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Integration of the loaded Korteweg-de Vries equation with a self-consistent source in the class of rapidly decreasing complex-valued functions

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Abstract. In this work, the inverse scattering method is applied to the integration of the loaded Korteweg-de Vries equation with a self-consistent source in the class of rapidly decreasing complex-valued functions. An example illustrating the described method is given.

Keywords. loaded Korteweg-de Vries equation, Sturm-Liouville operator, Jost solutions, scattering data, inverse problem of scattering theory, Gelfand-Levitan-Marchenko integral equation.

Mathematics Subject Classification (2010): 34L25, 35P25, 47A40, 37K15

1 Introduction

In 1967, American scientists C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura [1] showed that the solution of the Korteweg-de Vries (KdV) equation can be obtained for all "rapidly decreasing" initial conditions, that is, conditions which in a certain way vanish as the coordinate tends to infinity. This method is called the inverse scattering method (ISM), since it essentially uses the solution of the problem of reconstructing the potential of the Sturm-Liouville operator on the entire axis, from the scattering data. This inverse scattering problem was first solved by L.D. Faddeev [2], then in the works of V.A. Marchenko [3], B.M. Levitan [4] and others. Further, P. Lax [5] noticed the universality of the ISM and generalized the KdV equation by introducing the concept of the higher KdV equation. In this direction, the next important result was obtained by V.E. Zakharov and A.B. Shabat [6], who succeeded in integrating the non-linear Schrdinger equation (NLS). Soon M. Wadati [7], based on the ideas of [6], proposed a method for solving the Cauchy problem for the modified Korteweg-de Vries equation (mKdV). V.E. Zakharov, L.A. Takhtadzhyan, L.D. Faddeev [8], and M. Ablowitz, D. Kaup, A. Newell, H. Sigur [9] showed that the ISM can also be applied to the solution of the sine-Gordon equation. The application of the ISM to

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the NLS equation, mKdV and sine-Gordon equations is based on the scattering problem for the Dirac operator on the entire axis:

$$M = i\begin{pmatrix} \frac{d}{dx} & -q(x) \\ r(x) & -\frac{d}{dx} \end{pmatrix}, \quad x \in \mathbb{R}.$$

The inverse scattering problem for the Dirac operator on the entire axis was studied in [10], [11]. It is known that the operator M is not self-adjoint, it has a finite number of multiple complex eigenvalues in the "rapidly decreasing" case and may have spectral singularities that lie in the continuous spectrum. In [6], [7], [8], [9] in the case when all eigenvalues of the corresponding Dirac operator are simple and there are no spectral singularities, nonlinear equations such as NSE, mKdV, and sine-Gordon were integrated. In this regard, it is relevant to search for a solution of nonlinear evolution equations without a source and with a source corresponding to multiple eigenvalues of the Dirac operator. The articles [12], [13], [14], [15] are devoted to these problems.

In [16], V.K. Melnikov showed that the KdV equation with a self-consistent source can be solved using the ISM for the self-adjoint Sturm-Liouville operator on the whole line. Integrable non-linear evolution equations with sources have attracted a lot of attention in the modern scientific literature. They have important applications in plasma physics, hydrodynamics, solid state physics, etc. [17], [18], [19]. For example, the KdV equation with an integral source was considered in [20]. These equations can describe the interaction of long and short capillary-gravity waves [21].

For the first time the term "loaded equation" was used in the works of A.M. Nakhushev, where the most general definition of a loaded equation is given and various loaded equations are classified in detail, for example, loaded differential, integral, integro-differential, functional equations, etc., and numerous applications are described. In the literature, loaded differential equations are usually called equations containing in the coefficients or on the right side any functionals of the solution, in particular, the values of the solution or its derivatives on manifolds of lower dimension. The study of such equations is of interest both from the point of view of constructing a general theory of differential equations and from the point of view of applications. Among the works devoted to loaded equations, one should especially note the works of A.M. Nakhushev [22], [23], A.I. Kozhanov [24] and others.

The KdV equation without a loaded term is also encountered in applied mechanics. For example, in the works of A.A. Lugovtsov [25], [26], the system of equations describing the propagation of one-dimensional nonlinear waves in an inhomogeneous gas-liquid medium is reduced to one equation of the form

$$u_\tau + \alpha(\tau)uu_\eta + \beta(\tau)u_{\eta\eta\eta} - \mu(\tau)u_{\eta\eta} + \left[\frac{k}{2\tau} + \delta(\tau)\right]u = 0.$$

In particular, for $\mu = 0,\ k = 1,\ \delta = 0$, it is shown that under certain conditions, cylindrical waves can exist in the form of solitons.

Note that solutions of the KdV equation with a self-consistent source from the class of rapidly decreasing complex functions were considered in [27]. Integration of the loaded Korteweg-de Vries equation into the class of periodic functions was investigated in [28], [29].

In this paper, we consider a system of loaded nonlinear equations of the form

$$u_t - 6uu_x + u_{xxx} + \gamma(t)u(0,t)u_x = 2\sum_{j=1}^{N}\sum_{l=0}^{m_j-1} C^l_{m_j-1}\frac{\partial}{\partial x}\Big(\varphi^l_j\varphi_j^{m_j-1-l}\Big), \tag{1.1}$$

$$L(t)\varphi^l_j = k_j^2\varphi^l_j + l\varphi^{l-1}_j, \ \ (\mathrm{Im}k_j > 0), \ \ l = \overline{0, m_j-1}, \ \ j = \overline{1, N}, \tag{1.2}$$

where

$$L(t) := -\frac{d^2}{dx^2} + u(x,t), \;\; x \in \mathbb{R}, \;\; t > 0, \;\; C_n^l = \frac{n!}{l!(n-l)!},$$

and $\gamma(t)$ is a given continuously differentiable function. The functions $\varphi_j^l = \varphi_j^l(x,t)$ for each nonnegative t belong to the space $L^2(\mathbb{R})$, and $\varphi_j^0 = \varphi_j^0(x,t)$ is an eigenfunction of the operator $L(t)$ corresponding to the eigenvalue $\lambda_j(t) = k_j^2(t)$, $(\mathrm{Im}k_j > 0)$ of multiplicity $m_j(t), \;\; l = \overline{0, m_j - 1}, \;\; j = \overline{1, N}$.

