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Direct and inverse problems for a parabolic-hyperbolic equation involving Riemann–Liouville derivatives

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Abstract. *This work is devoted to the investigation of direct and inverse problems with nonlinear gluing condition for a mixed parabolic-hyperbolic equation involves Riemann–Liouville time fractional derivatives. The problem is reduced to study nonlinear Volterra integral equations. The methods of integral equations and successive approximations are used in proving theorems on existence and uniqueness.*

Keywords. Direct problem, inverse problem, parabolic-hyperbolic equation, Riemann–Liouville derivatives, nonlinear loaded terms, nonlinear integral equations, solvability.

Mathematics Subject Classification (2010): 34K37, 35A02, 35M10

1 Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of sciences and engineering. For instance, we can find numerous of applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [7], [17], [18], [20]). There has been a significant development in fractional differential equations in recent years (see the monographs [8]–[10], [16], [21]–[23], [26]–[28], [30], [32], [36] and the references therein). Notice, that some basic theory for the initial value problems of fractional differential equations involving Riemann–Liouville differential operator has been discussed in [24].

One hand, when modeling various processes, practical needs lead to problems of determining the right-hand side of a differential equation (source function) from some available data about the solution. These are the so-called inverse problems of determining sources of fractional partial differential equations. These types of inverse problems arise in various fields of human activity, such as seismology, biology, medicine, quality control of industrial

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goods, etc. Therefore, inverse problems among the important problems of modern mathematics (see the recent survey paper Liu, Li and Yamamoto [25] and references therein; see also [5], [31], [33]). Inverse problems of various type for partial differential equations of fractional order were studied in the many works (see, for examples, [1]–[6], [11]–[15], [19], [29], [34]–[47]).

In this article, we consider the direct and inverse problems for parabolic-hyperbolic equation of fractional order involves nonlinear loaded terms. The fractional part of considered equation will be defined through the Riemann–Liouville time fractional derivative D_{0t}^{α} of order $0<\alpha<1$.

2 Formulation of the problem

We consider the equation

$$\begin{cases} u_{xx}-D_{0t}^{\alpha}u+p_1(x,t,z_1(x))=f_1(x,t), & t>0,\\ u_{xx}-u_{tt}+p_2(x,t,z_2(x))=f_2(x,t), & t<0,\end{cases} \tag{2.1}$$

where D_{at}^{α} is Riemann–Liouville differential operator of fractional order $0<\alpha<1$,

$$z_i(x)=\lim_{(-1)^{i-1}t\to+0} D_{0t}^{(2-j)(\alpha-1)}u(x,t),$$

$$f_i(x,t)\in C(\Omega_i),\ \ t^{1-\alpha}p_1(x,t,z_1)\in C\left(\bar{\Omega}_1\times\mathbb{R}\right),\ \ p_2(x,t,z_2)\in C\left(\bar{\Omega}_2\times\mathbb{R}\right). \tag{2.2}$$

Notice, the functions $p_i(x,t,z_i(x))$ are nonlinear loaded parts of equation (2.1).

Let Ω be one connected domain bounded with segments: $A_1A_2=\{(x,t):\ x=l,\ 0<t<h\},\ B_1B_2=\{(x,t):\ x=0,\ 0<t<h\},\ B_2A_2=\{(x,y):\ t=h,\ 0<x<l\}$ at $t>0$ and by characteristics $B_1C:\ x+t=0,\ A_1C:\ x-t=l$ of equation (2.1) at $t<0$.

We enter designations $\Omega_1=\Omega\cap\{t>0\},\ \Omega_2=\Omega\cap\{t<0\}$ and $I=\{(x,t):\ t=0,\ 0<x<l\}$.

Direct Problem. To find a function $u(x,t)$ for equation (2.1) in domain $\Omega\setminus I$ with the following properties:

1. $u(x,t)\in W_1$, where $W_1=\Big\{u:\ D_{0t}^{\alpha-1}u\in C(\bar{\Omega}_1), u\in C(\bar{\Omega}_2)\cap C^2(\Omega_2),$ $u_{xx}\in C\left(\Omega_1\right), u_t\in C\left(\Omega_2\cup I\right), D_{0t}^{\alpha}u\in C\left(\Omega_1\cup I\right), D_{0t}^{\alpha-1}u_x\in C\left(\Omega_1\setminus\overline{A_2B_2}\right)\Big\},$
2. $u(x,t)$ satisfies boundary conditions:

$$u_x\left(0,t\right)=\varphi_1(t),\ \ u_x\left(l,t\right)=\varphi_2(t),\ \ 0\le t<h, \tag{2.3}$$

$$u\left(\frac{x}{2},-\frac{x}{2}\right)=\delta_1(x)u(x,-0)+\delta_2(x),\ \ 0\le x\le l, \tag{2.4}$$

matching gluing conditions:

$$\lim_{t\to+0} t^{1-\alpha}u(x,t)=\mu_1(x)u(x,-0)+\mu_2(x),\ 0\le x\le l \tag{2.5}$$

and for all x from the open interval $0<x<l$ it is true that

$$\lim_{t\to+0} t^{1-\alpha}\left(t^{1-\alpha}u(x,t)\right)_t=\lambda_1(x)u_t(x,-0)+\lambda_2(x)u_x(x,-0)+r(x,u(x,-0)), \tag{2.6}$$

where $r(x,z),\ \delta_i(x),\ \mu_i(x),\ \lambda_i(x),\ \varphi_i(t)\ (i=1,2)$ are given functions, and

$$t^{1-\alpha}\varphi_1(t),\ t^{1-\alpha}\varphi_2(t)\in C[0,h],\ \ \delta_i(x),\mu_i(x)\in C\left[0,l\right]\cap C^2\left(0,l\right), \tag{2.7}$$

$$\lambda_i(x) \in C\,[0,l] \cap C^1(0,l), \;\; r(x,z) \in C\left([0,l] \times \mathbb{R}\right) \cap C^1\left((0,l) \times \mathbb{R}\right). \tag{2.8}$$

