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Commutator of anisotropic maximal function with BMO functions on total anisotropic Morrey spaces

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Abstract. *In this paper, we consider the total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ introduced by Guliyev in [22] in the isotropic case $d = (1,\dots,1)$. These spaces generalize the anisotropic Morrey spaces so that $L^d_{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L^d_{p,\lambda}(\mathbb{R}^n)$ and the modified anisotropic Morrey spaces so that $L^d_{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$. We give basic properties of the spaces $L^d_{p,\lambda,\lambda}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. We also give necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator operator M^d_b and commutator of anisotropic maximal operator $[b, M^d]$ on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.*

Keywords. total anisotropic Morrey spaces, anisotropic maximal operator, commutator, BMO spaces

Mathematics Subject Classification (2010): Primary 42B20 · 42B25 · 42B35

1 Introduction

Let T be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result due to Coifman, Rochberg and Weiss [9] states that $[b, T]$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$ when $b \in BMO(\mathbb{R}^n)$. They also gave a characterization of $BMO(\mathbb{R}^n)$ in virtue of the L_p-boundedness of the above commutator. It is well-known that the commutator is an important integral operator and plays a key role in harmonic analysis. Maximal commutator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for example [9,16]). The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [10,11,15,24, 25]).

Let $\mathbb{R}^n$ be the n-dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$, S^{n-1} denotes the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $\mathcal{E}(x,r)$ denote the open ellipsoid centered at x of radius r and ${}^{\complement}\mathcal{E}(x,r)$ denote the set $\mathbb{R}^n \setminus \mathcal{E}(x,r)$. Let $d = (d_1,\dots,d_n)$,

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$d_i \geq 1, i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \left(t^{d_1}x_1, \dots, t^{d_n}x_n\right)$. By [6,14], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points $x,\ y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space (see, for example, [6,7,14]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let also $\Pi_d(x,r) = \{y \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelopiped, ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \setminus \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1,\dots,1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = \mathcal{E}(x,r)$. Note that in the standard parabolic case $d = (1,\dots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The anisotropic maximal operator M^d is given by

$$M^d f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)| dy$$

and the anisotropic maximal commutator of M^d with a locally integrable function b is defined by

$$M_b^d f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(x) - b(y)||f(y)| dy,$$

where $|\mathcal{E}(x,t)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x,t)$. If $d = \mathbf{1}$, then $M \equiv M^{\mathbf{1}}$ and $M_b \equiv M_b^{\mathbf{1}}$ are the classical Hardy-Littlewood maximal operator and maximal commutator, respectively. The operators M^d and M_b^d play an important role in real and harmonic analysis (see, for example, [29]).

On the other hand, we can define the (nonlinear) commutator of the anisotropic maximal operator M^d with a locally integrable function b by

$$[b, M^d] f(x) = b(x) M^d f(x) - M^d(bf)(x).$$

Obviously, operators M_b^d and $[b, M^d]$ essentially differ from each other since M_b^d is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear.

The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance [9, 17, 27–29]).

The boundedness of the Hardy-Littlewood maximal operator M on $L_p(\mathbb{R}^n)$ is one of the most fundamental results in harmonic analysis. It has been extended to a range of other function spaces, and to many variations of the standard maximal operator. In particular, one can study commutators of M with BMO functions b. These turn out to be L_p bounded for $1 < p < \infty$ if and only if $b \in BMO$ and $b^- \equiv -\min\{b, 0\} \in L_\infty(\mathbb{R}^n)$ [5]. This is useful, for instance, when studying the product of an H^1 function with a BMO function. Note that, the boundedness of the operator M_b on L_p spaces was proved by Garcia-Cuerva et al. [16].

Morrey spaces, introduced by C. B. Morrey [26], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In [22] Guliyev introduce a variant of Morrey spaces called total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We consider the total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, give basic properties of the spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. We also give necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator operator M^d_b and commutator of anisotropic maximal operator $[b, M^d]$ on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

The structure of the paper is as follows. In Section 2 we give basic properties of the spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. In Section 3 we find necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator M^d_b on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ spaces. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Definition and basic properties of total anisotropic Morrey spaces

Definition 2.1 *Let $d = (d_1, \dots, d_n)$, $d_i \ge 1$, $i = 1, \dots, n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L^d_{p,\lambda}(\mathbb{R}^n)$ the anisotropic Morrey space, by $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ the modified anisotropic Morrey space [21,23], and by $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ the total Morrey space the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{L^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

$$\|f\|_{\widetilde{L}^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

$$\|f\|_{L^d_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

respectively.

Definition 2.2 *Let $d = (d_1, \dots, d_n)$, $d_i \ge 1$, $i = 1, \dots, n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak anisotropic Morrey space $WL^d_{p,\lambda}(\mathbb{R}^n)$, the weak modified anisotropic Morrey space $W\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ [21,23] and the weak total anisotropic Morrey space $WL^d_{p,\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norms*

$$\|f\|_{WL^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

$$\|f\|_{W\widetilde{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

$$\|f\|_{WL^d_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

respectively.

