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The Riemann function of the Cauchy problem for a second-order hyperbolic equation with a growing coefficient

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Abstract. *The paper investigates the properties of the Riemann function of the Cauchy problem for a second-order hyperbolic equation with a growing coefficient. The existence and uniqueness of the Riemann function are proved. Estimates are found for the Riemann function and its derivatives.*

Keywords. The Cauchy problem · a second-order hyperbolic equation · the Riemann function · transformation operator · method of successive approximations.

Mathematics Subject Classification (2010): 34A55

1 Introduction and main result

Let $U = U\,(x, y)$ be a twice continuously differentiable solution of the Cauchy problem

$$\frac{\partial^2 U}{\partial x^2} - q_1(x)U = \frac{\partial^2 U}{\partial y^2} - q_2(y)U\,,\ -\infty < x < +\infty,\, 0 \le y < +\infty, \tag{1.1}$$

$$U|_{y=0} = \varphi(x),\ \left.\frac{\partial U}{\partial y}\right|_{y=0} = \psi(x). \tag{1.2}$$

We shall assume that functions $q_1(x)$, $x \in (-\infty, \infty)$ and $q_2(y)$, $y \in [0, \infty)$ are continuous. It is known that one of the main tools for studying the Cauchy problem for a second-order hyperbolic equation is the application of the Riemann function method. The value of the function $U\,(x, y)$ at the point (x_0, y_0) may be thought of as the value of a linear functional $T_{x_0}^{y_0}$ on the vector $(\varphi(x),\, \psi(x))$:

$$U\,(x_0, y_0) = \mathrm{T}_{x_0}^{y_0}\,[\varphi(x),\, \psi(x)]\,.$$

An expression for this functional was first found by B. Riemann [11]. Let $R\,(x, y, x_0,\, y_0)$ denote the twice continuously differentiable solution of the equation

$$\frac{\partial^2 R}{\partial x^2} - q_1(x)R = \frac{\partial^2 R}{\partial y^2} - q_2(y)R,$$

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taking the value 1 on the characteristics $x-x_0 = \pm (y - y_0)$ of this equation. It is known [3] that the solution of this equation with such properties exists and is unique. It is well known [3,8] that, using the Riemann function$R\left(x, y, x_0,\, y_0\right)$, one can represent the solution of problem (1.1), (1.2) as

$$U\left(x_0, y_0\right) = \frac{\varphi\left(x_0 + y_0\right) + \psi\left(x_0 - y_0\right)}{2} +$$

$$+\frac{1}{2}\int_{x_0-y_0}^{x_0+y_0} \left\{ \psi(x) R\left(x, 0, x_0, y_0\right) - \varphi(x) \frac{\partial R\left(x, 0, x_0, y_0\right)}{\partial y} \right\} dx.$$

Note that there exists no general method for constructing the Riemann function. In this connection, we mention the paper [3], in which six known methods for constructing the Riemann function for special types of hyperbolic equations were analyzed.

In this paper, we study the Riemann function of the equation

$$\frac{\partial^2 R}{\partial x^2} - x^\alpha R = \frac{\partial^2 R}{\partial y^2} - y^\alpha R,\ 0 < x < y, \tag{1.3}$$

where $\alpha \geq 1$. It should be noted that an equation of the form (1.3) arises when studying the following Goursat problem for a second-order hyperbolic equation:

$$\frac{\partial U}{\partial x^2} - x^\alpha U = \frac{\partial U}{\partial y^2} - y^\alpha U,\ 0 < x < y,$$

$$U\left(x, x\right) = \varphi(x),$$

$$\lim_{x+y\to\infty} U\left(x, y\right) = 0.$$

The latter problem, in turn, occurs when constructing transformation operators with a condition at infinity (see [1], [4]-[10], [12]).

