phi(\mathbb{R}, dm_\nu)$, and the inequality

$$\|M_\nu f\|_{L_{\Phi,\nu}} \le C_0 \|f\|_{L_{\Phi,\nu}} \tag{3.2}$$

holds with constant C_0 independent of f if and only if $\Phi \in \nabla_2$.

The following theorems were proved in [6].

Theorem 3.2 *[6] Let $b \in BMO(\mathbb{R}, dm_\nu)$ and $\Phi \in \mathcal{Y}$. Then the condition $\Phi \in \nabla_2$ is necessary and sufficient for the boundedness of $M_{b,\nu}$ on $L_\Phi(\mathbb{R}, dm_\nu)$.*

Theorem 3.3 *[6] Let Φ be a Young function with $\Phi \in \nabla_2$. Then the condition $b \in BMO(\mathbb{R}, dm_\nu)$ is necessary and sufficient for the boundedness of $M_{b,\nu}$ on $L_\Phi(\mathbb{R}, dm_\nu)$.*

We recall the definition of the space $BMO(\mathbb{R}, dm_\nu)$.

Definition 3.1 Suppose that $b \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$, let

$$\|b\|_{BMO(\nu)} := \sup_{x\in\mathbb{R}, r>0} \frac{1}{m_\nu(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r)}(x)|\, dm_\nu(y),$$

where

$$b_{B(x,r)} := \frac{1}{m_\nu(B(x,r))} \int_{B(x,r)} b(y)\, dm_\nu(y).$$

Define

$$BMO(\mathbb{R}, dm_\nu) := \{b \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu) \ : \ \|b\|_{BMO(\nu)} < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}, dm_\nu)$ is a Banach space with respect to the norm $\|\cdot\|_{BMO(\nu)}$.

We will need the following properties of BMO-functions (see [10]):

$$\|b\|_{BMO(\nu)} \approx \sup_{x\in\mathbb{R}, r>0} \left(\frac{1}{m_\nu(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dm_\nu(y) \right)^{\frac{1}{p}}, \tag{3.3}$$

where $1 \le p < \infty$ and the positive equivalence constants are independent of b, and

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_{BMO(\nu)} \ \ln\frac{t}{r} \quad \text{for any} \quad 0 < 2r < t, \tag{3.4}$$

where the positive constant C does not depend on b, x, r and t.

For any measurable set E with $m_\nu(E) < \infty$ and any suitable function f, the norm $\|f\|_{L(\log L),E}$ is defined by

$$|f\|_{L(\log L),E} = \inf\left\{\lambda > 0 : \frac{1}{m_\nu(E)} \int_E \frac{|f(x)|}{\lambda}\Big(2 + \frac{|f(x)|}{\lambda}\Big) dm_\nu(x) \le 1\right\}.$$

The norm $\|f\|_{\exp L,E}$ is defined by

$$|f\|_{\exp L,E} = \inf\left\{\lambda > 0 : \frac{1}{m_\nu(E)} \int_E \exp\Big(\frac{|f(x)|}{\lambda}\Big) dm_\nu(x) \le 2\right\}.$$

Then for any suitable functions f and g the generalized Hölders inequality holds (see [21])

$$\frac{1}{m_\nu(E)} \int_E |f(x)||g(x)| dm_\nu(x) \lesssim \|f\|_{\exp L,E}\, \|g\|_{L(\log L),E}. \tag{3.5}$$

The following John-Nirenberg inequalities on spaces of homogeneous type come from [13, Propositions 6, 7].

Lemma 3.1 *Let $b \in BMO(\mathbb{R}, dm_\nu)$. Then there exist constants $C_1, C_2 > 0$ such that for every ball $B \subset \mathbb{R}$ and every $\alpha > 0$, we have*

$$m_\nu\big(\{x \in B : |b(x) - b_B| > \alpha\}\big) \le C_1\, m_\nu(B) \exp\Big\{-\frac{C_2}{\|b\|_{BMO(\nu)}}\,\alpha\Big\}.$$

By the generalized Hölder's inequality in Orlicz spaces (see [21, page 58]) and John-Nirenberg's inequality, we get (see also [14, (2.14)]).

$$\frac{1}{|B|} \int_B \big|b(x) - b_B\big||g(x)| dm_\nu(x) \lesssim \|b\|_{BMO(\nu)}\, \|g\|_{L(\log L),B}. \tag{3.6}$$

We refer for instance to [11] and [15] for details on this space and properties.