Note that

$$\begin{aligned}
&L^d_{p,0,0}(\mathbb{R}^n) = \widetilde{L}^d_{p,0}(\mathbb{R}^n) = L^d_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n),\\
&WL^d_{p,0,0}(\mathbb{R}^n) = W\widetilde{L}^d_{p,0}(\mathbb{R}^n) = WL^d_{p,0}(\mathbb{R}^n) = WL_p(\mathbb{R}^n),\\
&L^d_{p,\lambda,\lambda}(\mathbb{R}^n) = L^d_{p,\lambda}(\mathbb{R}^n), \quad L^d_{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}^d_{p,\lambda}(\mathbb{R}^n),\\
&\|f\|_{WL^d_{p,\lambda,\mu}} \le \|f\|_{L^d_{p,\lambda,\mu}} \ \text{ and therefore } \ L^d_{p,\lambda,\mu}(\mathbb{R}^n) \subset WL^d_{p,\lambda,\mu}(\mathbb{R}^n)
\end{aligned}$$

and

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) \subset\!\!\succ L^d_{p,\lambda}(\mathbb{R}^n) \text{ and } \|f\|_{L^d_{p,\lambda}} \le \|f\|_{L^d_{p,\lambda,\mu}}, \tag{2.1}$$

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) \subset\!\!\succ L^d_{p,\mu}(\mathbb{R}^n) \text{ and } \|f\|_{L^d_{p,\mu}} \le \|f\|_{L^d_{p,\lambda,\mu}} \tag{2.2}$$

$$\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n) \subset\!\!\succ L_p(\mathbb{R}^n) \text{ and } \|f\|_{L_p} \le \|f\|_{\widetilde{L}^d_{p,\lambda}} \tag{2.3}$$

and if $\lambda < 0$ or $\lambda > |d|$, then $L^d_{p,\lambda}(\mathbb{R}^n) = \widetilde{L}^d_{p,\lambda}(\mathbb{R}^n) = WL^d_{p,\lambda}(\mathbb{R}^n) = W\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n) = \Theta$, where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 2.1 *If $0 < p < \infty$, $0 \le \lambda \le n$ and $0 \le \mu \le n$, then*

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n) \cap L_{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^d_{p,\lambda}}, \|f\|_{L^d_{p,\mu}}\right\}.$$

Proof. Let $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. Then from (2.1) and (2.2) we have that $f \in L^d_{p,\lambda}(\mathbb{R}^n) \cap L^d_{p,\mu}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L^d_{p,\lambda}}, \|f\|_{L^d_{p,\mu}}\right\} \le \|f\|_{L^d_{p,\lambda,\mu}}$.

Now let $f \in L^d_{p,\lambda}(\mathbb{R}^n) \cap L^d_{p,\mu}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|f\|_{L^d_{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda}\ [1/t]_1^{\mu} \int_{\mathcal{E}(x,t)} |f(y)|^p dy\right)^{1/p}\\
&= \max\left\{\sup_{x\in\mathbb{R}^n, 0<t\le 1}\left(t^{-\lambda}\int_{\mathcal{E}(x,t)} |f(y)|^p dy\right)^{1/p}, \sup_{x\in\mathbb{R}^n, t>1}\left(t^{-\mu}\int_{\mathcal{E}(x,t)} |f(y)|^p dy\right)^{1/p}\right\}\\
&\le \max\left\{\|f\|_{L^d_{p,\lambda}}, \|f\|_{L^d_{p,\mu}}\right\}.
\end{aligned}$$

Therefore, $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and the embedding $L^d_{p,\lambda}(\mathbb{R}^n) \cap L^d_{p,\mu}(\mathbb{R}^n) \subset\!\!\succ L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ is valid.

Thus $L^d_{p,\lambda,\mu}(\mathbb{R}^n) = L^d_{p,\lambda}(\mathbb{R}^n) \cap L^d_{p,\mu}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L^d_{p,\lambda}}, \|f\|_{L^d_{p,\mu}}\right\} = \|f\|_{L^d_{p,\lambda,\mu}}$.

Corollary 2.1 *If $0 < p < \infty$, $0 \le \lambda \le |d|$, then*

$$\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n) = L^d_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{L}^d_{p,\lambda}} = \max\left\{\|f\|_{L^d_{p,\lambda}}, \|f\|_{L_p}\right\}.$$

Lemma 2.2 *If* $0 < p < \infty$, $0 \le \lambda \le |d|$ *and* $0 \le \mu \le |d|$, *then*

$$WL^d_{p,\lambda,\mu}(\mathbb{R}^n) = WL^d_{p,\lambda}(\mathbb{R}^n) \cap WL^d_{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL^d_{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^d_{p,\lambda}}, \|f\|_{WL^d_{p,\mu}}\right\}.$$

Remark 2.1 If $0 < p < \infty$, and $\lambda < 0$ or $\lambda > |d|$ or $\mu < 0$ or $\mu > |d|$, then

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) = WL^d_{p,\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n).$$

