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Limit theorems for the Markov random walks describes by the generalization of autoregressive process of order one $(AR(1))$

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Abstract. In this paper are proved the strong law of large numbers and the central limit theorem for the Markov random walks describes by the generalization autoregressive process of order one.

Keywords. Markov random walks, autoregressive process of order one, strong law of large numbers, central limit theorem

Mathematics Subject Classification (2010): 62M10, 60F15

1 Introduction and problem statement

Let (Ω, F, P) be a probability space, and $\{\xi_n,\ n \geq 1\}$ be a sequence of independent random variables with $E\xi_n = 0$ and $E\xi_n = \sigma_n^2$. Define the sequence random variables $\{X_n\}$ by

$$X_n = \theta_0 X_{n-1} + \xi_n \tag{1.1}$$

for some fixed number $\theta_0 \in (-\infty, \infty)$, where initial value X_0 is independent of $\{\xi_n\}$.

In the case of independent and identically distributed random variables ξ_n (i.i.d) the process is call a autoregressive process of order one $(AR(1))$ ([1]-[9]).

We know that ([8], [9]) the least-squares estimator θ_0 for θ (1.1) gives

$$\theta_n = \frac{\sum\limits_{i=1}^{n} \frac{X_i X_{i-1}}{\sigma_i^2}}{\sum\limits_{i=1}^{n} \left(\frac{X_{i-1}}{\sigma_i}\right)^2}. \tag{1.2}$$

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In [9] was shown that under conditions

$$\sup_i \frac{\sigma_{i+1}^2}{\sigma_i^2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} E\left(\frac{\xi_n^2}{\sigma_n^2} \wedge 1\right) = \infty, \tag{1.3}$$

where $(a \wedge 1) = min(a, 1)$ convergence almost surely $\theta_n \overset{a.s.}{\to} \theta_0$ as $n \to \infty$ is true.

Set

$$A_n = \sum_{i=1}^{n} \frac{X_i X_{i-1}}{\sigma_i^2}, \;\; M_n = \sum_{i=1}^{n} \frac{\xi_i X_{i-1}}{\sigma_i^2}, \;\; B_n = \sum_{i=1}^{n} \left(\frac{X_{i-1}}{\sigma_i}\right)^2.$$

Then we have from (1.2)

$$\theta_n = \frac{A_n}{B_n} = \theta_0 + \frac{M_n}{B_n}.$$

It follows that convergence almost surely

$$\frac{M_n}{B_n} \overset{a.s.}{\to} 0 \;\text{ of }\; n \to \infty \tag{1.4}$$

is the necessary and sufficient condition for $\theta_n \overset{a.s.}{\to} \theta_0$ of $n \to \infty$.

In was shown that conditions (2.3) are sufficient for (1.4).

Note that in work [1] for the case of (i.i.d) random variables ξ_n with $E\xi_1 = 0$ and $E\xi_1^2 = 1$ was shown that if $|\theta_0| < 1$ and $EX_0^2 < \infty$ as $n \to \infty$

$$\frac{A_n}{n} \overset{a.s.}{\to} \frac{\theta_0}{1-\theta_0^2}, \;\; \frac{B_n}{n} \overset{a.s.}{\to} \frac{1}{1-\theta_0^2}$$

$$D_n = \frac{A_n^2}{B_n} \overset{a.s.}{\to} \frac{\theta_0^2}{1-\theta_0^2}. \tag{1.5}$$

Furthermore, in [8] for the case of i.i.d. random variables ξ_n with $E\xi_1 = 0$ and $E\xi_1^2 < \infty$ is obtained the following result: if $|\theta_0| < 1$ and $EX_0^2 < \infty$, then

$$\lim_{n\to\infty} P\left(\sqrt{n}\,(\theta_n - \theta_i) \le x\right) = \Phi\left(\frac{x}{\sqrt{1-\theta_0^2}}\right), \;\; x \in R = (-\infty, \infty), \tag{1.6}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-y^2/2} dy.$

Next we prove the strong law a of large numbers of the form (1.5) and with some additional assumptions the central limit theorem for A_n , B_n and D_n for the independent nonidentically random variables.

Note that the statistics A_n , B_n and D_n playes important roule in nonlinear Markov renewal theory and in sequential analysis ([1]-[7]).