Lemma 3.2 *[17] Let $b \in BMO(\mathbb{R}, dm_\nu)$ and Φ be a Young function with $\Phi \in \Delta_2$, then*

$$\|b\|_{BMO(\nu)} \approx \sup_{x\in\mathbb{R}, r>0} \Phi^{-1}\big(m_\nu(B(x,r)^{-1}\big)\, \big\|b(\cdot) - b_{B(x,r)}\big\|_{L_{\Phi,\nu}(B(x,r))}, \tag{3.7}$$

where the positive equivalence constants are independent of b.

Lemma 3.3 *Let $f \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$. Then*

$$M_\nu\big(M_\nu f\big)(x) \approx \sup_{B\ni x} \|f\chi_B\|_{L(1+\log^+ L),\nu}. \tag{3.8}$$

Proof. Let B be a ball in $\mathbb{R}$. We are going to use weak type estimates (see [24], for instance): there exist positive constants $c > 1$ such that for every $f \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$ and for every $t > \big(1/m_\nu(B)\big) \int_B |f(x)| dm_\nu(x)$ we have

$$\frac{1}{c\,t} \int_{\{x\in B: |f(x)|>t\}} |f(x)| dm_\nu(x) \le m_\nu(\{x \in B : M_\nu(f\,\chi_B)(x) > t\}) \le \frac{c}{t} \int_{\{x\in B: |f(x)|>t/2\}} |f(x)| dm_\nu(x).$$

Then

$$\int_B M_\nu(f\chi_B)(x)dm_\nu(x) = \int_0^\infty m_\nu(\{x\in B : M_\nu(f\chi_B)(x) > \lambda\})d\lambda$$
$$= \int_0^{|f|_B} m_\nu(\{x\in B : M_\nu(f\chi_B)(x) > \lambda\})d\lambda$$
$$+ \int_{|f|_B}^\infty m_\nu(\{x\in B : M_\nu(f\chi_B)(x) > \lambda\})d\lambda$$
$$= m_\nu(B)\,|f|_B + \int_{|f|_B}^\infty m_\nu(\{x\in B : M_\nu(f\chi_B)(x) > \lambda\})d\lambda$$
$$\geq m_\nu(B)\,|f|_B + \frac{1}{c}\int_{|f|_B}^\infty \Big(\int_{\{x\in B:|f(x)|>\lambda\}} |f(x)|dm_\nu(x)\Big)\frac{d\lambda}{\lambda}$$
$$= m_\nu(B)\,|f|_B + \frac{1}{c}\int_{\{x\in B:|f(x)|>|f|_B\}} \Big(\int_{|f|_B}^{|f(x)|}\frac{d\lambda}{\lambda}\Big)|f(x)|dm_\nu(x)$$
$$= m_\nu(B)\,|f|_B + \frac{1}{c}\int_{\{x\in B:|f(x)|>|f|_B\}} |f(x)|\log\frac{|f(x)|}{|f|_B}\,dm_\nu(x)$$
$$\geq \frac{1}{c}\int_B |f(x)|\Big(1+\log^+\frac{|f(x)|}{|f|_B}\Big)dm_\nu(x).$$

On the other hand,

$$\int_B M_\nu(f\chi_B)(x)dm_\nu(x) = \int_0^\infty m_\nu(\{x\in B : M(f\chi_B)(x) > \lambda\})d\lambda$$
$$\approx \int_0^\infty m_\nu(\{x\in B : M_\nu(f\chi_B)(x) > 2\lambda\})d\lambda$$
$$= \int_0^{|f|_B} m_\nu(\{x\in B : M_\nu(f\chi_B)(x) > 2\lambda\})d\lambda$$
$$+ \int_{|f|_B}^\infty m_\nu(\{x\in B : M_\nu(f\chi_B)(x) > 2\lambda\})d\lambda$$
$$\leq m_\nu(B)\,|f|_B + c\int_{|f|_B}^\infty \Big(\int_{\{x\in B:|f(x)|>\lambda\}} |f(x)|dm_\nu(x)\Big)\frac{d\lambda}{\lambda}$$
$$= m_\nu(B)\,|f|_B + c\int_{\{x\in B:|f(x)|>|f|_B\}} |f(x)|\log\frac{|f(x)|}{|f|_B}\,dm_\nu(x)$$
$$\leq c\int_B |f(x)|\Big(1+\log^+\frac{|f(x)|}{|f|_B}\Big)dm_\nu(x).$$