Lemma 2.3 *If* $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le n$ *and* $0 \le \mu_1 \le \mu_2 \le |d|$, *then*

$$L^d_{p,\lambda_1,\mu_1}(\mathbb{R}^n) \subset_{\succ} L^d_{p,\lambda_2,\mu_2}(\mathbb{R}^n)$$

and

$$\|f\|_{L^d_{p,\lambda_2,\mu_2}} \le \|f\|_{L^d_{p,\lambda_1,\mu_1}}.$$

Proof. Let $f \in L^d_{p,\lambda,\mu}$, $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le |d|$, $0 \le \mu_1 \le \mu_2 \le |d|$. Then

$$\|f\|_{L^d_{p,\lambda_2,\mu_2}} = \max\Big\{\sup_{x\in\mathbb{R}^n,\, 0<t\le 1}\Big(t^{\lambda_1-\lambda_2}t^{-\lambda_1}\int_{\mathcal{E}(x,t)}|f(y)|^p dy\Big)^{1/p},$$
$$\sup_{x\in\mathbb{R}^n,\, t\ge 1}\Big(t^{\mu_1-\mu_2}t^{-\mu_1}\int_{\mathcal{E}(x,t)}|f(y)|^p dy\Big)^{1/p}\Big\} \le \|f\|_{L^d_{p,\lambda_1,\mu_1}}.$$

Lemma 2.4 *If* $0 < p < \infty$, $0 \le \lambda \le |d|$ *and* $0 \le \mu \le |d|$, *then*

$$L^d_{p,|d|,\mu}(\mathbb{R}^n) \subset_{\succ} L_\infty(\mathbb{R}^n) \subset_{\succ} L^d_{p,\lambda,|d|}(\mathbb{R}^n)$$

and

$$\|f\|_{L^d_{p,\lambda,|d|}} \le v_n^{1/p}\|f\|_{L_\infty} \le \|f\|_{L^d_{p,|d|,\mu}}.$$

Proof. Let $f \in L_\infty(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$ and $0 < t \le 1$

$$\Big(t^{-\lambda}\int_{\mathcal{E}(x,t)}|f(y)|^p dy\Big)^{1/p} \le v_n^{1/p}\,\|f\|_{L_\infty}, \quad 0 \le \lambda \le |d|$$

and for all $x \in \mathbb{R}^n$ and $t \ge 1$

$$\Big(t^{-|d|}\int_{\mathcal{E}(x,t)}|f(y)|^p dy\Big)^{1/p} \le v_n^{1/p}\,\|f\|_{L_\infty}.$$

Therefore $f \in L^d_{p,\lambda,|d|}(\mathbb{R}^n)$ and

$$\|f\|_{L^d_{p,\lambda,|d|}} \le v_n^{1/p}\|f\|_{L_\infty}.$$

Let $f \in L^d_{p,|d|,\mu}(\mathbb{R}^n)$. By the Lebesgue's differentiation theorem we have (see [29])

$$\lim_{t\to 0}|\mathcal{E}(x,t)|^{-1}\int_{\mathcal{E}(x,t)}|f(y)|^p dy = |f(x)|^p \quad \text{for} \quad a.e.\ x \in \mathbb{R}^n.$$

Then for a.e. $x \in \mathbb{R}^n$

$$|f(x)| = \Big(\lim_{t\to 0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)|^p dy\Big)^{1/p}$$
$$\le v_n^{-1/p} \sup_{x\in\mathbb{R}^n,\, 0<t\le 1} \Big(t^{-|d|} \int_{\mathcal{E}(x,t)} |f(y)|^p dy\Big)^{1/p} \le v_n^{-1/p} \|f\|_{L^d_{p,|d|,\mu}}.$$

Therefore $f \in L_\infty(\mathbb{R}^n)$ and

$$\|f\|_{L_\infty} \le v_n^{-1/p} \|f\|_{L^d_{p,|d|,\mu}}.$$

Corollary 2.2 *If $0 < p < \infty$, then*

$$L^d_{p,|d|}(\mathbb{R}^n) = \widetilde{L}^d_{p,|d|}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$$

and

$$\|f\|_{L^d_{p,|d|}} = \|f\|_{\widetilde{L}^d_{p,|d|}} = v_n^{1/p} \|f\|_{L_\infty}.$$

Lemma 2.5 *If $0 \le \lambda < |d|$, $0 \le \mu < |d|$, $0 \le \alpha < |d| - \lambda$ and $0 \le \beta < |d| - \mu$, then for $\frac{|d|-\lambda}{\alpha} \le p \le \frac{|d|-\mu}{\beta}$*

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) \subset\!\!\succ L_{1,|d|-\alpha,|d|-\beta}(\mathbb{R}^n)$$

and for $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ the following inequality

$$\|f\|_{L^d_{1,|d|-\alpha,|d|-\beta}} \le v_n^{1/p'} \|f\|_{L^d_{p,\lambda,\mu}}$$

is valid.