Moreover, we suppose that $\mu_1(x) \neq 0$ for all $x \in [0,l]$, $\delta_1(0) \neq 1$.

We put

$$\begin{cases} f_1(x,t) = f(x)g_1(t), \text{ for } (x,t) \in \Omega_1, \\ f_2(x,t) = f(x)g_2(t), \text{ for } (x,t) \in \Omega_2, \end{cases}$$

where $g_i(t)$ $(j = 1, 2)$ are given functions.

Inverse Problem. To find a pair of functions $\{u(x,t),\, f(x)\}$ for equation (2.1) in $\Omega \setminus I$ with the following properties:

1. $f(x) \in C(0,l) \cap L_1(0,l)$;
2. $u(x,t) \in W_2$ satisfies boundary conditions (2.3) and (2.4), gluing conditions (2.5) and (2.6), and following additional conditions

$$\frac{\partial u(x,t)}{\partial n}|_{y=x} = \psi_1(x), \;\; 0 < x \leq \frac{l}{2}; \;\; \frac{\partial u(x,t)}{\partial n}|_{y=x-l} = \psi_2(x), \;\; \frac{l}{2} \leq x < l, \tag{2.9}$$

where $W_2 = W_1 \cap W_0$, $W_0 = \left\{u : u \in C^1\left(\overline{\Omega}_2 \setminus \overline{I}\right)\right\}$, $\psi_j(x)\,(j = 1, 2)$ are given functions, and

$$\psi_1(x) \in C\left[0, \frac{l}{2}\right] \cap C^1\left(0, \frac{l}{2}\right], \; \psi_2(x) \in C\left[\frac{l}{2}, l\right] \cap C^1\left[\frac{l}{2}, l\right). \tag{2.10}$$

3 Direct Problem

31 Main functional relation and reduction of the problem

Notice, that the solution of Cauchy problem for equation (2.1) with the initial data

$$u(x,-0) = \tau^-(x), \; 0 \leq x \leq l, \;\; u_t(x,-0) = \nu^-(x), \; 0 < x < l \tag{3.1}$$

has a form:

$$u(x,t) = \frac{1}{2}\left(\tau^-(x+t) + \tau^-(x-t)\right) - \frac{1}{4}\int_{x+t}^{x-t} d\eta \int_{x+t}^{\eta} f_2\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) d\xi$$
$$+ \frac{1}{4}\int_{x+t}^{x-t} d\eta \int_{x+t}^{\eta} p_2\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}, \tau^-\left(\frac{\xi+\eta}{2}\right)\right) d\xi. \tag{3.2}$$

By virtue of conditions (2.4), from the solution (3.2) we obtain

$$\nu^-(x) = (1 - 2\delta_1(x))\tau'^-(x) - 2\delta_1'(x)\tau^-(x)$$
$$+ \frac{1}{2}\int_0^x p_2\left(\frac{t+x}{2}, \frac{t-x}{2}, \tau^-\left(\frac{t+x}{2}\right)\right) dt$$
$$- \frac{1}{2}\int_0^x f_2\left(\frac{t+x}{2}, \frac{t-x}{2}\right) dt - 2\delta_2'(x). \tag{3.3}$$

On the other hand, introducing notations

$$\lim_{t \to +0} D_{0t}^{\alpha-1} u(x,t) = \tau^+(x), \;\; \lim_{t \to +0} t^{1-\alpha}\left(t^{1-\alpha} u(x,t)\right)_t = \nu^+(x), \tag{3.4}$$

and applying operator $D_{0t}^{\alpha-1}$ to both sides of (2.1) at $t\to+0$, due to equality $\lim\limits_{t\to+0} D_{0t}^{\alpha-1}f(t)=\Gamma(\alpha)\lim\limits_{t\to+0} t^{1-\alpha}f(t)$, we have

$$\tau''^{+}(x)-\Gamma^2(\alpha)\nu^{+}(x)+\Gamma(\alpha)p_{11}\left(x,\tau^{+}(x)\right)=\Gamma(\alpha)\bar f_1(x), \tag{3.5}$$

where $\Gamma(\alpha)\bar f_1(x)=\lim\limits_{t\to+0} D_{0t}^{\alpha-1}f_1(x,t),\ \Gamma(\alpha)p_{11}(x,\tau^{+}(x))=\lim\limits_{t\to+0} D_{0t}^{\alpha-1}p_1(x,t,z_1)$.