Therefore, for all $f\in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$ we get

$$M_\nu\big(M_\nu f\big)(x) \approx \sup_{B\ni x} m_\nu(B)^{-1}\int_B |f(x)|\Big(1+\log^+\frac{|f(x)|}{|f|_B}\Big)dm_\nu(x). \qquad (3.9)$$

Since

$$1 \leq \frac{1}{m_\nu(B)}\int_B |f(x)|\Big(1+\log^+\frac{|f(x)|}{|f|_B}\Big)dm_\nu(x),$$

then

$$|f|_B \le \|f\chi_B\|_{L(1+\log^+ L),\nu}.$$

Using the inequality $\log^+(ab) \le \log^+ a + \log^+$ with $a, b > 0$, we get

$$\begin{aligned}
&\frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{|f|_B}\Big) dm_\nu(x) \\
&= \frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \Big(\frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L),\nu}} \frac{\|f\chi_B\|_{L(1+\log^+ L),\nu}}{|f|_B}\Big)\Big) dm_\nu(x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L),\nu}}\Big) dm_\nu(x) \\
&+ \frac{1}{m_\nu(B)} \int_B |f(x)| \log^+ \frac{\|f\chi_B\|_{L(1+\log^+ L),\nu}}{|f|_B} dm_\nu(x) \\
&\le \|f\chi_B\|_{L(1+\log^+ L),\nu} + |f|_B \log^+ \frac{\|f\chi_B\|_{L(1+\log^+ L),\nu}}{|f|_B}.
\end{aligned}$$

Since $\frac{\|f\chi_B\|_{L(1+\log^+ L),\nu}}{|f|_B} \ge 1$ and $\log t \le t$ when $t \ge 1$, we get

$$\frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{|f|_B}\Big) dm_\nu(x) \le 2\|f\chi_B\|_{L(1+\log^+ L),\nu}. \tag{3.10}$$

On the other hand, since

$$\|f\chi_B\|_{L(1+\log^+ L),\nu} = \frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L),\nu}}\Big) dm_\nu(x),$$

on using

$$|f|_B \le \|f\chi_B\|_{L(1+\log^+ L),\nu},$$

we get that

$$\|f\chi_B\|_{L(1+\log^+ L),\nu} \lesssim \frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{|f|_B}\Big) dm_\nu(x). \tag{3.11}$$

Therefore, from (3.9), (3.10) and (3.11) we have (3.8).

For proving our main results, we need the following estimate.

Lemma 3.4 *Let $b \in BMO(\mathbb{R}, dm_\nu)$ Then there exists a positive constant C such that*

$$M_{b,\nu} f(x) \le C\|b\|_{BMO(\nu)}\, M_\nu\big(M_\nu f\big)(x) \tag{3.12}$$

for almost every $x \in \mathbb{R}$ and for all functions $f \in L_1^{\mathrm{loc}}(\mathbb{R}, dm_\nu)$.

Proof. Let $x \in \mathbb{R}$, $r > 0$, $B = B(x,r)$ and $\lambda B = B(x, \lambda r)$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{3B}(y)$, $f_2(y) = f(y)\chi_{\complement_{(3B)}}(y)$, and χ_{3B} denotes the characteristic function of $3B$. Then for any $y \in \mathbb{R}$

$$\begin{aligned} M_{b,\nu}f(y) &= M_\nu\big((b-b(y))f\big)(y) = M_\nu\big((b - b_{3B} + b_{3B} - b(y))f\big)(y) \\ &\le M_\nu\big((b-b_{3B})f\big)(y) + M_\nu\big((b_{3B}-b(y))f\big)(y) \\ &\le M_\nu\big((b-b_{3B})f_1\big)(y) + M_\nu\big((b-b_{3B})f_2\big)(y) + \big|b_{3B}-b(y)\big| M_\nu f(y). \end{aligned}$$