Proof. Let $0 < \alpha < |d|$, $0 \le \lambda < |d|$, $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and $\frac{|d|-\lambda}{\alpha} \le p \le \frac{|d|-\mu}{\beta}$. By the Hölder's inequality we have

$$\|f\|_{L^d_{1,|d|-\alpha,|d|-\beta}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{\alpha-|d|}\, [1/t]_1^{|d|-\beta} \int_{\mathcal{E}(x,t)} |f(y)| dy$$
$$\le v_n^{1/p'} \sup_{x\in\mathbb{R}^n,\, t>0} \big([t]_1\, t^{-1}\big)^{-|d|/p'}\, [t]_1^{\alpha-\frac{|d|-\lambda}{p}}\, [1/t]_1^{|d|-\beta-\frac{\mu}{p}} \left([t]_1^{-\lambda}\, [1/t]_1^{\mu} \int_{\mathcal{E}(x,t)} |f(y)|^p dy\right)^{1/p}$$
$$\le v_n^{1/p'} \|f\|_{L^d_{p,\lambda,\mu}} \sup_{t>0} \big([t]_1\, t^{-1}\big)^{\frac{|d|-\mu}{p}-\beta}\, [t]_1^{\alpha-\frac{|d|-\lambda}{p}}.$$

Note that

$$\sup_{t>0} \big([t]_1\, t^{-1}\big)^{\frac{|d|-\mu}{p}-\beta}\, [t]_1^{\alpha-\frac{|d|-\lambda}{p}} = \max\Big\{\sup_{0<t\le 1} t^{\alpha-\frac{|d|-\lambda}{p}}, \sup_{t>1} t^{\beta-\frac{|d|-\mu}{p}}\Big\} < \infty$$
$$\Longleftrightarrow \frac{|d|-\lambda}{\alpha} \le p \le \frac{|d|-\mu}{\beta}.$$

Therefore $f \in L^d_{1,|d|-\alpha,|d|-\beta}(\mathbb{R}^n)$ and

$$\|f\|_{L_{1,|d|-\alpha,|d|-\beta}} \le v_n^{1/p'} \|f\|_{L^d_{p,\lambda,\mu}}.$$

From Lemma 2.5 we get the following:

Corollary 2.3 *If* $0 \le \mu \le \lambda < |d|$, $0 \le \alpha < |d| - \lambda$, *then for* $\frac{|d|-\lambda}{\alpha} \le p \le \frac{|d|-\mu}{\alpha}$

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) \subset\!\!\succ L^d_{1,|d|-\alpha}(\mathbb{R}^n)$$

and for $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ *the following inequality*

$$\|f\|_{L^d_{1,|d|-\alpha}} \le v_n^{1/p'} \|f\|_{L^d_{p,\lambda,\mu}}$$

is valid.

Corollary 2.4 *If* $0 \le \lambda < |d|$ *and* $0 \le \alpha < |d| - \lambda$, *then for* $p = \frac{|d|-\lambda}{\alpha}$

$$L^d_{p,\lambda}(\mathbb{R}^n) \subset L_{1,|d|-\alpha}(\mathbb{R}^n) \quad and \quad \|f\|_{L^d_{1,|d|-\alpha}} \le v_n^{1/p'} \|f\|_{L^d_{p,\lambda}}.$$

Corollary 2.5 *If* $0 \le \lambda < |d|$ *and* $0 \le \alpha < |d| - \lambda$, *then for* $\frac{|d|-\lambda}{\alpha} \le p \le \frac{|d|}{\alpha}$

$$\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n) \subset L^d_{1,|d|-\alpha}(\mathbb{R}^n) \quad and \quad \|f\|_{L^d_{1,|d|-\alpha}} \le v_n^{1/p'} \|f\|_{\widetilde{L}^d_{p,\lambda}}.$$

Remark 2.2 Note that, in the isotropic case $d = (1, \dots, 1)$ Lemmas 2.1, 2.2, 2.3, 2.4 and 2.5 was proved [22, Lemmas 2, 3, 4, 5 and 6].

3 $L^d_{p,\lambda,\mu}$-boundedness of the anisotropic maximal commutator operator M^d_b

In this section, we find necessary and sufficient conditions for the anisotropic maximal commutator M^d_b to be bounded on the spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

The following Guliyev type local estimates are valid (see also [18–20]).

Lemma 3.1 *[1] Let* $1 \le p < \infty$ *and* $\mathcal{E}(x, r)$ *be any ellipsoid in* $\mathbb{R}^n$. *If* $p > 1$, *then the inequality*

$$\|M^d f\|_{L_p(\mathcal{E}(x,r))} \lesssim r^{\frac{|d|}{p}} \sup_{t>2r} t^{-\frac{|d|}{p}} \|f\|_{L_p(\mathcal{E}(x,t))} \tag{3.1}$$

holds for all $f \in L^{\rm loc}_p(\mathbb{R}^n)$.

Moreover if $p = 1$, *then the inequality*

$$\|M^d f\|_{WL_1(\mathcal{E}(x,r))} \lesssim r^{|d|} \sup_{t>2r} t^{-|d|} \|f\|_{L_1(\mathcal{E}(x,t))} \tag{3.2}$$

holds for all $f \in L^{\rm loc}_1(\mathbb{R}^n)$.

Theorem 3.1 *1. If* $f \in L^d_{1,\lambda,\mu}(\mathbb{R}^n)$, $0 \le \lambda < |d|$ *and* $0 \le \mu < |d|$, *then* $M^d f \in WL^d_{1,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|M^d f\|_{WL^d_{1,\lambda,\mu}} \le C_{1,\lambda,\mu} \|f\|_{L^d_{1,\lambda,\mu}}, \tag{3.3}$$

where $C_{1,\lambda,\mu}$ *is independent of* f.