Considering designation (3.1) and (3.4), from (2.5) and (2.6) we derive

$$\nu^{+}(x)=\lambda_1(x)\nu^{-}(x)+\lambda_2(x)\tau'^{-}(x)+r(x,\tau^{-}(x)), \tag{3.6}$$

$$\tau^{+}(x)=\Gamma(\alpha)\mu_1(x)\tau^{-}(x)+\Gamma(\alpha)\mu_2(x). \tag{3.7}$$

Taking (3.6), (3.7) and (3.3) into account, from equation (3.5) we come to a nonlinear integro-differential equation

$$\mu_1(x)\tau''(x)+\left(2\mu_1'(x)-\Gamma(\alpha)\lambda_1(x)(1-2\delta_1(x))-\Gamma(\alpha)\lambda_2(x)\right)\tau'(x)$$

$$+(\mu_1''(x)+2\Gamma(\alpha)\lambda_1(x)\delta_1'(x))\tau(x)-\Gamma(\alpha)r(x,\tau(x))+p_{11}\left(x,\Gamma(\alpha)\mu_1(x)\tau(x)+\Gamma(\alpha)\mu_2(x)\right)$$

$$-\frac{\Gamma(\alpha)\lambda_1(x)}{2}\int_0^x p_2\left(\frac{t+x}{2},\frac{t-x}{2},\tau^{-}\left(\frac{t+x}{2}\right)\right)dt$$

$$=\bar f_1(x)-\mu_2''(x)-\frac{\Gamma(\alpha)\lambda_1(x)}{2}\int_0^x f_2\left(\frac{t+x}{2},\frac{t-x}{2}\right)dt-2\Gamma(\alpha)\lambda_1(x)\delta_2'(x), \tag{3.8}$$

where $\tau(x)=\tau^{-}(x)$.

Hence, owing to condition $\mu_1(x)\neq0,\ \forall x\in[0,l]$, equation (3.8) we will rewrite as:

$$\tau''(x)+a(x)\tau'(x)+b(x)\tau(x)+A(x,\tau(x))$$

$$-\frac{\Gamma(\alpha)\lambda_1(x)}{2}\int_0^x p_2\left(\frac{t+x}{2},\frac{t-x}{2},\tau^{-}\left(\frac{t+x}{2}\right)\right)dt=F(x), \tag{3.9}$$

where

$$a(x)=\frac{2\mu_1'(x)-\Gamma(\alpha)\lambda_1(x)(1-2\delta_1(x))-\Gamma(\alpha)\lambda_2(x)}{\mu_1(x)}, \tag{3.10}$$

$$b(x)=\frac{\mu_1''(x)+2\Gamma(\alpha)\lambda_1(x)\delta_1'(x)}{\mu_1(x)}, \tag{3.11}$$

$$A(x,\tau(x))=\frac{-1}{\mu_1(x)}\left(\Gamma(\alpha)r(x,\tau(x))-p_{11}\left(x,\Gamma(\alpha)\mu_1(x)\tau(x)+\Gamma(\alpha)\mu_2(x)\right)\right), \tag{3.12}$$

$$F(x)=\frac{\bar f_1(x)-\mu_2''(x)-2\Gamma(\alpha)\lambda_1(x)\delta_2'(x)}{\mu_1(x)}-\frac{\Gamma(\alpha)\lambda_1(x)}{2\mu_1(x)}\int_0^x f_2\left(\frac{t+x}{2},\frac{t-x}{2}\right)dt.$$

Based on the class of given functions (2.2), (2.7) and (2.8), easy to spot, that

$$|a(x)|\le a_0,\ |b(x)|\le b_0,\ |A(x,z)|\le A_0,\ |F(x)|\le F_0, \tag{3.13}$$

where $a_0,\ b_0,\ A_0,\ F_0=\text{const}>0$.

Lemma 3.1 *If $\mu_1(x) \in C[0,l] \cap C^2(0,l)$, and the conditions*

$$r(x,z), p_{11}(x,z) \in C\left([0,l] \times \mathbb{R}\right) \cap C^1\left((0,l) \times \mathbb{R}\right), \tag{3.14}$$

$$|r(x,z_1) - r(x,z_2)| \leq L_0\,|z_1 - z_2|,\ \forall z_1, z_2 \in \mathbb{R}, \tag{3.15}$$

$$|p_{11}(x,z_1) - p_{11}(x,z_2)| \leq L_1\,|z_1 - z_2|,\ \forall z_1, z_2 \in \mathbb{R} \tag{3.16}$$

are true, then takes place the inequality

$$|A(x,z_1) - A(x,z_2)| \leq L\,|z_1 - z_2|,\ \forall z_1, z_2 \in \mathbb{R}, \tag{3.17}$$

where $L_0,\ L_1 = \text{const} > 0,\ L = \Gamma(\alpha)\mu_0 L_0 + \Gamma^2(\alpha) L_1,\ \mu_0 = \max\limits_{x\in[0,l]} \frac{1}{|\mu_1(x)|}$.