Theorem 1.1 *Let $\{\xi_n,\; n \ge 1\}$ be a sequence of independent random variables with $E\xi_n = 0$ and $E\xi_n^2 = 1$. Suppose that $\sum\limits_{n=1}^{\infty} E\left(\xi_n^2 \wedge 1\right) = \infty$ and $|\theta_0| < 1,\;\; EX_0^2 < \infty.$*

Then as $n \to \infty$

$$\frac{B_n}{n} \overset{a.s.}{\to} \frac{1}{1-\theta_0^2}, \tag{1.7}$$

$$\frac{A_n}{n} \overset{a.s.}{\to} \frac{\theta_0}{1-\theta_0^2}, \tag{1.8}$$

and

$$\frac{D_n}{n} \overset{a.s.}{\to} \frac{\theta_0^2}{1-\theta_0^2}. \tag{1.9}$$

Proof. By equality

$$X_i^2 = \theta_0^2 X_{i-1} + 2\theta_0 X_{i-1}\xi_i + \xi_i^2$$

and

$$\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_{i-1}^2 + X_n^2 - X_0^2.$$

We have

$$\sum_{i=1}^{n} X_{i-1}^2 + X_n^2 - X_0^2 = \theta_0^2 \sum_{i=1}^{n} X_{i-1} + 2\theta_0 \sum_{i=1}^{n} X_{i-1}\xi_i + \sum_{i=1}^{n} \xi_1^2$$

or

$$\left(1-\theta_0^2\right) \sum_{i=1}^{n} X_{i-1}^2 = \sum_{i=1}^{n} \xi_1^2 + 2\theta_0 \sum_{i=1}^{n} \xi_1 X_{i-2} + \left(X_0^2 - X_n^2\right). \tag{1.10}$$

In view of $X_n^2 = \theta_0^2 X_n^2 + 2\theta_0 \xi_n X_{n-1} + \xi_n^2$ by (1.1). We have

$$EX_n^2 = 1 + \theta_0^2 EX_{n-1}^2, \tag{1.11}$$

since random variables ξ_n and X_{n-1} are independent and $E\xi_n = 0,\ E\xi_n^2 = 1$.

From (1.11) we obtain

$$EX_n^2 = 1 + \theta_0^2 + \theta_0^4 + ... + \theta_0^{2n-2} + \theta_0^{2n} EX_0^2,$$

wich by asumptions $|\theta_0| < 1$ and $EX_0^2 < \infty$ yield

$$EX_n^2 \to \frac{1}{1-\theta_0} \text{ as } n \to \infty. \tag{1.12}$$

By (1.12) and Chebyshev's inequality we have

$$\frac{X_0^2 - X_n^2}{n} \overset{P}{\to} 0 \text{ as } n \to \infty. \tag{1.13}$$

We have by (1.12)

$$E\left(\frac{1}{n}\sum_{i=1}^{n} \xi_1 X_{i-1}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} E\xi_1^2 \cdot EX_{i-2}^2$$
$$= \frac{1}{n^2}\sum_{i=1}^{n} EX_{i-1}^2 \to 0 \text{ as } n \to \infty.$$

Hence

$$\frac{1}{n}\sum_{i=1}^{n} \xi_1 X_{i-1} \overset{P}{\to} 0 \text{ as } n \to \infty. \tag{1.14}$$

Applying Kolmoqorov's theorem on the strong law of large numbers for random variable ξ_n^2.

We have

$$\frac{1}{n}\sum_{i=1}^{n}\xi_1^2 \overset{a.s.}{\to} 1 \text{ as } n\to\infty. \tag{1.15}$$

By (1.13), (1.14) and (1.15) from (10) it follows that

$$\frac{B_n}{n} = \frac{1}{n}\sum_{i=1}^{n}X_{i-1}^2 \overset{P}{\to} \frac{1}{1-\theta_0^2} \text{ as } n\to\infty. \tag{1.16}$$

It is easy to see (1.16) is true for almost sure convergence, sense the sequence $B_n,\ n\geq 1$ increating to infinity $B_n \overset{a.s.}{\to} \infty$ as $n\to\infty$. Consequently,

$$\frac{B_n}{n} \overset{a.s.}{\to} \frac{1}{1-\theta_0^2} \quad as\ n\to\infty. \tag{1.17}$$

Now we prove (1.8).