2. If $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le \lambda < |d|$ *and* $0 \le \mu < |d|$, *then* $M^d f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|M^d f\|_{L^d_{p,\lambda,\mu}} \le C_{p,\lambda,\mu} \|f\|_{L^d_{p,\lambda,\mu}}, \tag{3.4}$$

where $C_{p,\lambda,\mu}$ *depends only on* p, λ, μ *and* n.

Proof. Let $p = 1$. From the inequality (3.2) we get

$$\begin{aligned}
\|M^d f\|_{WL^d_{1,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, \|M^d f\|_{WL_1(\mathcal{E}(x,t))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, t^{|d|} \sup_{\tau>2t} \tau^{-|d|}\, \|f\|_{L_1(\mathcal{E}(x,\tau))} \\
&\lesssim \|f\|_{L^d_{1,\lambda,\mu}} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, t^{|d|} \sup_{\tau>t} \tau^{-|d|}\, [\tau]_1^{\lambda}\, [1/\tau]_1^{-\mu} \\
&= \|f\|_{L^d_{1,\lambda,\mu}} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{|d|-\lambda}\, [1/t]_1^{\mu-|d|} \sup_{\tau>t}\, [\tau]_1^{\lambda-|d|}\, [1/\tau]_1^{|d|-\mu} \\
&= \|f\|_{L^d_{1,\lambda,\mu}}
\end{aligned}$$

which implies that the operator M^d is bounded from $L^d_{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL^d_{1,\lambda,\mu}(\mathbb{R}^n)$.

Let $1 < p < \infty$. From the inequality (3.1) we get

$$\begin{aligned}
\|M^d f\|_{L^d_{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}}\, \|M^d f\|_{L_p(\mathcal{E}(x,t))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}}\, t^{\frac{|d|}{den}} \sup_{\tau>2t} \tau^{-\frac{|d|}{p}}\, \|f\|_{L_p(\mathcal{E}(x,\tau))} \\
&\lesssim \|f\|_{L^d_{p,\lambda,\mu}} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}}\, t^{\frac{|d|}{p}} \sup_{\tau>t} \tau^{-\frac{|d|}{p}}\, [\tau]_1^{\frac{\lambda}{p}}\, [1/\tau]_1^{-\frac{\mu}{p}} \\
&= \|f\|_{L^d_{p,\lambda,\mu}} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{\frac{|d|-\lambda}{p}}\, [1/t]_1^{\frac{\mu-|d|}{p}} \sup_{\tau>t}\, [\tau]_1^{\frac{\lambda-|d|}{p}}\, [1/\tau]_1^{\frac{n-\mu}{p}} \\
&= \|f\|_{L^d_{p,\lambda,\mu}}
\end{aligned}$$

which implies that the operator M^d is bounded in $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

From Theorem 3.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3.1 *[12] 1. If $f \in L^d_{1,\lambda}(\mathbb{R}^n)$ and $0 \le \lambda < |d|$, then $M^d f \in WL^d_{1,\lambda}(\mathbb{R}^n)$ and*

$$\|M^d f\|_{WL^d_{1,\lambda}} \le C_{1,\lambda}\, \|f\|_{L^d_{1,\lambda}},$$

where $C_{1,\lambda}$ is independent of f.

2. If $f \in L^d_{p,\lambda}(\mathbb{R}^n)$, $1 < p < \infty$ and $0 \le \lambda < |d|$, then $M^d f \in L^d_{p,\lambda}(\mathbb{R}^n)$ and

$$\|M^d f\|_{L^d_{p,\lambda}} \le C_{p,\lambda}\, \|f\|_{L^d_{p,\lambda}},$$

where $C_{p,\lambda}$ depends only on p, λ and n.

Corollary 3.2 *[13] 1. If $f \in \widetilde{L}^d_{1,\lambda}(\mathbb{R}^n)$ and $0 \le \lambda < |d|$, then $M^d f \in W\widetilde{L}^d_{1,\lambda}(\mathbb{R}^n)$ and*

$$\|M^d f\|_{W\widetilde{L}^d_{1,\lambda}} \le C_{1,\lambda}\, \|f\|_{\widetilde{L}^d_{1,\lambda}},$$

where $C_{1,\lambda}$ is independent of f.

2. If $f \in \widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$, $1 < p < \infty$ and $0 \le \lambda < |d|$, then $M^d f \in \widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ and

$$\|M^d f\|_{\widetilde{L}^d_{p,\lambda}} \le C_{p,\lambda}\, \|f\|_{\widetilde{L}^d_{p,\lambda}},$$

where $C_{p,\lambda}$ depends only on p, λ and n.

Definition 3.1 *We define the space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norm*

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y) - f_{\mathcal{E}(x,t)}| dy < \infty,$$

where $f_{\mathcal{E}(x,t)} = |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} f(y)dy$.