Proof. Using condition (3.16), we obtain

$$\big|p_{11}\big(x, \Gamma(\alpha)\mu_1(x)z_1 + \Gamma(\alpha)\mu_2(x)\big) - p_{11}\big(x, \Gamma(\alpha)\mu_1(x)z_2 + \Gamma(\alpha)\mu_2(x)\big)\big|$$

$$\leq L_1\big|\Gamma(\alpha)\mu_1(x)z_1 - \Gamma(\alpha)\mu_1(x)z_2\big| \leq L_1\Gamma(\alpha)|\mu_1(x)|\,|z_1 - z_2|. \tag{3.18}$$

Further, taking (3.15), (3.18) into account, from (3.12), we get

$$|A(x,z_1) - A(x,z_2)| \leq \left|\frac{\Gamma(\alpha)}{\mu_1(x)}\right| |r(x,z_1) - r(x,z_2)|$$

$$+\left|\frac{\Gamma(\alpha)}{\mu_1(x)}\right| \big|p_{11}\big(x, \Gamma(\alpha)\mu_1(x)z_1 + \Gamma(\alpha)\mu_2(x)\big) - p_{11}\big(x, \Gamma(\alpha)\mu_1(x)z_2 + \Gamma(\alpha)\mu_2(x)\big)\big|$$

$$\leq \left|\frac{\Gamma(\alpha)}{\mu_1(x)}\right| \Big(L_0|z_1 - z_2| + L_1\Gamma(\alpha)|\mu_1(x)| \cdot |z_1 - z_2|\Big).$$

Taking the class of function $\mu_1(x)$ into account and due to $\mu_1(x) \neq 0,\ \forall x \in [0,l]$, we obtain $\frac{1}{|\mu_1(x)|} \leq \mu_0$, where $\mu_0 = \text{const} > 0$. Hence, we derive, that

$$|A(x,z_1) - A(x,z_2)| \leq L\,|z_1 - z_2|,$$

where $L = \Gamma(\alpha)\mu_0 L_0 + \Gamma^2(\alpha) L_1$. Lemma 3.1 is proved.

Since $\mu_1(0) \neq 0,\ \delta_1(0) \neq 1$,

$$\tau'^{+}(0) = \Gamma(\alpha) \lim_{t\to+0} t^{1-\alpha}\varphi_1(t) = \varphi_0\, \tau(0) = \tau^{-}(0) = \frac{\delta_2(0)}{1 - \delta_1(0)} = \tau_{01},$$

from

$$\tau'^{+}(x) = \Gamma(\alpha)\mu_1'(x)\tau^{-}(x) + \Gamma(\alpha)\mu_1(x)\tau'^{-}(x) + \Gamma(\alpha)\mu_2'(x)$$

we have

$$\tau'(0) = \tau'^{-}(0) = \frac{1}{\Gamma(\alpha)\mu_1(0)}\left(\varphi_0 - \Gamma(\alpha)\mu_2'(0) - \frac{\Gamma(\alpha)\mu_1'(0)\delta_2(0)}{1 - \delta_1(0)}\right) = \tau_{02}.$$

Thus, we found initial conditions $\tau(0) = \tau_{01}$ and $\tau'(0) = \tau_{02}$ to nonlinear integral equation (3.9). Using these initial conditions, we integrate the equation (3.9) two times with respected to x and we get a nonlinear integral equation

$$\tau(x) = \int_0^x K(x,t)\tau(t)dt - \int_0^x (x-t)A(t,\tau(t))dt$$

$$+\frac{\Gamma(\alpha)}{2}\int_0^x \lambda_1(t)(x-t)dt\int_0^t p_2\left(\frac{z+t}{2},\frac{z-t}{2},\tau\left(\frac{z+t}{2}\right)\right)dz+F_1(x), \qquad (3.19)$$

where

$$K(x,t)=(a(t)(x-t))'-b(t)(x-t), \qquad (3.20)$$

$$F_1(x)=(a(0)x+1)\tau_{01}+x\tau_{02}+\int_0^x (x-t)F(t)dt.$$

Hence, by virtue of (3.13), we obtain

$$|K(x,t)|\leq k_0,\ |F_1(x)|\leq F_{01},\ k_0,F_{01}=\text{const}>0. \qquad (3.21)$$

Theorem 3.1 *If all conditions of the Lemma 3.1 and* (2.2)*,* (2.7)*,* (2.8) *are true and takes place the Lipschitz condition*

$$|p_2(x,t,z_1)-p_2(x,t,z_2)|\leq L_2\,|z_1-z_2|,\ L_2=\text{const}>0,$$

then a solution of the direct problem exist and unique.

Proof. We would like to note, that the investigation of the direct problem symmetrically reduced to the unique solvability of the nonlinear integral equation (3.19), i.e. existence and uniqueness of solution of the direct problem follows from the unique solvability to equation (3.19).

Unique solvability nonlinear integral equation (3.19) we prove by the method successive approximations. With the aid of recurrent equation

$$\tau_n(x)=\int_0^x K(x,t)\tau_{n-1}(t)dt-\int_0^x (x-t)A(t,\tau_{n-1}(t))dt$$

$$+\frac{\Gamma(\alpha)}{2}\int_0^x \lambda_1(t)(x-t)dt\int_0^t p_2\left(\frac{z+t}{2},\frac{z-t}{2},\tau_{n-1}\left(\frac{z+t}{2}\right)\right)dz+F_1(x), \qquad (3.22)$$

we construct functional sequence $\{\tau_n(x)\}$ with zero approximations $\tau_0(x)=F_1(x)$.