For $0 < \delta < 1$ we have

$$\begin{aligned} &\Big(\frac{1}{m_\nu(B)}\int_B \big(M_{b,\nu}f(y)\big)^\delta dm_\nu(y)\Big)^{\frac{1}{\delta}} \le \Big(\frac{1}{m_\nu(B)}\int_B \big(M_\nu\big((b-b_{3B})f_1\big)(y)\big)^\delta dm_\nu(y)\Big)^{\frac{1}{\delta}} \\ &+ \Big(\frac{1}{m_\nu(B)}\int_B \big(M_\nu\big((b-b_{3B})f_2\big)(y)\big)^\delta dm_\nu(y)\Big)^{\frac{1}{\delta}} \\ &+ \Big(\frac{1}{m_\nu(B)}\int_B \big|b(y)-b_{3B}\big|\big(M_\nu f)(y)\big)^\delta dm_\nu(y)\Big)^{\frac{1}{\delta}} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We first estimate I_1. Recall that M_ν is weak-type $(1,1)$, (cf. [5]). We have

$$\begin{aligned} I_1^\delta &\le \frac{1}{m_\nu(B)}\int_B \big|M_\nu\big((b-b_{3B})f_1\big)(y)\big|^\delta dm_\nu(y) \\ &\le \frac{1}{m_\nu(B)}\int_0^{m_\nu(B)} \big[\big(M_\nu\big((b-b_{3B})f_1\big)\big)^*(t)\big]^\delta dt \\ &\le \frac{1}{m_\nu(B)}\Big[\sup_{0<t<m_\nu(B)} t\,\big(M_\nu\big((b-b_{3B})f_1\big)\big)^*(t)\Big]^\delta \int_0^{m_\nu(B)} t^{-\delta}dt \\ &\lesssim \frac{1}{m_\nu(B)}\big\|(b-b_{3B})\,f_1\big\|_{L_{1,\nu}}^\delta m_\nu(B)^{-\delta+1} \\ &\lesssim \big\|(b-b_{3B})\,f\chi_{3B}\big\|_{L_{1,\nu}}^\delta m_\nu(B)^{-\delta}. \end{aligned}$$

Thus

$$I_1 \le m_\nu(B)^{-1}\int_{3B} |b(y)-b_{3B}|\,|f(y)|dm_\nu(y).$$

Then, by (3.5) and Lemmas 3.1 and 3.4 , we obtain

$$\begin{aligned} I_1 &\le \|b-b_{3B}\|_{\exp L,3B}\,\|f\|_{L(\log L),3B} \\ &\lesssim \|b\|_{BMO(\nu)}\,\|f\|_{L(\log L),3B} \\ &\le \|b\|_{BMO(\nu)}\; M_\nu\big(M_\nu f\big)(x). \end{aligned}$$

Let us estimate I_2. Since for any two points $x, y \in B$, we have

$$M_\nu\big((b-b_{3B})f\big)(y) \le CM_\nu\big((b-b_{3B})f\big)(x)$$

with C an absolute constant (see, for example, [2, p. 160]).

Therefore, by (3.5) and Lemma 3.4 we obtain

$$
\begin{aligned}
I_2 &= \Big(\frac{1}{m_\nu(B)}\int_B \big(M_\nu\big((b-b_{3B})f_2\big)(y)\big)^\delta dm_\nu(y)\Big)^{\frac{1}{\delta}} \\
&\lesssim M_\nu\big((b-b_{3B})f\big)(x) \\
&= \sup_{B\ni x} m_\nu(B)^{-1}\int_B |b(y)-b_{3B}|\,|f(y)|dm_\nu(y) \\
&\le \sup_{B\ni x}\|b-b_{3B}\|_{\exp L,3B}\,\|f\|_{L(\log L),3B} \\
&\lesssim \|b\|_{BMO(\nu)}\sup_{B\ni x}\|f\|_{L(\log L),3B} \\
&\le \|b\|_{BMO(\nu)}\,M_\nu\big(M_\nu f\big)(x).
\end{aligned}
$$

Therefore we get

$$I_2 \lesssim \|b\|_{BMO(\nu)}\,M_\nu\big(M_\nu f\big)(x).$$

Finally, for estimate I_3, applying Hölders inequality with exponent $a = 1/\delta$, $0 < \delta < 1$, by Lemmas 3.2 for $\Phi(t) = t^a$, $1 < a < \infty$ we get

$$
\begin{aligned}
I_3 &\le \Big(\frac{1}{m_\nu(B)}\int_B \big|b(y)-b_{3B}\big|^a dm_\nu(y)\Big)^{\frac{1}{a}}\frac{1}{m_\nu(B)}\int_B M_\nu f(y)dm_\nu(y) \\
&\lesssim \|b\|_{BMO(\nu)}\,M_\nu\big(M_\nu f\big)(x).
\end{aligned}
$$

Lemma 3.4 is proved by the estimate of I_1, I_2, I_3 and the Lebesgue differentiation theorem.