Theorem 3.2 *[2, Corollary 1.11] If $b \in BMO(\mathbb{R}^n)$, then there exists a positive constant C such that*

$$M_b^d f(x) \le C\|b\|_* \big(M^d\big)^2 f(x) \tag{3.5}$$

for almost every $x \in \mathbb{R}^n$ and for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Theorem 3.3 *Let $1 < p < \infty$, $0 \le \lambda \le |d|$ and $0 \le \mu \le |d|$. The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$.

(ii) The operator M_b^d is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Theorems 3.1 and 3.2, we get

$$\|M_b^d f\|_{L^d_{p,\lambda,\mu}} \lesssim \|b\|_* \|\big(M^d\big)^2 f\|_{L^d_{p,\lambda,\mu}} \lesssim \|b\|_* \|M^d f\|_{L^d_{p,\lambda,\mu}} \lesssim \|b\|_* \|f\|_{L^d_{p,\lambda,\mu}}.$$

$(ii) \Rightarrow (i)$. Assume that M_b is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. Let $\mathcal{E} = \mathcal{E}(x,r)$ be a fixed ellipsoid . We consider $f = \chi_{\mathcal{E}}$. It is easy to compute that

$$\begin{aligned}
\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}} &\approx \sup_{y\in\mathbb{R}^n, t>0} \Big([t]_1^{-\lambda}\, [1/t]_1^{\mu} \int_{\mathcal{E}(y,t)} \chi_{\mathcal{E}}(z) dz \Big)^{\frac{1}{p}} \\
&= \sup_{y\in\mathbb{R}^n, t>0} \Big(|\mathcal{E}(y,t) \cap \mathcal{E}|\, [t]_1^{-\lambda}\, [1/t]_1^{\mu} \Big)^{\frac{1}{p}} \\
&= \sup_{\mathcal{E}(y,t)\subseteq\mathcal{E}} \Big(|\mathcal{E}(y,t)|\, [t]_1^{-\lambda}\, [1/t]_1^{\mu} \Big)^{\frac{1}{p}} = r^{\frac{|d|}{p}}\, [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}.
\end{aligned} \tag{3.6}$$

On the other hand, since

$$M_b^d(\chi_{\mathcal{E}})(x) \gtrsim \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}| dz \quad \text{for all} \quad x \in \mathcal{E},$$

we have

$$\begin{aligned}
\|M_b^d(\chi_{\mathcal{E}})\|_{L^d_{p,\lambda,\mu}} &\approx \sup_{\mathcal{E}(y,t)} \Big([t]_1^{-\lambda}\, [1/t]_1^{\mu} \int_{\mathcal{E}(y,t)} |M_b^d(\chi_{\mathcal{E}})(z)|^p dz \Big)^{\frac{1}{p}} \\
&\gtrsim r^{\frac{|d|}{p}}\, [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}| dz.
\end{aligned} \tag{3.7}$$

Since by assumption

$$\|M_b^d(\chi_{\mathcal{E}})\|_{L^d_{p,\lambda,\mu}} \lesssim \|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}},$$

by (3.6) and (3.7), we get that

$$\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}| dz \lesssim 1.$$

From Theorem 3.3 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries, see [3, 4].

Corollary 3.3 *[3] Let $1 < p < \infty$ and $0 \le \lambda \le |d|$. The following assertions are equivalent:*
(i) $b \in BMO(\mathbb{R}^n)$.
(ii) *The operator M_b^d is bounded on $L^d_{p,\lambda}(\mathbb{R}^n)$.*

Corollary 3.4 *Let $1 < p < \infty$ and $0 \le \lambda \le |d|$. The following assertions are equivalent:*
(i) $b \in BMO(\mathbb{R}^n)$.
(ii) *The operator M_b^d is bounded on $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$.*

Remark 3.1 Note that, in the isotropic case $d = (1, \dots, 1)$ Theorems 3.1, 3.3 was proved [22, Theorems 1, 3].

4 $L^d_{p,\lambda,\mu}$-boundedness of the commutator of anisotropic maximal operator $[b, M^d]$

In this section we find necessary and sufficient conditions for the commutator of the anisotropic maximal operator $[b, M^d]$ to be bounded on the spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M^d]$ and M_b^d are valid:

Let b be any non-negative locally integrable function. Then for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned} &\big|[b, M^d]f(x)\big| = \big|b(x)M^d f(x) - M^d(bf)(x)\big| \\ &= \big|M^d(b(x)f)(x) - M^d(bf)(x)\big| \le M^d(|b(x) - b|f)(x) = M_b^d f(x). \end{aligned}$$

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M^d]f(x)| \le M_b^d f(x) + 2b^-(x)\, M^d f(x), \qquad x \in \mathbb{R}^n \tag{4.1}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$ (see, for example [22,30]).

Obviously, the M_b^d and $[b, M^d]$ operators are essentially different from each other because M_b^d is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear.

Applying Theorem 3.3, we obtain the following result.