By virtue of the classes (see (2.2), (2.7) and (2.8)) for given functions, we have

$$|\Gamma(\alpha)\lambda_1(x)|\leq\lambda_{01},\ \ |p_i(x,t,z)|\leq p_{0i}\ (i=1,2). \qquad (3.23)$$

By virtue of (3.13), from (3.22) we have

$$|\tau_1(x)-\tau_0(x)|\leq f_0k_0x+a_0\frac{x^2}{2!}+\lambda_{01}p_{02}\frac{x^3}{3!}. \qquad (3.24)$$

It is easy to check, that

$$x^2l\leq xl,\ \ x^3\leq x^2l\leq xl^2\ \textit{for}\ \ 0<x\leq l.$$

Therefore, the inequality (3.24) we can write as

$$|\tau_1(x)-\tau_0(x)|\leq c_0x,\ c_0=f_0k_0+A_0\frac{l}{2}+\lambda_{01}p_{02}\frac{l^2}{6}. \qquad (3.25)$$

Similarly, taking (3.25) into account, from (3.22) we obtain

$$|\tau_2(x)-\tau_1(x)|\leq k_0c_0\frac{x^2}{2!}+Lc_0\frac{x^3}{3!}+\lambda_{01}L_2c_0\frac{x^4}{4!}\leq c_0(k_0+L\frac{l}{3}+L_2\frac{l^2}{6})\frac{x^2}{2!}=c_0c_1\frac{x^2}{2!},$$

$$|\tau_3(x)-\tau_2(x)|\leq k_0c_0c_1\frac{x^3}{3!}+Lc_0c_1\frac{x^4}{4!}+\lambda_{01}L_2c_0c_1\frac{x^5}{5!}\leq c_0c_1(k_0+L\frac{l}{3}+L_2\frac{l^2}{6})\frac{x^3}{3!}=c_0c_1^2\frac{x^3}{3!}.$$

Consequently, applying the method of mathematical induction, it is easy to show that

$$|\tau_n(x) - \tau_{n-1}(x)| \le c_0 c_1^{n-1} \frac{x^n}{n!}. \tag{3.26}$$

Thus, we conclude that the right-hand side of equation (3.19) is a compressive operator, and there exist a unique fixed point for this operator. Hence, the nonlinear integral equation (3.19) has a unique solution from the class of functions $C[0,1] \cap C^2(0,1)$.

After determining $\tau(x) = \tau^-(x)$, unknown functions $\tau^+(x)$ and $\nu^-(x)$ we will find from (3.7) and (3.3), respectively. Consequently, a solution of the direct problem in domain Ω_2 we will restore as a solution of the Cauchy problem (see (3.2)), and in domain Ω_1 as a solution of the second boundary-value problem [29]. Thus, existence of solution of the direct problem is proved.

Now we prove uniqueness of solution of equation (3.19). We suppose the opposite: let us $\tau_1(x)$ and $\tau_2(x)$ are two different solutions of equation (3.19). Then from the equation (3.19), we have

$$\begin{aligned}\tau_1(x) - \tau_2(x) = \int_0^x K(x,t)\big(\tau_1(t) - \tau_1(t)\big)dt - \int_0^x (x-t)\left(A\big(t,\tau_1(t)\big) - A\big(t,\tau_2(t)\big)\right) dt \\ +\frac{\Gamma(\alpha)}{2}\int_0^x \lambda_1(t)(x-t)dt\int_0^t \left[p_2\left(\frac{z+t}{2}, \frac{z-t}{2}, \tau_1\left(\frac{z+t}{2}\right)\right)\right. \\ \left. -p_2\left(\frac{z+t}{2}, \frac{z-t}{2}, \tau_2\left(\frac{z+t}{2}\right)\right)\right] dz.\end{aligned} \tag{3.27}$$

Entering designation $\tau(x) \equiv \tau_1(x) - \tau_2(x)$ and considering (4.23), (3.23) and inequality (3.17), from (3.27) we find

$$\begin{aligned}|\tau(x)| \le \left|\int_0^x K(x,t)\tau(t)dt\right| + \left|\int_0^x (x-t)\big|A\big(t,\tau_1(t)\big) - A\big(t,\tau_2(t)\big)\big|dt\right| \\ +\frac{\Gamma(\alpha)}{2}\left|\int_0^x \lambda_1(t)(x-t)dt\int_0^t \left|p_2\left(\frac{z+t}{2}, \frac{z-t}{2}, \tau_1\left(\frac{z+t}{2}\right)\right)\right.\right. \\ \left.\left. -p_2\left(\frac{z+t}{2}, \frac{z-t}{2}, \tau_2\left(\frac{z+t}{2}\right)\right)\right| dz\right| \\ \le k_0\left|\int_0^x |\tau(t)|dt\right| + L\left|\int_0^x (x-t)|\tau(t)|dt\right| + \frac{\lambda_{01}L_2}{2}\left|\int_0^x (x-t)dt\int_0^t \left|\tau\left(\frac{z+t}{2}\right)\right| dz\right| \\ \le \left|k_0 + Lx + \frac{\lambda_{01}L_2}{2}\frac{x^2}{2}\right| \cdot \left|\int_0^x |\tau(t)|dt\right| \le c_2\int_0^x |\tau(t)|dt,\end{aligned} \tag{3.28}$$