The system of equations (1.1) - (1.2) is considered under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.3}$$

where the initial function $u_0(x)$ is complex-valued and has the following properties:
1) for some $\varepsilon > 0$

$$\int_{-\infty}^{\infty} |u_0(x)|\, e^{\varepsilon|x|} dx < \infty; \tag{1.4}$$

2) non-self-adjoint operator $L(0)$ has N complex eigenvalues $\lambda_1(0), \lambda_2(0), \ldots, \lambda_N(0)$ with multiplicities $m_1(0), m_2(0), \ldots, m_N(0)$ respectively, and has no spectral singularities.

It is assumed that

$$\frac{1}{(m_j - 1 - l)!} \int_{-\infty}^{\infty} \varphi_j^{m_j-1}(x,t) \varphi_j^{m_j-1-l}(x,t) dx = A^j_{m_j-1-l}(t), \qquad l = \overline{0, m_j - 1}, \;\; j = \overline{1, N}. \tag{1.5}$$

Here, $A^j_{m_j-1-l}(t)$ are initially given continuous functions.

It is required to find a complex-valued function $u(x,t)$ that is sufficiently smooth and tends to its limits as $x \to \pm\infty$ rather quickly, i.e.

$$\int_{-\infty}^{\infty} \left| \frac{\partial^j u(x,t)}{\partial x^j} \right| e^{\varepsilon|x|} dx < \infty, \quad j = 0, 1, 2, 3. \tag{1.6}$$

The main goal of this work is to find the equation of dynamics in time t for the scattering data of a non-self-adjoint operator $L(t)$ with a potential that is a solution to the loaded KdV equation with a self-consistent source in the class of rapidly decreasing complex-valued functions.

2 Preliminaries. Scattering data for a non-self-adjoint Sturm-Liouville operator

Consider the equation

$$L(0)y := -y'' + u_0(x)y = k^2 y, \quad x \in \mathbb{R}, \tag{2.1}$$

where the potential $u_0(x)$ is assumed to be complex-valued and satisfies condition (1.4). In this section, we present the information necessary for the further presentation concerning the direct and inverse scattering problems for equation (2.1). We denote by $f_+(x,k)$ and $f_-(x,k)$ the solutions of Eq. (2.1) with conditions at infinity for $\mathrm{Im}k > -\frac{\varepsilon}{2}$:

$$f_+(x,k) = e^{ikx} + o(1), \;\; x \to +\infty; \quad f_-(x,k) = e^{-ikx} + o(1), \;\; x \to -\infty.$$

These solutions are called Jost solutions and the following representations are valid for them:

$$f_{\pm}(x,k) = e^{\pm ikx} \pm \int_x^{\pm\infty} K_{\pm}(x,y)e^{\pm iky}dy. \tag{2.2}$$

These solutions, under condition (1.4), exist, are unique, and holomorphic in k in the half-plane $\mathrm{Im}k > -\frac{\varepsilon}{2}$. Moreover, the kernels $K_{\pm}(x,y)$ have continuous derivatives that satisfy inequalities

$$|K_+(x,y)| \leq C_a^+ e^{-\varepsilon\frac{x+y}{2}}, \quad y \geq x \geq a,$$

$$|K_-(x,y)| \leq C_a^- e^{\varepsilon\frac{x+y}{2}}, \quad y \leq x \leq a,$$

$$\left|K'_{+x}(x,y)\right|, \left|K'_{+y}(x,y)\right| \leq \frac{1}{4}\left|u_0\left(\frac{x+y}{2}\right)\right| + C_a^+ e^{-\varepsilon\left(\frac{3x}{2}+y\right)}, \quad x > a.$$

In addition, the kernals $K_{\pm}(x,y)$ are related to the potential $u_0(x)$ as follows:

$$u_0(x) = \mp 2\frac{dK_{\pm}(x,x)}{dx}. \tag{2.3}$$

We also note that pairs of functions $\{f_{\pm}(x,k), f_{\pm}(x,-k)\}$ form a system of fundamental solutions in the strip $|\mathrm{Im}k| < \frac{\varepsilon}{2}$ whose Wronskians are equal to $W\{f_{\pm}(x,k), f_{\pm}(x,-k)\} = \mp 2ik$.

We denote by $w(k)$ and $v(k)$ Wronskians

$$w(k) := f_-(x,k)f'_+(x,k) - f'_-(x,k)f_+(x,k),$$
$$v(k) := f_+(x,-k)f'_-(x,k) - f_-(x,k)f'_+(x,-k).$$

The function $w(k)$ extends analytically to the half-plane $\mathrm{Im}k > -\frac{\varepsilon}{2}$ and has the asymptotics

$$w(k) = 2ik\left[1 + O\left(\frac{1}{k}\right)\right], \quad |k| \to \infty, \tag{2.4}$$

uniformly in each half-plane $\mathrm{Im}k \geq \eta,\ \eta > -\frac{\varepsilon}{2}$. It follows from the asymptotics (2.4) and the analyticity of $w(k)$ that in the half-plane $\mathrm{Im}k \geq 0$ the function $w(k)$ has a finite number of zeros (in the general case, multiple ones). The requirement that there are no spectral singularities for the operator $L(0)$ means that the function $w(k)$ does not have real zeros, that is, $w(k) \neq 0$, $k \in \mathbb{R}$. Let the non-real zeros $w(k)$ be $k_1,\ k_2, \ldots, k_N$ ($\mathrm{Im}k_j > 0$, $j = \overline{1,N}$), then $\lambda_j = k_j^2$, $j = \overline{1,N}$ are the eigenvalues of the operator $L(0)$. The multiplicity of the root k_j of the equation $w(k) = 0$ is denoted by $m_j,\ j = \overline{1,N}$.