Theorem 4.1 *Let $1 < p < \infty$, $0 \le \lambda \le |d|$ and $0 \le \mu \le |d|$. Suppose that b is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*
(i) $b \in BMO(\mathbb{R}^n)$ *such that* $b^- \in L_\infty(\mathbb{R}^n)$.
(ii) *The operator $[b, M^d]$ is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Theorems 3.1 and 3.3, and inequality (4.1), we get

$$\begin{aligned} \|[b, M^d]f\|_{L^d_{p,\lambda,\mu}} &\le \|M_b^d f + 2b^-\, M^d f\|_{L^d_{p,\lambda,\mu}} \\ &\le \|M_b^d f\|_{L^d_{p,\lambda,\mu}} + \|b^-\|_{L_\infty}\, \|M^d f\|_{L^d_{p,\lambda,\mu}} \\ &\lesssim \big(\|b\|_* + \|b^-\|_{L_\infty}\big)\, \|f\|_{L^d_{p,\lambda,\mu}}. \end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that $[b, M^d]$ is bounded on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. Let $\mathcal{E} = \mathcal{E}(x, r)$ be a fixed ellipsoid . Denote by $M^d_b f$ the local maximal function of f:

$$M^d_b f(x) := \sup_{\mathcal{E}' \ni x:\, \mathcal{E}' \subset \mathcal{E}} \frac{1}{|\mathcal{E}'|} \int_{\mathcal{E}'} |f(y)|\, dy,\ x \in \mathbb{R}^n.$$

Since

$$M^d(b\chi_{\mathcal{E}})\chi_{\mathcal{E}} = M^d_b(b) \quad \text{and} \quad M^d(\chi_{\mathcal{E}})\chi_{\mathcal{E}} = \chi_{\mathcal{E}},$$

we have

$$\begin{aligned} |M^d_b(b) - b\chi_{\mathcal{E}}| &= |M^d(b\chi_{\mathcal{E}})\chi_{\mathcal{E}} - bM^d(\chi_{\mathcal{E}})\chi_{\mathcal{E}}| \\ &\le |M^d(b\chi_{\mathcal{E}}) - bM^d(\chi_{\mathcal{E}})| = |[b, M^d]\chi_{\mathcal{E}}|. \end{aligned}$$

Hence

$$\|M^d_b(b) - b\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)} \le \|[b, M^d]\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)}.$$

Thus from (3.6) we get

$$\begin{aligned} \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b - M^d_b(b)| &\le \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b - M^d_b(b)|^p\right)^{\frac{1}{p}} \\ &\le |\mathcal{E}|^{-\frac{1}{p}}\, [r]_1^{\frac{\lambda}{p}}\, [1/r]_1^{-\frac{\mu}{p}}\, \|b\chi_{\mathcal{E}} - M^d_b(b)\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)} \\ &\lesssim r^{-\frac{|d|}{p}}\, [r]_1^{\frac{\lambda}{p}}\, [1/r]_1^{-\frac{\mu}{p}}\, \|[b, M^d]\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)} \\ &\lesssim r^{-\frac{|d|}{p}}\, [r]_1^{\frac{\lambda}{p}}\, [1/r]_1^{-\frac{\mu}{p}}\, \|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}} \approx 1. \end{aligned}$$

Denote by

$$E := \{x \in \mathcal{E} :\ b(x) \le b_{\mathcal{E}}\}, \quad F := \{x \in \mathcal{E} :\ b(x) > b_{\mathcal{E}}\}.$$

Since

$$\int_E |b(t) - b_{\mathcal{E}}|\, dt = \int_F |b(t) - b_{\mathcal{E}}|\, dt,$$

in view of the inequality $b(x) \le b_{\mathcal{E}} \le M^d_b(b)$, $x \in E$, we get

$$\begin{aligned} &\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b - b_{\mathcal{E}}| = \frac{2}{|\mathcal{E}|}\int_E |b - b_{\mathcal{E}}| \\ &\le \frac{2}{|\mathcal{E}|}\int_E |b - M^d_b(b)| \le \frac{2}{|\mathcal{E}|}\int_{\mathcal{E}} |b - M^d_b(b)| \lesssim 1. \end{aligned}$$

Consequently, $b \in BMO(\mathbb{R}^n)$.

In order to show that $b^- \in L_\infty(\mathbb{R}^n)$, note that $M^d_b(b) \ge |b|$. Hence

$$0 \le b^- = |b| - b^+ \le M^d_b(b) - b^+ + b^- = M^d_b(b) - b.$$

Thus

$$(b^-)_{\mathcal{E}} \le c,$$

and by the Lebesgue differentiation theorem we get that

$$b^-(x) \le c \quad \text{for a.e. } x \in \mathbb{R}^n.$$

From Theorem 4.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 4.1 *[3] Let $1 < p < \infty$ and $0 \le \lambda \le |d|$. Suppose that b is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$ such that $b^- \in L_\infty(\mathbb{R}^n)$.*
(ii) *The operator $[b, M^d]$ is bounded on $L^d_{p,\lambda}(\mathbb{R}^n)$.*

Corollary 4.2 *Let $1 < p < \infty$ and $0 \le \lambda \le |d|$. Suppose that b is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$ such that $b^- \in L_\infty(\mathbb{R}^n)$.*
(ii) *The operator $[b, M^d]$ is bounded on $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$.*

Remark 4.1 Note that, in the isotropic case $d = (1, \dots, 1)$ Theorem 4.1 was proved [22, Theorem 4].