where $c_2 = k_0 + Ll + \frac{\lambda_{01}L_2}{2}\frac{l^2}{2}$. Reinforcing inequality (3.28), we derive

$$|\tau(x)| \le c_2^2\left|\int_0^x dt\int_0^t |\tau(z)|dz\right| \le c_2^2\left|\int_0^x |\tau(t)||x-t|dt\right|,$$

$$|\tau(x)| \le c_2^3\left|\int_0^x |\tau(t)|\frac{(x-t)^2}{2!}dt\right|, \ \ldots,$$

$$|\tau(x)| \le c_2^{n+1}\left|\int_0^x |\tau(t)|\frac{(x-t)^n}{n!}dt\right| \le \frac{c_2^{n+1}l^n}{n!}\|\tau(x)\|_0, \tag{3.29}$$

where $\|\tau(x)\|_0 = \max\limits_{x\in[0,l]} \int\limits_0^x |\tau(t)|dt$.

Considering $\|\tau(x)\|_0 \le \text{const} < \infty$, $\forall x \in [0,l]$, from (3.29) at $n \to \infty$ we get $|\tau(x)| \equiv 0$. Hence, we conclude that $\tau_1(x) \equiv \tau_2(x)$, i.e. solution of equation (3.19) is unique. Thus, Theorem 3.1 is proved.

4 Inverse Problem

We note,that the general solution of equation (2.1) in Ω_2 has a form:

$$u(x,t)=F_1(x+t)+F_2(x-t)+g_2(t)w(x)$$
$$+\frac{1}{4}\int_{x+t}^{x-t}d\eta\int_{x+t}^{\eta}p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2},z_2\left(\frac{\xi+\eta}{2}\right)\right)d\xi, \tag{4.1}$$

where $F_1(x+t)$ and $F_2(x-t)$ are arbitrary two times continuous-differentiable functions, and $w(x)$ is any solution of the equation

$$g_2(t)w''(x)-g_2''(t)w(x)=f(x)g_2(t),\ (x,t)\in\Omega_2. \tag{4.2}$$

Using (2.9), from the solution (4.1) we find

$$2F_1'(0)+g_2(-x)w'(x)+g_2'(-x)w(x)$$
$$-\frac{1}{2}\int_0^{2x}p_2\left(\frac{l_1+\eta}{2},\frac{l_1-\eta}{2},z_2\left(\frac{\xi+\eta}{2}\right)\right)d\eta=\sqrt{2}\psi_1(x),\ 0<x\le\frac{l}{2}, \tag{4.3}$$
$$2F_2'(l)+g_2(x-l)w'(x)-g_2'(x-l)w(x)$$
$$+\frac{1}{2}\int_{2x-l}^{l}p_2\left(\frac{\xi+l_2}{2},\frac{\xi-l_2}{2},z_2\left(\frac{\xi+\eta}{2}\right)\right)d\xi=\sqrt{2}\psi_2(x),\ \frac{l}{2}\le x<l. \tag{4.4}$$

We assume, that $f(x)=\begin{cases}f_1(x), & \text{at } 0<x\le\frac{l}{2},\\ f_2(x), & \text{at } \frac{l}{2}\le x<l,\end{cases}$ then, differentiating (4.3) and (4.4), owing to (4.2), we find

$$f_1(x)g_2(-x)=p_2(x,-x,z_2(x))+\sqrt{2}\psi_2'(x),\ 0<x\le\frac{l}{2}, \tag{4.5}$$

and

$$f_2(x)g_2(x-l)=p_2(x,x-l,z_2(x))-\sqrt{2}\psi_3'(x),\ \frac{l}{2}\le x<l, \tag{4.6}$$

respectively, besides $f_1\left(\frac{l}{2}\right)=f_2\left(\frac{l}{2}\right)$.

Notice, that the solution of Cauchy problem for equation (2.1) with the initial dates

$$u(x,-0)=\tau^-(x),\ 0\le x\le l,\ \ u_t(x,-0)=\nu^-(x),\ 0<x<l; \tag{4.7}$$

has a form:

$$u(x,t)=\frac{1}{2}\big(\tau^-(x+t)+\tau^-(x-t)\big)-\frac{g_2(0)}{2}\big(w(x+t)+w(x-t)\big)+g(t)w(x)$$
$$-\frac{1}{2}\int_{x+t}^{x-t}\nu^-(z)dz+\frac{g_2'(0)}{2}\int_{x+t}^{x-t}w(z)dz$$
$$+\frac{1}{4}\int_{x+t}^{x-t}d\eta\int_{x+t}^{\eta}p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2},\tau^-\left(\frac{\xi+\eta}{2}\right)\right)d\xi. \tag{4.8}$$

By virtue of condition (2.4), from the solution (4.8) we get

$$\nu^-(x)=(1-2\delta_1(x))\tau'^-(x)-2\delta_1'(x)\tau^-(x)-w'(x)g_2(0)$$
$$-g_2'\left(\frac{-x}{2}\right)w\left(\frac{x}{2}\right)+g_2\left(\frac{-x}{2}\right)w'\left(\frac{x}{2}\right)$$

$$+g_2'(0)w(x)-2\delta'(x)+\frac{1}{2}\int_0^x p_2\left(\frac{t+x}{2},\frac{t-x}{2},\tau^-\left(\frac{t+x}{2}\right)\right)dt. \qquad (4.9)$$

We would like to not that, if $g_2(t)\neq 0,\ \forall t\leq 0$, then from the equation (4.2) follows that $\frac{g_2''(t)}{g_2(t)}=\text{const}$.