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M_{b,\nu}$ on $L_\Phi(\mathbb{R}, dm_\nu)$, when b belongs to the $BMO(\nu)$ space.

Theorem 3.4 *Let $b \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$ and $\Phi \in \mathcal{Y}$ be a Young function.*
1. *If $\Phi \in \nabla_2$, then the condition $b \in BMO(\mathbb{R}, dm_\nu)$ is sufficient for the boundedness of $M_{b,\nu}$ on $L_\Phi(\mathbb{R}, dm_\nu)$.*
2. *The condition $b \in BMO(\mathbb{R}, dm_\nu)$ is necessary for the boundedness of $M_{b,\nu}$ on $L_\Phi(\mathbb{R}, dm_\nu)$.*
3. *If $\Phi \in \nabla_2$, then the condition $b \in BMO(\mathbb{R}, dm_\nu)$ is necessary and sufficient for the boundedness of $M_{b,\nu}$ on $L_\Phi(\mathbb{R}, dm_\nu)$.*

Proof. 1. Let $b \in BMO(\mathbb{R}, dm_\nu)$. Then from Lemma 3.12 we have

$$M_{b,\nu}f(x) \lesssim \|b\|_{BMO(\nu)}\,M_\nu\big(M_\nu f\big)(x) \tag{3.13}$$

for almost every $x \in \mathbb{R}$ and for all functions from $f \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$.

Combining Theorem 3.1, Lemma 3.4 and from (3.13), we get

$$
\begin{aligned}
\|M_{b,\nu}f\|_{L_{\Phi,\nu}} &\lesssim \|b\|_{BMO(\nu)}\|M_\nu\big(M_\nu f\big)\|_{L_{\Phi,\nu}} \\
&\lesssim \|b\|_{BMO(\nu)}\|M_\nu f\|_{L_{\Phi,\nu}} \\
&\lesssim \|b\|_{BMO(\nu)}\|f\|_{L_{\Phi,\nu}}.
\end{aligned}
$$

.

2. We shall now prove the second part. Suppose that $M_{b,\nu}$ is bounded from $L_\Phi(\mathbb{R}, dm_\nu)$ to $L_\Psi(\mathbb{R}, dm_\nu)$. Choose any ball $B = B(x,r)$ in $\mathbb{R}$, by Lemma 2.1 and (2.1)

$$\frac{1}{m_\nu(B)}\int_B |b(y)-b_B|dm_\nu(y) = \frac{1}{m_\nu(B)}\int_B \Big|\frac{1}{m_\nu(B)}\int_B (b(y)-b(z))dm_\nu(z)\Big|dm_\nu(y)$$
$$\leq \frac{1}{m_\nu(B)^2}\int_B\int_B |b(y)-b(z)|dm_\nu(z)dm_\nu(y)$$
$$= \frac{1}{m_\nu(B)^1}\int_B \frac{1}{m_\nu(B)}\int_B |b(y)-b(z)|\chi_B(z)dm_\nu(z)dm_\nu(y)$$
$$\leq \frac{1}{m_\nu(B)}\int_B M_{b,\nu}(\chi_B)(y)dm_\nu(y)$$
$$\leq \frac{2}{m_\nu(B)}\,\|M_{b,\nu}(\chi_B)\|_{L_\Phi(B)}\|1\|_{L_{\widetilde{\Phi}}(B)} \leq C.$$

Thus $b \in BMO(\mathbb{R}, dm_\nu)$.

3. The third statement of the theorem follows from the first and second parts of the theorem.

If we take $\Phi(t) = t^p$ in Theorem 3.4 we get the following corollary.

Corollary 3.1 *Let $1 < p < \infty$ and $b \in L_1^{\mathrm{loc}}(\mathbb{R}, dm_\nu)$. Then $M_{b,\nu}$ is bounded on $L_p(\mathbb{R}, dm_\nu)$ if and only if $b \in BMO(\mathbb{R}, dm_\nu)$.*

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