Unlike $w(k)$, the function $v(k)$ is defined only in the strip $|\mathrm{Im}k| < \frac{\varepsilon}{2}$. In the strip $|\mathrm{Im}k| < \frac{\varepsilon}{2}$ functions $w(k)$ and $v(k)$ satisfy the equality

$$w(k)w(-k) - v(k)v(-k) = 4k^2. \tag{2.5}$$

In addition, in the strip $|\mathrm{Im}k| < \frac{\varepsilon}{2}$ the following equality holds:

$$f_-(x,k) = \frac{v(k)}{2ik}f_+(x,k) + \frac{w(k)}{2ik}f_+(x,-k). \tag{2.6}$$

There exist so-called normalizing chains of numbers $\{\chi_0^j,\ \chi_1^j,\dots,\chi_{m_j-1}^j\}$ and $\{\theta_0^j,\ \theta_1^j,\dots,$ $\theta_{m_j-1}^j\}$, $j=\overline{1,N}$ such that the following relations hold

$$\frac{1}{s!}\Big(\Big(\frac{d}{dk}\Big)^s f_-(x,k)\Big)\Big|_{k=k_j}=\sum_{\nu=0}^{s}\chi_{s-\nu}^j\frac{1}{\nu!}\Big(\Big(\frac{d}{dk}\Big)^\nu f_+(x,k)\Big)\Big|_{k=k_j},$$
$$\frac{1}{s!}\Big(\Big(\frac{d}{d\lambda}\Big)^s f_-(x,\sqrt{\lambda})\Big)\Big|_{\lambda=k_j^2}=\sum_{\nu=0}^{s}\theta_{s-\nu}^j\frac{1}{\nu!}\Big(\Big(\frac{d}{d\lambda}\Big)^\nu f_+(x,\sqrt{\lambda})\Big)\Big|_{k=k_j^2},\tag{2.7}$$
$$s=\overline{0,m_j-1},\quad j=\overline{1,N},$$

while $\chi_0^j\neq 0,\ \theta_0^j\neq 0$.

Normalizing chains of numbers $\{\chi_0^j,\chi_1^j,...,\chi_{m_j-1}^j\}$ and $\{\theta_0^j,\theta_1^j,\dots,\theta_{m_j-1}^j\}$, $j=\overline{1,N}$ are interconnected by means of recurrence relations.

As it is known in [30], [31], the kernel $K_+(x,y)$ of the transformation operator (2.2) satisfies the Gelfand-Levitan-Marchenko integral equation

$$K_+(x,y)+F_+(x+y)+\int_x^\infty K_+(x,s)F_+(s+y)ds=0,\quad x\le y,\tag{2.8}$$

where

$$F_+(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(k)e^{ikx}dk+\sum_{j=1}^{N}\sum_{\nu=0}^{m_j-1}\chi_{m_j-\nu-1}^j\frac{1}{\nu!}\frac{d^\nu}{dk^\nu}\left(\frac{2k(k-k_j)^{m_j}}{\omega(k)}e^{ikx}\right),\tag{2.9}$$

$$S(k):=\frac{v(k)}{\omega(k)},\tag{2.10}$$

herewith the potential $u_0(x)$ is found by formula (2.3).

Definition 2.1 *The set $\{S(k),\ \lambda_j,\ \chi_0^j,\dots,\chi_{m_j-1}^j,\ j=\overline{1,N}\}$ or $\{S(k),\ \lambda_j,\ \theta_0^j,\dots,\theta_{m_j-1}^j,$ $j=\overline{1,N}\}$ is called the scattering data for the operator $L(0)$.*

The problem that implies the determination of the complex-valued potential $u_0(x)$ from scattering data is called the inverse problem.

The following theorem is true [30].

Theorem 2.1 *The scattering data uniquely determine operator $L(0)$.*

In what follows, we will often use the results of the following lemmas.

Lemma 2.1 *If the functions $y(x,\zeta)$ and $z(x,\eta)$ are solutions of the equations $Ly=\zeta^2y$ and $Lz=\eta^2z$, then the equality*

$$\frac{d}{dx}W\{y,z\}=(\zeta^2-\eta^2)yz,$$

is true.

Lemma 2.2 *Let the functions f_-, φ_j^l, $l=0,1,\dots,m_j-1$ be solutions of the following equations*

$$Le_-=\lambda e_-;\quad L\varphi_j^l=\lambda_j\varphi_j^l+l\varphi_j^{l-1},\quad l=0,1,\dots,m_j-1,\quad \lambda=k^2.$$

Then the equality

$$\frac{d}{dx}W\{\varphi_j^l,f_j\}=(\lambda_j-\lambda)\varphi_j^lf_-+l\varphi_j^{l-1}f_-,$$

is true.

Corollary 2.1 *Under the conditions of Lemma 2.2 and $\lambda \neq \lambda_j$, the following equalities*

$$\begin{aligned} \varphi_j^l f_- &= \sum_{r=0}^{l} \frac{1}{(\lambda-\lambda_j)^{r+1}} \cdot \frac{l!}{(l-r)!} \frac{d}{dx} W\{f_-, \varphi_j^{l-r}\}, \\ \varphi_j^{m_j-1-l} f_- &= \sum_{r=0}^{m_j-1-l} \frac{(m_j-1-l)!}{(\lambda-\lambda_j)^{m_j-r}(m_j-1-l-r)!} \frac{d}{dx} W\{f_-, \varphi_j^{m_j-1-l-r}\}, \end{aligned} \tag{2.11}$$

hold.

Differentiating equalities (2.11) n times with respect to λ, and setting $\lambda = \lambda_j$, we obtain the following corollary.

Corollary 2.2 *The following equalities take place*

$$\begin{gathered} \varphi_j^{l-1} \cdot f_-^{(n)}(x,k_j) = \frac{n}{l} \varphi_j^l \cdot f_-^{(n-1)}(x,k_j) - \frac{1}{l} \frac{d}{dx} W\{f_-^{(n)}(x,k_j), \varphi_j^l(x,k_j)\}, \\ l = \overline{1, m_j-1}. \end{gathered} \tag{2.12}$$

The following lemma can be proved by direct verification.

Lemma 2.3 *If φ_j is the eigenfunction of the operator $L(0)$ with the potential $u_0(x)$ that corresponds to the eigenvalue k_j^2, then the equalities*

$$\int_{-\infty}^{\infty} u_0(x)\varphi_j'\varphi_j dx = 0, \qquad \int_{-\infty}^{\infty} u_0'(x)\varphi_j^2 dx = 0,$$

hold.