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References

1. Abasova, G.A.: *Parabolic maximal operator and its commutators in parabolic generalized Orlicz-Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **38** (1), Mathematics, 3-12 (2018).
2. Agcayazi, M., Gogatishvili, A., Koca, K., Mustafayev, R.: *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan, **67** (2), 581-593 (2015).
3. Akbulut, A., Burenkov, V.I., Guliyev, V.S.: *Anisotropic fractional maximal commutators with BMO functions on anisotropic Morrey-type spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **40** (4), Mathematics, 13-32 (2020).
4. Armutcu, H., Omarova, M.N.: *Some characterizations of BMO spaces via maximal commutators in Orlicz spaces over Carleson curves*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **42** (1), Mathematics, 47-60 (2022).
5. Bastero, J., Milman, M., Ruiz, F.J.: *Commutators for the maximal and sharp functions*, Proc. Amer. Math. Soc., **128** (11), 3329-3334 (2000) (electronic).
6. Besov, O.V., Il'in, V.P., Lizorkin, P.I.: *The L_p-estimates of a certain class of non-isotropically singular integrals,* (Russian) Dokl. Akad. Nauk SSSR, **169**, 1250-1253 (1966).
7. Bramanti, M., Cerutti, M.C.: *Commutators of singular integrals on homogeneous spaces*, Boll. Un. Mat. Ital. B, **10** (7), 843-883 (1996).
8. Bonami, A., Iwaniec, T., Jones, P., Zinsmeister, M.: *On the product of functions in BMO and H_1*, Ann. Inst. Fourier Grenoble, **57** (5), 1405-1439 (2007).
9. Coifman, R.R., Rochberg, R., Weiss, G.: *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (3), 611-635 (1976).
10. Chiarenza, F., Frasca, M., Longo, P.: *Interior $W^{2,p}$ estimates for non-divergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40**, 149-168 (1991).
11. Chiarenza, F., Frasca, M., Longo, P.: *$W^{2,p}$-solvability of the Dirichlet problem for non-divergence elliptic equations with* VMO *coefficients*, Trans. Amer. Math. Soc. **336**, 841-853 (1993).
12. Dzhabrailov, M.S., Khaligova, S.Z.: *Anisotropic fractional maximal operator in anisotropic generalized Morrey spaces*, J. Math. Res. **4** (6), 109-120 (2012).
13. Dzhabrailov, M.S., Khaligova, S.Z.: *Some embeddings into the anisotropic Morrey and modified anisotropic Morrey spaces. Some applications*, An. tiin. Univ. Al. I. Cuza Iai. Mat. (N.S.) **60** (1), 245-257 (2014).

14. Fabes, E.B., Rivère, N.: *Singular integrals with mixed homogeneity*, Studia Math. **27**, 19-38 (1966).
15. Di Fazio, G., Palagachev, D.K., Ragusa, M.A.: *Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients*, J. Funct. Anal. **166** (2), 179-196 (1999).
16. Garcia-Cuerva, J., Harboure, E., Segovia, C., Torrea, J.L.: *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J., **40**, 1397-1420 (1991).
17. Grafakos, L.: *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, Vol. 250, Springer, New York, (2009).
18. Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$, *Doctor of Sciences, Moscow, Mat. Inst. Steklova* 1-329 (1994). (in Russian)
19. Guliyev, V.S.: Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups, *Some Applications, (Russian), ELM, Baku*, 1996.
20. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J Inequal Appl., 2009, Art. ID 503948, 20 pp.
21. Guliyev, V.S., Hasanov, J.J., Zeren, Y.: *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal., **5** (4), 491-506 (2011).
22. Guliyev, V.S.: *Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal., **16** (4), 1509-1524 (2022).
23. Guliyev, V.S., Rahimova, K.R.: *Parabolic fractional integral operator in modified parabolic Morrey spaces*, Proc. A. Razmadze Math. Inst. **163**, 59-80 (2013).
24. Guliyev, V.S., Softova, L.: *Global regularity in generalized Morrey spaces of solutions to nondivergence elliptic equations with VMO coefficients*, Potential Anal. **38** (4), 843-862 (2013).
25. Guliyev, V.S., Omarova, M.N.: *Estimates for operators on generalized weighted Orlicz-Morrey spaces and their applications to non-divergence elliptic equations*, Positivity **26** (2), Paper No. 40, 27 pp. (2022).
26. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
27. Sawano, Y., Di Fazio, G., Hakim, D.I.: *Morrey SpacesIntroduction and Applications to Integral Operators and PDE's*, Vol. I. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 479 pp. (2020).
28. Sawano, Y., Di Fazio, G., Hakim, D.I.: *Morrey SpacesIntroduction and Applications to Integral Operators and PDE's*, Vol. II. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 409 pp. (2020).
29. Stein, E.M.: *Harmonic analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, New Jersey (1993).
30. Zhang, P., Wu, J., Sun, J.: *Commutators of some maximal functions with Lipschitz function on Orlicz spaces*, Mediterr. J. Math., **15** (6), Paper No. 216, 13 pp. (2018).