If $\frac{g_2''(t)}{g_2(t)}=k^2,\ k\in\mathbb{R}\setminus\{0\}$, then the equation (4.2) has a form

$$w''(x)-k^2w(x)=f(x),\ 0<x<l. \qquad (4.10)$$

Notice, that one of the solutions of equation (4.10) is presented as follows:

$$w(x)=\int_0^x f(t)\left[(x-t)+k\int_t^x(z-t)\sinh k(x-z)dz\right]dt. \qquad (4.11)$$

Considering (4.5) and (4.6) at $g_2(x)\neq 0$, from (4.11) we receive

$$w(x)=\int_0^x \bar{H}(x,t)p_2(t,\tau(t))dt+\int_0^x\psi(t)\bar{H}(x,t)dt, \qquad (4.12)$$

where

$$\bar{H}(x,t)=\begin{cases}\frac{H(x,t)}{g_2(-t)} & \text{for } 0\leq t\leq x\leq \frac{l}{2},\\ \frac{H(x,t)}{g_2(l-t)} & \text{for } \frac{l}{2}\leq t\leq x\leq l,\end{cases} \qquad (4.13)$$

$$H(x,t)=(x-t)+k\int_t^x(z-t)\sinh k(x-z)dz,$$

$$p_2(x,\tau(x))=\begin{cases}p_2(x,-x,\tau(x)), & \text{for } 0\leq x\leq\frac{l}{2},\\ p_2(x,l-x,\tau(x)), & \text{for } \frac{l}{2}\leq x\leq l,\end{cases} \qquad (4.14)$$

$$\psi(x)=\begin{cases}\sqrt{2}\frac{\psi_2'(x)}{g_2(-x)}, & \text{for } 0\leq x\leq\frac{l}{2},\\ -\sqrt{2}\frac{\psi_3'(x)}{g_2(x-l)}, & \text{for } \frac{l}{2}\leq x\leq l.\end{cases}$$

Thus, substituting function (4.11) into functional relation (4.9), we get

$$\nu^-(x)=(1-2\delta_1(x))\tau'^-(x)-2\delta_1'(x)\tau^-(x)+\int_0^x p_2(t,\tau(t))H_3(x,t)dt$$

$$+\int_0^{\frac{x}{2}}p_2(t,\tau(t))H_4(x,t)dt+\frac{1}{2}\int_0^x p_2\left(\frac{t+x}{2},\frac{t-x}{2},\tau^-\left(\frac{t+x}{2}\right)\right)dt$$

$$+\int_0^x\psi(t)H_3(x,t)dt-\sqrt{2}\int_0^{\frac{x}{2}}\frac{\psi_3'(t)}{g_2(t-l)}H_4(x,t)dt-2\delta'(x), \qquad (4.15)$$

where

$$H_3(x,t)=g_2'(0)\bar{H}(x,t)-g_2(0)\bar{H}_x(x,t),$$

$$H_4(x,t)=g_2\left(-\frac{x}{2}\right)\frac{H_x(x,t)}{g_2(-t)}-g_2'\left(-\frac{x}{2}\right)\frac{H(x,t)}{g_2(-t)}.$$

Further, as the direct problem, considering designations (3.4) and applying operator $D_{0t}^{\alpha-1}$ to both sides of (2.1) at $t\to+0$, due to equality

$$\lim_{t\to+0}D_{0t}^{\alpha-1}f(t)=\Gamma(\alpha)\lim_{t\to+0}t^{1-\alpha}f(t),$$

we have

$$\tau''^{+}(x)-\Gamma^{2}(\alpha)\nu^{+}(x)+\Gamma(\alpha)p_{11}\left(x,\tau^{+}(x)\right)=\Gamma(\alpha)f(x)g_{1}, \tag{4.16}$$

where $\Gamma(\alpha)g_1=\lim\limits_{t\to+0}D_{0t}^{\alpha-1}g_1(t)$.

Taking (3.6), (3.7) into account, from equation (4.16) we obtain a nonlinear integro-differential equation

$$\begin{aligned}
&\mu_1(x)\tau''(x)+\left(2\mu_1'(x)-\Gamma(\alpha)\left(\lambda_2(x)+\lambda_1(x)(1-2\delta_1(x))\right)\right)\tau'(x)\\
&+\left(\mu_1''(x)+2\Gamma(\alpha)\lambda_1(x)\delta_1'(x)\right)\tau(x)-\Gamma(\alpha)r(x,\tau(x))\\
&-\Gamma(\alpha)\lambda_1(x)\int_0^x p_2(t,\tau(t))H_3(x,t)dt-\Gamma(\alpha)\lambda_1(x)\int_0^{\frac{x}{2}} p_2(t,\tau(t))H_4(x,t)dt\\
&-\frac{\Gamma(\alpha)\lambda_1(x)}{2}\int_0^x p_2\left(\frac{t+x}{2},\frac{t-x}{2},\tau^{-}\left(\frac{t+x}{2}\right)\right)dt\\
&+p_{11}\left(x,\Gamma(\alpha)\mu_1(x)\tau(x)+\Gamma(\alpha)\mu_2(x)\right)=f(x)g_1-\mu_2''(x)+2\Gamma(\alpha)\lambda_1(x)\delta_2'(x)\\
&-\Gamma(\alpha)\lambda_1(x)\int_0^x\psi(t)H_3(x,t)dt+\sqrt{2}\Gamma(\alpha)\lambda_1(x)\int_0^{\frac{x}{2}}\frac{\psi_3'(t)}{g_2(t-l)}H_4(x,t)dt,
\end{aligned} \tag{4.17}$$