3 Evolution of the scattering data of a non-self-adjoint Sturm-Liouville operator

Consider the following KdV equation with the right-hand side

$$u_t - 6uu_x + u_{xxx} = G(x,t), \tag{3.1}$$

where

$$G(x,t) = -\gamma(t)u(0,t)u_x + 2\sum_{j=1}^{N}\sum_{l=0}^{m_j-1} C_{m_j-1}^l \frac{\partial}{\partial x}\left(\varphi_j^l \varphi_j^{m_j-1-l}\right). \tag{3.2}$$

For equation (3.1), we look for the Lax pair [32] in the following form

$$-\Phi_{xx} + (u-\lambda)\Phi = 0, \tag{3.3}$$

$$\Phi_t = (-u_x + 4i\lambda\sqrt{\lambda})\Phi + (2u+4\lambda)\Phi_x + F(x,t). \tag{3.4}$$

Using identity $\Phi_{xxt} = \Phi_{txx}$, based on equalities (3.1) - (3.4), we obtain

$$-F_{xx} + (u(x,t)-\lambda)F = -G\Phi. \tag{3.5}$$

Assuming $\Phi(x,t) = f_-(x,\sqrt{\lambda};t)$, we are looking for a solution to equation (3.5) in the form

$$F = B(x)f_-(x,\sqrt{\lambda};t) + C(x)f_-(x,-\sqrt{\lambda};t).$$

Then, to determine $B(x)$ and $C(x)$, we obtain the system of equations

$$B'(x)f_-(x,\sqrt{\lambda};t)+C'(x)f_-(x,-\sqrt{\lambda};t)=0,$$

$$B'(x)f'_-(x,\sqrt{\lambda};t)+C'(x)f'_-(x,-\sqrt{\lambda};t)=Gf_-(x,\sqrt{\lambda};t),$$

whose the solution has the form

$$B(x)=-\frac{1}{2i\sqrt{\lambda}}\int_{-\infty}^{x}f_-(s,\sqrt{\lambda};t)f_-(s,-\sqrt{\lambda};t)Gds,$$

$$C(x)=\frac{1}{2i\sqrt{\lambda}}\int_{-\infty}^{x}f_-^2(s,\sqrt{\lambda};t)Gds.$$

Therefore, in this case, the second equation of the Lax pair has the form

$$\begin{aligned}\frac{\partial f_-(x,\sqrt{\lambda};t)}{\partial t}&=(-u_x+4i\lambda\sqrt{\lambda})f_-(x,\sqrt{\lambda};t)+(2u+4\lambda)\frac{\partial f_-(x,\sqrt{\lambda};t)}{\partial x}\\&-\frac{f_-(x,\sqrt{\lambda};t)}{2i\sqrt{\lambda}}\int_{-\infty}^{x}f_-(s,\sqrt{\lambda};t)f_-(s,-\sqrt{\lambda};t)Gds\\&+\frac{f_-(x,-\sqrt{\lambda};t)}{2i\sqrt{\lambda}}\int_{-\infty}^{x}f_-^2(s,\sqrt{\lambda};t)Gds.\end{aligned}\tag{3.6}$$

Passing to the limit $x\to\infty$ in equality (3.6), by virtue of (2.5), (2.6) and the asymptotics of the Jost solution, we derive

$$\begin{aligned}\frac{dw(\sqrt{\lambda};t)}{dt}&=-\frac{w(\sqrt{\lambda};t)}{2i\sqrt{\lambda}}\int_{-\infty}^{\infty}f_-(x,\sqrt{\lambda};t)f_-(x,-\sqrt{\lambda};t)G(x,t)dx\\&-\frac{v(-\sqrt{\lambda};t)}{2i\sqrt{\lambda}}\int_{-\infty}^{\infty}f_-^2(x,\sqrt{\lambda};t)G(x,t)dx,\end{aligned}\tag{3.7}$$

$$\begin{aligned}\frac{dv(\sqrt{\lambda};t)}{dt}&=8i\lambda\sqrt{\lambda}v(\sqrt{\lambda};t)-\frac{v(\sqrt{\lambda};t)}{2i\sqrt{\lambda}}\int_{-\infty}^{\infty}f_-(x,\sqrt{\lambda};t)f_-(x,-\sqrt{\lambda};t)G(x,t)dx\\&-\frac{w(-\sqrt{\lambda};t)}{2i\sqrt{\lambda}}\int_{-\infty}^{\infty}f_-^2(x,\sqrt{\lambda};t)G(x,t)dx.\end{aligned}\tag{3.8}$$

Multiplying (3.8) by w and subtracting it from equality (3.7) multiplied by v, according to (2.10), we obtain

$$\frac{dS(\sqrt{\lambda};t)}{dt}=8i\lambda\sqrt{\lambda}S(\sqrt{\lambda};t)-\frac{2i\sqrt{\lambda}}{w(\sqrt{\lambda};t)}\int_{-\infty}^{\infty}f_-^2(x,\sqrt{\lambda};t)G(x,t)dx.$$

Lemma 3.1 *The following identities hold*

$$\int_{-\infty}^{\infty}G(x,t)f_-^2(x,\sqrt{\lambda};t)dx=-\gamma(t)u(0,t)v(\sqrt{\lambda};t)w(\sqrt{\lambda};t),$$

$$\int_{-\infty}^{\infty}G(x,t)f_-(x,\sqrt{\lambda};t)f_-(x,-\sqrt{\lambda};t)dx=\gamma(t)u(0,t)v(\sqrt{\lambda};t)v(-\sqrt{\lambda};t),\tag{3.9}$$

where the function $G(x,t)$ is defined by equality (3.2).