We write equation (1.3) in the new variables $\frac{y+x}{2} = \xi$, $\frac{y-x}{2} = \eta$, $\frac{y_0+x_0}{2} = \xi_0$, $\frac{y_0-x_0}{2} = \eta_0$, setting $r\left(\xi, \eta; \xi_0, \eta_0\right) = R\left(\xi - \eta, \xi + \eta, \xi_0 - \eta_0, \xi_0 + \eta_0\ \right)$. This yields the following equation for the function $r\left(\xi, \eta\right) = r\left(\xi, \eta; \xi_0, \eta_0\right)$ in the domain
$G = \{(\xi, \eta) :\ 0 < \eta < \eta_0 \leq \xi_0 < \xi\}$:

$$\frac{\partial^2 r\left(\xi,\, \eta\right)}{\partial \xi\, \partial \eta} - \left[(\xi + \eta)^\alpha - (\xi - \eta)^\alpha\right]\, r\left(\xi,\, \eta\right) = 0\,, \tag{1.4}$$

together with the following conditions on the characteristics:

$$r(\xi, \eta; \xi_0, \eta_0)\left|_{\xi=\xi_0}\right. = 1, 0 \leq \eta \leq \eta_0, \tag{1.5}$$

$$r(\xi, \eta; \xi_0, \eta_0)\left|_{\eta=\eta_0}\right. = 1,\ \xi_0 \leq \xi < \infty. \tag{1.6}$$

The main result of the present paper is as follows.

Theorem 1.1 *Problem (1.4)-(1.6) has a unique solution r and the following estimate is valid:*

$$\left|r\left(\xi, \eta,\, \xi_0, \eta_0\right)\right| \leq e^{\sigma(\xi)}, \tag{1.7}$$

where

$$\sigma\left(\xi\right) = \left(\alpha + 1\right)^{-\frac{1}{2}} \left(2\xi\right)^{\frac{\alpha+2}{2}}\ . \tag{1.8}$$

2 Proof of the theorem

Problem (1.4)-(1.6) is obviously equivalent to the integral equation

$$r(\xi,\eta,\,\xi_0,\eta_0)=1-\int_{\xi_0}^{\xi}du\int_{\eta}^{\eta_0}p\,(u,v)\,r(u,v,\,\xi_0,\eta_0)dv, \tag{2.1}$$

where

$$p\,(u,v)=(u+v)^{\alpha}-(u-v)^{\alpha}\,. \tag{2.2}$$

This equation has a unique continuous solution which can be obtained by the method of successive approximations. In fact, set

$$r_0(\xi,\eta,\,\xi_0,\eta_0)=1\,,$$

$$r_n(\xi,\eta,\,\xi_0,\eta_0)=-\int_{\xi_0}^{\xi}du\int_{\eta}^{\eta_0}p\,(u,v)\,r_{n-1}(u,v,\,\xi_0,\eta_0)dv. \tag{2.3}$$

Note that in the domain $G=\{(\xi,\eta):\;0<\eta<\eta_0\leq\xi_0<\xi\}$ the function $p\,(\xi,\eta)$ satisfies the estimate

$$|p\,(\xi,\eta)|\leq 2^{\alpha}\xi^{\alpha}.$$

Then we will have

$$|r_1\,(\xi,\eta,\,\xi_0,\eta_0)|\leq 2^{\alpha}\int_{\xi_0}^{\xi}u^{\alpha}du\int_{\eta}^{\eta_0}dv=\frac{2^{\alpha}}{\alpha+1}\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)(\eta_0-\eta)\,.$$

Taking this estimate into account, we obtain

$$|r_2\,(\xi,\eta,\,\xi_0,\eta_0)|\leq 2^{\alpha}\int_{\xi_0}^{\xi}du\int_{\eta}^{\eta_0}\frac{2^{\alpha}}{\alpha+1}u^{\alpha}\left(u^{\alpha+1}-\xi_0^{\alpha+1}\right)(\eta_0-v)\,dv=$$

$$=\left(\frac{2^{\alpha}}{\alpha+1}\right)^2\left.\frac{\left(u^{\alpha+1}-\xi_0^{\alpha+1}\right)^2}{2}\right|_{\xi_0}^{\xi}\left.\frac{(\eta_0-v)^2}{2}\right|_{\eta_0}^{\eta}=\left(\frac{2^{\alpha}}{\alpha+1}\right)^2\frac{\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)^2(\eta_0-\eta)^2}{(2!)^2}.$$

Using induction, we next prove that

$$|r_n\,(\xi,\eta,\,\xi_0,\eta_0)|\leq\left(\frac{2^{\alpha}}{\alpha+1}\right)^n\frac{\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)^n(\eta_0-\eta)^n}{(n!)^2},n=1,2,.... \tag{2.4}$$