Note that

$$A_n = \sum_{i=1}^{n}X_iX_{i-1} = \theta_0\sum_{i=1}^{n}X_{i-1}^2 + \sum_{i=1}^{n}\xi_iX_{i-1}$$

or

$$\frac{A_n}{n} = \frac{\theta_0}{n}\sum_{i=1}^{n}X_{i-1}^2 + \frac{1}{n}\sum_{i=1}^{n}\xi_iX_{i-1}. \tag{1.18}$$

Under conditions of theorem 1.1 (see [9])

$$\frac{M_n}{B_n} = \frac{1}{B_n}\sum_{i=1}^{n}\xi_iX_{i-1} \overset{a.s.}{\to} 0 \text{ as } n\to\infty.$$

By (1.17) we have

$$\frac{1}{n}\sum_{i=1}^{n}\xi_iX_{i-1} = \frac{M_m}{n} = \frac{B_n}{n}\frac{M_n}{B_n} \overset{a.s.}{\to} 0 \text{ as } n\to\infty.$$

Then by (1.17) from (1.18) we obtain

$$\frac{A_n}{n} \overset{a.s.}{\to} \frac{\theta_0}{1-\theta_0^2} \text{ as } n\to\infty.$$

Convergence (1.9) by (1.7) and (1.8) follows from equality

$$\frac{D_n}{n} = \frac{n}{B_n}\left(\frac{A_n}{n}\right)^2.$$

Next, we shall prove the central limit theorem for A_n, B_n and D_n.

Let $N(\mu,\sigma^2)$ be a random variable with normal distribution with mean μ and variance σ^2.

Theorem 1.2 *Suppose that the sequence* $\{\xi_n, n \geq 1\}$ *of independent random variables with* $E\xi_n = 0,\ \ E\xi_n^2 = 1$ *is uniformly inteqrable and* $\sum\limits_{n=1}^{\infty} E\left(\xi_n^2 \wedge 1\right) = \infty.$

If $0 < |\theta_0| < 1$ *and* $EX_0^2 < \infty$*, then as* $n \to \infty$.

1)

$$\sqrt{n}\left(\frac{B_n}{n} - \frac{1}{1-\theta_0^2}\right) \xrightarrow{d} N(0,\alpha_1),\ \ where\ \ \alpha_1 = \frac{1}{|\theta_0|^2\left(1-\theta_0^2\right)}. \tag{1.19}$$

2)

$$\sqrt{n}\left(\frac{A_n}{n} - \frac{\theta_0}{1-\theta_0^2}\right) \xrightarrow{d} N(0,\alpha_2),\ \ where\ \ \alpha_2 = \frac{1}{1-\theta_0^2}. \tag{1.20}$$

3)

$$\sqrt{n}\left(\frac{D_n}{n} - \frac{\theta_0^2}{1-\theta_0^2}\right) \xrightarrow{d} N(0,\alpha_3),\ \ where\ \ \alpha_3 = \frac{\theta_0^2}{1-\theta_0^2}. \tag{1.21}$$

Proof. Under conditions of theorem 1.2 in paper [6] is obtained the following result (see (1.6))

$$\sqrt{n}\left(\theta_n - \theta_0\right) \xrightarrow{d} N\left(0, 1-\theta_0^2\right) \ \text{ as } \ n \to \infty.$$

or

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\theta_n - \theta\right) \leq x\right) = \Phi\left(\frac{x}{\sqrt{1-\theta_0^2}}\right), \tag{1.22}$$

$$x \in R,\ \ \Phi(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-y^2/2} by.$$

Without of logs set $\sigma^2 = 1$.

Applying (1.22) we have in view of $\dfrac{B_n}{n\theta_n} \to \dfrac{1}{\theta_0(1-\theta_0^2)}$ for the case $0 < \theta_0 < 1$