Proof. Indeed, using expression (3.2), we have

$$\int_{-\infty}^{\infty} G f_-^2(x,\sqrt{\lambda};t)dx = -\gamma(t)u(0,t)\int_{-\infty}^{\infty} f_-^2(x,\sqrt{\lambda};t)u_x(x,t)dx$$

$$+2\int_{-\infty}^{\infty}\sum_{j=1}^{N}\sum_{l=0}^{m_j-1} C_{m_j-1}^{l}\frac{\partial}{\partial x}\left(\varphi_j^l\varphi_j^{m_j-1-l}\right) f_-^2(x,\sqrt{\lambda};t)dx$$

$$= 2\gamma(t)u(0,t)\int_{-\infty}^{\infty}\left(f_-''(x,\sqrt{\lambda};t)+\lambda f_-(x,\sqrt{\lambda};t)\right) f_-'(x,\sqrt{\lambda};t)dx$$

$$+\int_{-\infty}^{\infty}\left(\sum_{j=1}^{N}\sum_{l=0}^{m_j-1} C_{m_j-1}^{l}\Big[\varphi_j^l f_-^2\frac{\partial}{\partial x}\varphi_j^{m_j-1-l}+\varphi_j^{m_j-1-l} f_-^2\frac{\partial}{\partial x}\varphi_j^{m_j-1-l}\varphi_j^l\right.$$

$$\left.-2\varphi_j^l\varphi_j^{m_j-1-l}\frac{\partial}{\partial x}f_-\Big]\right)dx = \gamma(t)u(0,t)\int_{-\infty}^{\infty}\left[\left(\left(f_-'(x,\sqrt{\lambda};t)\right)^2\right)' + k^2\left(f_-^2(x,\sqrt{\lambda};t)\right)'\right]dx$$

$$= \int_{-\infty}^{\infty}\left(\sum_{j=1}^{N}\sum_{l=0}^{m_j-1} C_{m_j-1}^{l}\Big[\varphi_j^l f_- W\{f_-,\varphi_j^{m_j-1-l}\}+\varphi_j^{m_j-1-l} f_- W\{f_-,\varphi_j^l\}\Big]\right)dx.$$

According to (2.11), we obtain

$$\int_{-\infty}^{\infty} G f_-^2(x,\sqrt{\lambda};t)dx = \gamma(t)u(0,t)\lim_{R\to\infty}\left[k^2 f_-^2(x,k;t)+\left(f_-'(x,k;t)\right)^2\right]\Big|_{-R}^{R}$$

$$+\int_{-\infty}^{\infty}\left[\sum_{j=1}^{N}\sum_{l=0}^{m_j-1} C_{m_j-1}^{l}\sum_{r=0}^{l}\frac{l!}{(l-r)!(\lambda-\lambda_j)^{r+1}}\frac{d}{dx}\left(W\left\{f_-,\varphi_j^{l-r}\right\}\right)\right.$$

$$\left.+\sum_{r=0}^{m_j-1-l}\frac{(m_j-1-l)!}{(m_j-1-l-r)!(\lambda-\lambda_j)^{r+1}} W\left\{f_-,\varphi_j^l\right\}\frac{d}{dx}\left(W\{f_-,\varphi_j^{m_j-1-l-r}\}\right)\right]dx$$

$$= \gamma(t)u(0,t)\lim_{R\to\infty}\left[\lambda\left(\frac{v(\sqrt{\lambda};t)}{2i\sqrt{\lambda}}e^{i\sqrt{\lambda}R}+\frac{w(\sqrt{\lambda};t)}{2i\sqrt{\lambda}}e^{-i\sqrt{\lambda}R}\right)^2-\lambda e^{2i\sqrt{\lambda}R}\right.$$

$$\left.+\left(\frac{v(\sqrt{\lambda};t)}{2}e^{i\sqrt{\lambda}R}-\frac{w(\sqrt{\lambda};t)}{2}e^{-i\sqrt{\lambda}R}\right)^2+\lambda e^{2i\sqrt{\lambda}R}\right]$$

$$+\int_{-\infty}^{\infty}\left(\sum_{j=1}^{N}\sum_{l=0}^{m_j-1} C_{m_j-1}^{l}\sum_{r=0}^{m_j-1-l}\frac{(m_j-1-l)!}{(m_j-1-l-r)!(\lambda-\lambda_j)^{r+1}}\right.$$

$$\left.\times\frac{d}{dx}\Big(W\{f_-,\varphi_j^l\}W\{f_-,\varphi_j^{m_j-1-l-r}\}\Big)\right)dx = -\gamma(t)u(0,t)v(\sqrt{\lambda};t)w(\sqrt{\lambda};t).$$

Equality (3.9) is proved similarly. This is complete the proof.

According to Lemma 3.1 and equality (3.7), we have $w_t(\sqrt{\lambda},t)=0$. Therefore, we deduce that

$$\frac{d\lambda_j(t)}{dt}=0, \tag{3.10}$$

$$S_t(\sqrt{\lambda},t)=\left[8i\lambda\sqrt{\lambda}-2i\sqrt{\lambda}\gamma(t)u(0,t)\right]S(\sqrt{\lambda},t). \tag{3.11}$$

Now we turn to finding the evolution of the normalization chain $\{\theta_0^n,\theta_1^n,\ldots,\theta_{m_n-1}^n\}$ corresponding to $\lambda_n, n=\overline{1,N}$. For this, we rewrite equality (3.6) in the following form

$$\frac{\partial f_-(x,\sqrt{\lambda};t)}{\partial t}=(-u_x+4i\lambda\sqrt{\lambda})f_-(x,\sqrt{\lambda};t)+(2u+4\lambda)\frac{\partial f_-(x,\sqrt{\lambda};t)}{\partial x}$$
$$-\frac{1}{2i\sqrt{\lambda}}\left[f_-(x,\sqrt{\lambda};t)\int_{-\infty}^{x}f_-(s,\sqrt{\lambda};t)f_-(s,-\sqrt{\lambda};t)G(s,t)ds\right.$$
$$\left.-f_-(x,\sqrt{\lambda};t)\int_{-\infty}^{x}f_-^2(s,\sqrt{\lambda};t)G(s,t)ds\right]=(-u_x+4i\lambda\sqrt{\lambda})f_-+(2u+4\lambda)\frac{\partial f_-(x,\sqrt{\lambda};t)}{\partial x}$$
$$+\frac{\gamma(t)u(0,t)f_-(x,\sqrt{\lambda};t)}{2i\sqrt{\lambda}}\left[f_-(x,\sqrt{\lambda};t)f_-(x,-\sqrt{\lambda};t)u(x,t)\right.$$
$$\left.-\int_{-\infty}^{x}u(s,t)\left(f_-'(s,\sqrt{\lambda};t)f_-(s,-\sqrt{\lambda};t)+f_-(s,\sqrt{\lambda};t)f_-'(s,-\sqrt{\lambda};t)\right)ds\right]$$
$$-\frac{\gamma(t)u(0,t)f_-(x,-\sqrt{\lambda};t)}{2i\sqrt{\lambda}}\left[f_-^2(x,\sqrt{\lambda};t)u(x,t)-\int_{-\infty}^{x}2f_-'(s,\sqrt{\lambda};t)f_-(s,\sqrt{\lambda};t)u(s,t)ds\right]$$
$$+\int_{-\infty}^{x}\sum_{j=1}^{N}\sum_{l=0}^{m_j-1}C_{m_j-1}^{l}\sum_{r=0}^{m_j-1-l}\frac{(m_j-1-l)!}{(m_j-1-l-r)!(\lambda-\lambda_j)^{r+1}}$$
$$\times\frac{d}{dx}(W\{f_-(s,\sqrt{\lambda};t),\varphi_j^{m_j-1-l-r}\})ds\cdot\varphi_j^l=(-u_x+4i\lambda\sqrt{\lambda})f_-(x,\sqrt{\lambda};t)$$
$$+(2u+4\lambda)\frac{\partial f_-(x,\sqrt{\lambda};t)}{\partial x}-\gamma(t)u(0,t)f_-'(x,\sqrt{\lambda};t)-i\sqrt{\lambda}\gamma(t)u(0,t)f_-(x,\sqrt{\lambda};t)$$
$$+\varphi_j^l\int_{-\infty}^{x}\left(\sum_{j=1}^{N}\sum_{l=0}^{m_j-1}C_{m_j-1}^{l}\varphi_j^{m_j-1-l}f_-(s,\sqrt{\lambda};t)\right)ds. \tag{3.12}$$