The estimates (2.4) show that the series $r\,(\xi,\eta,\,\xi_0,\eta_0)=\sum\limits_{n=0}^{\infty}r_n\,(\xi,\eta,\,\xi_0,\eta_0)$, of continuous functions $r_n\,(\xi,\eta,\,\xi_0,\eta_0)$, converges absolutely and uniformly in $\{(\xi,\eta):0\leq\eta\leq\eta_0\leq\xi_0\leq\xi\}$ and that its sum satisfies equation (2.1). Moreover, due to (2.4) we have

$$|r\,(\xi,\eta,\,\xi_0,\eta_0)|\leq\sum_{n=0}^{\infty}\left(\frac{2^{\alpha}}{\alpha+1}\right)^n\frac{\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)^n(\eta_0-\eta)^n}{(n!)^2}=$$

$$=J_0\left(i\sqrt{\frac{2^{\alpha+2}}{\alpha+1}\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)(\eta_0-\eta)}\right), \tag{2.5}$$

where $J_0(z) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$ is the Bessel function of the first kind. Using the well-known [2] inequality $|J_0(z)| \le e^{|Imz|}$, from (2.5) we finally obtain (1.7).

In view of the continuity of the function $p(u,v)$, it follows from equation (2.1) that $r(\xi,\eta)$ continuously differentiable in $\{(\xi,\eta) : 0 \le \eta \le \eta_0 \le \xi_0 \le \xi\}$, and that

$$r_\xi(\xi,\eta,\,\xi_0,\eta_0) = -\int_\eta^{\eta_0} p(\xi,v)\,r(\xi,v,\,\xi_0,\eta_0)dv,$$

$$r_\eta(\xi,\eta,\,\xi_0,\eta_0) = \int_{\xi_0}^{\xi} p(u,\eta)\,r(u,\eta,\,\xi_0,\eta_0)du.$$

By virtue of these equalities and the continuous differentiability of the function $p(u,v)$, the function $r(\xi,\eta)$ is twice continuously differentiable and satisfies equation (1.4). In addition, from the last equations, the equations that are obtained from them by differentiation, and estimates (1.7), we obtain that for all $0 < \eta \le \eta_0 \le \xi_0 < \xi$the following relations hold:

$$|r_\xi(\xi,\eta,\,\xi_0,\eta_0)| \le 2^\alpha \xi^{\alpha+1} e^{\sigma(\xi)},$$

$$|r_\eta(\xi,\eta,\,\xi_0,\eta_0)| \le 2^\alpha \xi^{\alpha+1} e^{\sigma(\xi)},$$

$$|r_{\xi\eta}(\xi,\eta,\,\xi_0,\eta_0)| \le 2^\alpha \xi^{\alpha} e^{\sigma(\xi)},$$

$$|r_{\xi\xi}(\xi,\eta,\,\xi_0,\eta_0)| \le 2^{\alpha-1} \xi^{\alpha} e^{\sigma(\xi)} + (2^\alpha)^2\,\xi^{2\alpha+2} e^{\sigma(\xi)},$$

$$|r_{\eta\eta}(\xi,\eta,\,\xi_0,\eta_0)| \le 2^{\alpha-1} \xi^{\alpha} e^{\sigma(\xi)} + (2^\alpha)^2\,\xi^{2\alpha+2} e^{\sigma(\xi)}.$$

Thus, the proof of the theorem is complete.

Remark . Let $\alpha = 1$. Then from relations (2.1), (2.2) we have

$$p(u,v) = 2v,$$

$$r_n(\xi,\eta,\,\xi_0,\eta_0) = (-1)^n\,\frac{(\xi-\,\xi_0)^n}{n!}\,\frac{(\eta_0^2-\,\eta^2)^n}{n!}.$$

Whence it follows that

$$r(\xi,\eta,\,\xi_0,\eta_0) = \sum_{n=0}^{\infty} r_n(\xi,\eta,\,\xi_0,\eta_0) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{2\sqrt{(\xi-\,\xi_0)(\eta_0^2-\,\eta^2)}}{2}\right)^{2n} = J_0\left(2\sqrt{(\xi-\,\xi_0)(\eta_0^2-\,\eta^2)}\right).$$

Similarly, for $\alpha = 2$ from (2.1), (2.2) we find that

$$r(\xi,\eta,\,\xi_0,\eta_0) = J_0\left(2\sqrt{(\xi^2-\,\xi_0^2)(\eta_0^2-\,\eta^2)}\right).$$

Moreover, in both cases, instead of estimate (1.7), we obtain

$$|r(\xi,\eta,\,\xi_0,\eta_0)| \le 1.$$

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