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{B_n}{n} - \frac{1}{1-\theta_0^2}\right) \leq x\right) = \lim_{n\to\infty} P\left(\frac{B_n}{n} \leq \frac{x}{\sqrt{n}} + \frac{1}{1-\theta_0^2}\right)$$

$$= \lim_{n\to\infty} P\left(\theta_n \cdot \frac{B_n}{n\theta_n} \leq \frac{x}{\sqrt{n}} + \frac{1}{1-\theta_0^2}\right) = \lim_{n\to\infty} P\left(\theta_n \leq \frac{x\theta_0(1-\theta_0^2)}{\sqrt{n}} + \theta_0\right)$$

$$= \lim_{n\to\infty} P\left(\sqrt{n}\left(\theta_n - \theta_0\right) \leq x\theta_0\left(1-\theta_0^2\right)\right)$$

$$= \Phi\left(\frac{x\theta_0\left(1-\theta_0^2\right)}{\sqrt{1-\theta_0^2}}\right) = \Phi\left(x\theta_0\sqrt{1-\theta_0^2}\right). \tag{1.23}$$

For the case $-1 < \theta_0 < 0$ by the equality $\Phi(x) = 1 - \Phi(-x)$ we obtain

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{B_n}{n} - \frac{1}{1-\theta_0^2}\right) \leq x\right) = \Phi\left(-x\theta_0\sqrt{1-\theta_0^2}\right)$$

which together with (1.23) yields

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{B_n}{n} - \frac{1}{1-\theta_0^2}\right) \leq x\right) = \Phi\left(x\left|\theta_0\right|\sqrt{1-\theta_0^2}\right),$$

from which (1.19) follows.

For the proof of (1.20) by (1.22). We have in view of $\frac{A_n}{n\theta_n} \overset{a.s.}{\to} \frac{1}{1-\theta_0^2}$ as $n \to \infty$

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{A_n}{n} - \frac{\theta_0}{1-\theta_0^2}\right) \le x\right)$$
$$= \lim_{n\to\infty} P\left(\frac{A_n}{n} \le \frac{x}{\sqrt{n}} + \frac{\theta_0}{1-\theta_0^2}\right)$$
$$= \lim_{n\to\infty} P\left(\theta_n \frac{A_n}{n\theta_n} \le \frac{x}{\sqrt{n}} + \frac{\theta_0}{1-\theta_0^2}\right)$$
$$= \lim_{n\to\infty} P\left(\theta_n \le \frac{x\left(1-\theta_0^2\right)}{\sqrt{n}} + \theta_0\right)$$
$$= \lim_{n\to\infty} P\left(\sqrt{n}\left(\theta_n - \theta_0\right) \le x\left(1-\theta_0^2\right)\right)$$
$$= \Phi\left(x\sqrt{1-\theta_0^2}\right),$$

which proves (1.20).

Finally by (1.22), in view of $\frac{D_n}{n\theta_n} \overset{a.s.}{\to} \frac{\theta_0}{1-\theta_0^2}$ as $n \to \infty$ for the case $0 < \theta_0 < 1$, we have

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{D_n}{n} - \frac{\theta_0^2}{1-\theta_0^2}\right) \le x\right)$$
$$= \lim_{n\to\infty} P\left(\frac{D_n}{n} \le \frac{x}{\sqrt{n}} + \frac{\theta_0^2}{1-\theta_0}\right)$$
$$= \lim_{n\to\infty} P\left(\theta_n \frac{D_n}{n\theta_n} \le \frac{x}{\sqrt{n}} + \frac{\theta_0^2}{1-\theta_0}\right)$$
$$= \lim_{n\to\infty} P\left(\theta_n \le \frac{x(1-\theta_0^2)}{\theta_0\sqrt{n}} + \theta_0\right)$$
$$= \lim_{n\to\infty} P\left(\sqrt{n}\left(\theta_n - \theta_0\right) \le \frac{x\left(1-\theta_0^2\right)}{\theta_0}\right)$$
$$= \Phi\left(x\frac{\sqrt{1-\theta_0^2}}{\theta_0}\right). \tag{1.24}$$

For $-1 < \theta_0 < 0$, by equality $\Phi(x) = 1 - \Phi(x)$ we have

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{D_n}{n} - \frac{\theta_0^2}{1-\theta_0^2}\right) \le x\right) = \Phi\left(-x\frac{\sqrt{1-\theta_0^2}}{\theta_0}\right),$$

which together (1.24) yields

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{D_n}{n} - \frac{\theta_0^2}{1-\theta_0^2}\right) \le x\right) = \Phi\left(x\frac{\sqrt{1-\theta_0^2}}{|\theta_0|}\right),$$

from which (1.21) follows.

Note that the above results are of importance in nonlinear Markov renewal and in certain statistical applications ([1], [2], [9]).

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