Differentiating equality (3.12) m_n-1 times with respect to λ, setting $\lambda=\lambda_n$ and taking into account the asymptotics of the Jost solution at $x\to+\infty$, we obtain

$$\frac{\partial f_-^{(m_n-1)}(x,\sqrt{\lambda_n};t)}{\partial t}=4i\Big[(\lambda_n)^{\frac{3}{2}}f_-^{(m_n-1)}(x,\sqrt{\lambda_n};t)+\frac{3}{2}C_{m_n-1}^{1}(\lambda_n)^{\frac{1}{2}}f_-^{(m_n-2)}(x,\sqrt{\lambda_n};t)$$
$$+\frac{3}{4}C_{m_n-1}^{2}(\lambda_n)^{-\frac{1}{2}}f_-^{(m_n-3)}(x,\sqrt{\lambda_n};t)-\frac{3}{8}C_{m_n-1}^{3}(\lambda_n)^{-\frac{3}{2}}f_-^{(m_n-4)}(x,\sqrt{\lambda_n};t)$$
$$+3\sum_{r=4}^{m_n-1}C_{m_n-1}^{r}\frac{(-1)^r}{2^{r+1}}\frac{(2r-5)!}{(r-3)!}(\lambda_n)^{-\frac{(2r-3)}{2}}f_-^{(m_n-1-r)}(x,\sqrt{\lambda_n};t)\Big]$$
$$+4\lambda_n\frac{\partial}{\partial x}f_-^{(m_n-1)}(x,\sqrt{\lambda_n};t)+4(m_n-1)\frac{\partial}{\partial x}f_-^{(m_n-2)}(x,\sqrt{\lambda_n};t)$$

$$-\gamma(t)u(0,t)\Bigg[\frac{\partial}{\partial x}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)+i(\lambda_n)^{\frac{1}{2}}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)$$

$$+\frac{i}{2}C_{m_n-1}^{1}(\lambda_n)^{-\frac{1}{2}}f_{-}^{(m_n-2)}(x,\sqrt{\lambda_n};t)-\frac{i}{4}(\lambda_n)^{-\frac{3}{2}}C_{m_n-1}^{2}f_{-}^{(m_n-3)}(x,\sqrt{\lambda_n},t)$$

$$+\frac{3i}{8}(\lambda_n)^{-\frac{5}{2}}C_{m_n-1}^{3}f_{-}^{(m_n-4)}(x,\sqrt{\lambda_n},t)$$

$$-\sum_{r=4}^{m_n-1}C_{m_n-1}^{r}\frac{(-1)^r(2r-3)!}{2^{2r-2}(r-2)!}(\lambda_n)^{-\frac{(2r-1)}{2}}f_{-}^{(m_n-1-r)}(x,\sqrt{\lambda_n},t)\Bigg]$$

$$+\sum_{l=0}^{m_n-1}C_{m_n-1}^{l}\int_{-\infty}^{\infty}\varphi_n^{m_n-1-l}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)dx\cdot\varphi_n^{l}. \tag{3.13}$$

Using formulas (2.12), one can show that

$$\sum_{l=0}^{m_n-1}C_{m_n-1}^{l}\int_{-\infty}^{x}\varphi_n^{m_n-1-l}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)dx\cdot\varphi_n^{l}$$

$$=\sum_{l=0}^{m_n-1}C_{m_n-1}^{l}\int_{-\infty}^{x}\varphi_n^{m_n-1}f_{-}^{(m_n-1-l)}(x,\sqrt{\lambda_n};t)dx\cdot\varphi_n^{l}+\sum_{l=0}^{m_n-1}Q_l(x)\cdot\varphi_n^{l},$$

where $Q_l(x)$ is a linear combination of expressions of the form $W\{\varphi_n^r, f_-^{(q)}\}$, $(r-q=l)$, and therefore $\lim\limits_{x\to\infty}Q_l(x)=0$. According to the definition of the functions φ_n^s and f_-^s, $s=0,1,2,\ldots,m_n-1$, there are numbers $d_0,d_1,\ldots,d_{m_n-1}$ such that

$$\varphi_n^l=\sum_{s=0}^{l}C_l^s d_{l-s}f_-^s,\quad l=0,1,2,\ldots,m_n-1.$$

Therefore,

$$\sum_{l=0}^{m_n-1}C_{m_n-1}^{l}\int_{-\infty}^{\infty}\varphi_n^{m_n-1-l}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)dx\cdot\varphi_n^{l}$$

$$=\sum_{s=0}^{m_n-1}C_{m_n-1}^{s}\int_{-\infty}^{\infty}\varphi_n^{m_n-1}\varphi_n^{m_n-1-s}dx\cdot f_-^{(s)}.$$