where $\tau(x)=\tau^{-}(x)$.

Due to $g_2(x)\neq 0$, the nonlinear integro-differential equation (4.17) we can write as

$$\begin{aligned}
&\tau''(x)+a(x)\tau'(x)+b(x)\tau(x)+B(x,\tau(x))-\frac{\Gamma(\alpha)\lambda_1(x)}{\mu_1(x)}\int_0^{\frac{x}{2}}p_2(t,\tau(t))H_4(x,t)dt\\
&-\frac{\Gamma(\alpha)\lambda_1(x)}{2\mu_1(x)}\int_0^x p_2\left(\frac{t+x}{2},\frac{t-x}{2},\tau^{-}\left(\frac{t+x}{2}\right)\right)dt=F_2(x),
\end{aligned} \tag{4.18}$$

where $a(x),\ b(x)$ are defined from (3.10), and

$$\begin{aligned}
B(x,\tau(x))=&\frac{1}{\mu_1(x)}p_{11}\left(x,\Gamma(\alpha)\mu_1(x)\tau(x)+\Gamma(\alpha)\mu_2(x)\right)-\frac{\Gamma(\alpha)}{\mu_1(x)}r(x,\tau(x))\\
&-\frac{\Gamma(\alpha)\lambda_1(x)}{\mu_1(x)}\int_0^x p_2(t,\tau(t))H_3(x,t)dt,\\
F_2(x)=&\frac{f(x)g_1-\mu_2''(x)+2\Gamma(\alpha)\lambda_1(x)\delta_2'(x)}{\mu_1(x)}-\\
&-\frac{\Gamma(\alpha)\lambda_1(x)}{\mu_1(x)}\int_0^x\psi(t)H_3(x,t)dt+\frac{\sqrt{2}\Gamma(\alpha)\lambda_1(x)}{\mu_1(x)}\int_0^{\frac{x}{2}}\frac{\psi_3'(t)}{g_2(t-l)}H_4(x,t)dt.
\end{aligned} \tag{4.19}$$

Lemma 4.1 *If all conditions of the Lemma 3.1 and*

$$|p_2(x,z_1)-p_2(x,z_2)|\le L_2\,|z_1-z_2|,\ \forall z_1,z_2\in\mathbb{R} \tag{4.20}$$

are fulfilled, then there for $\forall z_1,z_2\in\mathbb{R}$ *takes place the inequality*

$$|B(x,z_1(x))-B(x,z_2(x))|\le L\ |z_1-z_2|+L_{21}\int_0^x|z_1(t)-z_2(t)|\,dt, \tag{4.21}$$

where $L_2,L_{21}=\text{const}>0$, *besides* $L_{21}=\frac{\Gamma(\alpha)}{2}L_2\mu_0\lambda_{01}h_{03}$, $h_{03}=\max\limits_{0\le t\le x\le l}\frac{1}{|H_3(x,t)|}$.

Lemma 4.1 proves similarly, as the case of Lemma 3.1. Therefore, we do not present its proof here.

Using initial conditions $\tau(0) = \tau_{01}$ and $\tau'(0) = \tau_{02}$, we integrate the equation (4.18) two times with respected to x and we obtain a nonlinear integral equation

$$\tau(x) = \int_0^x K(x,t)\tau(t)dt - \int_0^x (x-t)B(t,\tau(t))dt + \int_0^x (x-t)dt \int_0^{\frac{t}{2}} p_2(z,\tau(z))\bar{H}_4(t,z)dz$$

$$+\frac{\Gamma(\alpha)}{2}\int_0^x \lambda_1(t)(x-t)dt \int_0^t p_2\left(\frac{z+t}{2}, \frac{z-t}{2}, \tau\left(\frac{z+t}{2}\right)\right)dz + F_3(x), \qquad (4.22)$$

where $K(x,t)$ is defined from (3.20),

$$F_3(x) = (a(0)x+1)\tau_{01} + x\tau_{02} + \int_0^x (x-t)F_2(t)dt.$$

Hence, by virtue of (3.13), we have

$$|K(x,t)| \leq k_0, \quad |F_3(x)| \leq F_{03}, \quad k_0,\, F_{03} = \text{const} > 0. \qquad (4.23)$$

Theorem 4.1 *If all conditions of the Lemma 4.1, Theorem 3.1 and* (2.2)*,* (2.7)*,* (2.8)*,* (2.10) *are fulfilled, then solution of the inverse problem exist and unique.*

The Theorem 4.1 proves similarly as theorem 3.1.

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