Thus, equality (3.13) can be rewritten as

$$\frac{\partial f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)}{\partial t}=4i\Big[(\lambda_n)^{\frac{3}{2}}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)+\frac{3}{2}C_{m_n-1}^{1}(\lambda_n)^{\frac{1}{2}}f_{-}^{(m_n-2)}(x,\sqrt{\lambda_n};t)$$

$$+\frac{3}{4}C_{m_n-1}^{2}(\lambda_n)^{-\frac{1}{2}}f_{-}^{(m_n-3)}(x,\sqrt{\lambda_n};t)-\frac{3}{8}C_{m_n-1}^{3}(\lambda_n)^{-\frac{3}{2}}f_{-}^{(m_n-4)}(x,\sqrt{\lambda_n};t)$$

$$+3\sum_{r=4}^{m_n-1}C_{m_n-1}^{r}\frac{(-1)^r}{2^{r+1}}\frac{(2r-5)!}{(r-3)!}(\lambda_n)^{-\frac{(2r-3)}{2}}f_{-}^{(m_n-1-r)}(x,\sqrt{\lambda_n};t)\Big]$$

$$+4\lambda_n\frac{\partial}{\partial x}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)+4(m_n-1)\frac{\partial}{\partial x}f_{-}^{(m_n-2)}(x,\sqrt{\lambda_n};t)$$

$$-\gamma(t)u(0,t)\Bigg[\frac{\partial}{\partial x}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)+i(\lambda_n)^{\frac{1}{2}}f_{-}^{(m_n-1)}(x,\sqrt{\lambda_n};t)$$

$$+\frac{i}{2}C^1_{m_n-1}(\lambda_n)^{-\frac{1}{2}}f_-^{(m_n-2)}(x,\sqrt{\lambda_n};t)$$

$$-\frac{i}{4}(\lambda_n)^{-\frac{3}{2}}C^2_{m_n-1}f_-^{(m_n-3)}(x,\sqrt{\lambda_n},t)+\frac{3i}{8}(\lambda_n)^{-\frac{5}{2}}C^3_{m_n-1}f_-^{(m_n-4)}(x,\sqrt{\lambda_n},t)$$

$$-\sum_{r=4}^{m_n-1}C^r_{m_n-1}\frac{(-1)^r(2r-3)!}{2^{2r-2}(r-2)!}(\lambda_n)^{-\frac{(2r-1)}{2}}f_-^{(m_n-1-r)}(x,\sqrt{\lambda_n},t)\Bigg]$$

$$+\sum_{r=0}^{m_n-1}C^r_{m_n-1}\int_{-\infty}^{\infty}\varphi_n^{m_n-1}\varphi_n^{m_n-1-r}dx\cdot f_-^{(r)}(x,\sqrt{\lambda_n};t).$$

Using (2.7) and equating the coefficients at $(ix)^l\cdot e^{i\sqrt{\lambda_n}x},\ l=m_n-1,\ m_n-2,\dots,1,\ 0,$ we find an analogue of the Gardner-Greene-Kruskal-Miura equations

$$\frac{d\theta_0^n(t)}{dt}=\left(8i(\lambda_n)^{\frac{3}{2}}+A_0^n(t)-2i(\lambda_n)^{\frac{1}{2}}\gamma(t)u(0,t)\right)\theta_0^n(t),$$

$$\frac{d\theta_1^n(t)}{dt}=\left(8i(\lambda_n)^{\frac{3}{2}}+A_0^n(t)-2i(\lambda_n)^{\frac{1}{2}}\gamma(t)u(0,t)\right)\theta_1^n(t)$$

$$+\left(12i(\lambda_n)^{\frac{1}{2}}+A_1^n(t)-\frac{i}{2}\left((\lambda_n)^{-\frac{1}{2}}+2\right)\gamma(t)u(0,t)\right)\theta_0^n(t),$$

$$\frac{d\theta_2^n(t)}{dt}=\left(8i(\lambda_n)^{\frac{3}{2}}+A_0^n(t)-2i(\lambda_n)^{\frac{1}{2}}\gamma(t)u(0,t)\right)\theta_2^n(t)$$

$$+\left(12i(\lambda_n)^{\frac{1}{2}}+A_1^n(t)-\frac{i}{2}\left((\lambda_n)^{-\frac{1}{2}}+2\right)\gamma(t)u(0,t)\right)\theta_1^n(t)$$

$$+\left(3i(\lambda_n)^{-\frac{1}{2}}+A_2^n(t)+\frac{i}{8}(\lambda_n)^{-\frac{3}{2}}\gamma(t)u(0,t)\right)\theta_0^n(t),$$

$$\frac{d\theta_3^n(t)}{dt}=\left(8i(\lambda_n)^{\frac{3}{2}}+A_0^n(t)-2i(\lambda_n)^{\frac{1}{2}}\gamma(t)u(0,t)\right)\theta_3^n(t)$$

$$+\left(12i(\lambda_n)^{\frac{1}{2}}+A_1^n(t)-\frac{i}{2}\left((\lambda_n)^{-\frac{1}{2}}+2\right)\gamma(t)u(0,t)\right)\theta_2^n(t)$$

$$+\left(3i(\lambda_n)^{-\frac{1}{2}}+A_2^n(t)+\frac{i}{8}(\lambda_n)^{-\frac{3}{2}}\gamma(t)u(0,t)\right)\theta_1^n(t)$$

$$+\left(\frac{i}{2}(\lambda_n)^{-\frac{3}{2}}+A_3^n(t)-\frac{i}{16}(\lambda_n)^{\frac{5}{2}}\gamma(t)u(0,t)\right)\theta_0^n(t), \tag{3.14}$$

$$\frac{d\theta_p^n(t)}{dt}=\left(8i(\lambda_n)^{\frac{3}{2}}+A_0^n(t)-2i(\lambda_n)^{\frac{1}{2}}\gamma(t)u(0,t)\right)\theta_p^n(t)$$

$$+\left(12i(\lambda_n)^{\frac{1}{2}}+A_1^n(t)-\frac{i}{2}\left((\lambda_n)^{-\frac{1}{2}}+2\right)\gamma(t)u(0,t)\right)\theta_{p-1}^n(t)$$

$$+\left(3i(\lambda_n)^{-\frac{1}{2}}+A_2^n(t)+\frac{i}{8}(\lambda_n)^{-\frac{3}{2}}\gamma(t)u(0,t)\right)\theta_{p-2}^n(t)$$

$$+\left(\frac{i}{2}(\lambda_n)^{-\frac{3}{2}}+A_3^n(t)-\frac{i}{16}\lambda_n^{-\frac{5}{2}}\gamma(t)u(0,t)\right)\theta_{p-3}^n(t)$$

$$+\sum_{r=4}^{p}\left(\frac{24i(-1)^r}{2^{r+1}}\cdot\frac{(2r-5)!}{r!(r-3)!}\lambda_n^{-\frac{(2r-3)}{2}}+A_r^n(t)\right.$$
$$\left.+\frac{i(-1)^r}{2^{2r-2}}\cdot\frac{(2r-3)!}{r!(r-2)!}(\lambda_n)^{-\frac{(2r-1)}{2}}\gamma(t)u(0,t)\right)\theta_{p-r}^n(t),$$
$$p=4,5,\ldots,m_n-1;\ \ n=1,2,\ldots,N.$$

Thus, we have proved the following theorem.

Theorem 3.1 *If the system of functions* $u(x,t), \varphi_j^l(x,t),\ l=0,\ 1,\ldots,m_j-1,\ j=\overline{1,N}$ *is a solution to problem* (1.1) - (1.6)*, then the scattering data* $\{S(\sqrt{\lambda},t),\ \lambda_n(t),\ \theta_0^n(t),\ \theta_1^n(t),\ \ldots,\ \theta_{m_n-1}^n(t),\ n=\overline{1,N}\}$ *of the operator* $L(t)$ *with potential* $u(x,t)$ *satisfy differential equations* (3.10)*,* (3.11)*, and* (3.14).

Remark 3.1 Consider the kernel of the Gelfand-Levitan-Marchenko integral equation

$$F_+(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(k,t)e^{ikx}dk+\sum_{j=1}^{N}\sum_{\nu=0}^{m_j-1}\chi_{m_j-\nu-1}^{j}(t)\frac{1}{\nu!}\frac{d^\nu}{dk^\nu}\left(\frac{2k(k-k_j)^{m_j}}{w(k,t)}e^{ikx}\right)$$

with the scattering data from Theorem 3.1. Then the data $\{S(k,t),\lambda_j(t),\chi_0^j(t),\chi_1^j(t),\ldots,\chi_{m_j-1}^j(t), j=\overline{1,N}\}$ satisfies all the necessary conditions given in the second paragraph of this article. Therefore, according to Theorem 2.1, the potential $u(x,t)$ in the operator $L(t)$ is uniquely determined.

Remark 3.2 The obtained relations completely determine the evolution of the scattering data for the operator $L(t)$ and thus allow us to apply the inverse scattering method to solve problem (1.1) - (1.4).

Let the function $|u_0(x)|\,e^{\varepsilon|x|}\in L^1(\mathbb{R})$ be given. The solution to the problem is found by the following algorithm.

We solve the direct scattering problem with the initial function $u_0(x)$ and obtain the scattering data $\{S(k),\lambda_j,\chi_0^j,\ldots,\chi_{m_j-1}^j, j=\overline{1,N}\}$ for the non-self-adjoint operator $L(0)$.

Using Theorem 3.1, we find the scattering data for $t>0$:

$$\{S(k,t),\ \ \lambda_j(t),\ \ \chi_0^j(t),\chi_1^j(t),\ldots,\chi_{m_j-1}^j(t),\ \ j=\overline{1,N}\}.$$

Using the method based on the Gel'fand-Levitan-Marchenko integral equation, we solve the inverse scattering problem, i.e. find $u(x,t)$ from the scattering data for $t>0$ obtained in the previous step.

Example 3.1 Consider the following problem

$$\begin{cases} u_t-6uu_x+u_{xxx}=-\gamma(t)u(0,t)u_x+2\frac{\partial}{\partial x}(\varphi_0^2(x,t)),\\ L\varphi_0=k_0^2\varphi_0,\end{cases}\tag{3.15}$$

$$u(x,0)=\frac{8a^2e^{2iax}}{(1+e^{2iax})^2},\quad \mathrm{Im}a>0,\quad x\in\mathbb{R}.\tag{3.16}$$

Here

$$A(t)=\int_{-\infty}^{\infty}\varphi_0^2(x,t)dx=\frac{i}{2a}e^{-2arcsht},\ \ \gamma(t)=2(t^2+1)+\frac{(t^2+1)e^{-2arcsht}}{8a^4}-\frac{\sqrt{t^2+1}}{2ia^3}.$$

It is easy to find the scattering data for the operator $L(0) = -\frac{d^2}{dx^2} + u_0(x),\ x \in \mathbb{R}:$

$$\lambda(0) = k_0^2 = a^2, \quad v(k,0) = 0, \quad S(k,t) = 0, \quad \theta_0(0) = \chi_0(0) = 1.$$

By Theorem 3.1, we have

$$\lambda(t) = \lambda(0) = a^2; \quad S(k,t) = 0, \quad \chi_0(t) = e^{\beta(t)},$$

where

$$\beta(t) = 8ia^3 t + \int_0^t A(\tau)d\tau - 2ia\int_0^t \gamma(\tau)u(0,\tau)d\tau.$$

Substituting these data into formula (2.9), we find the kernel of the Gelfand-Levitan-Marchenko integral equation:

$$F_+(x,t) = -2iae^{iax+\beta(t)}.$$

Further, solving the integral equation

$$K_+(x,y;t) - 2iae^{\beta(t)} \cdot e^{ia(x+y)} - 2iae^{\beta(t)} \cdot e^{iay}\int_x^\infty K_+(x,s;t)e^{ias}ds = 0,$$

we get

$$K_+(x,y;t) = \frac{2iae^{\beta(t)} \cdot e^{ia(x+y)}}{1 + e^{\beta(t)} \cdot e^{2iax}}.$$

From where we find the solution of the Cauchy problem (3.15) - (3.16)

$$u(x,t) = \frac{8a^2 e^{2iax+2\mathrm{arcsh}t}}{(1+e^{2iax+2\mathrm{arcsh}t})^2}, \quad \varphi_0(x,t) = \frac{e^{iax}}{1+e^{2iax+2\mathrm{arcsh}t}}.$$

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