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Polynomial stability of a transmission problem with second sound and distributed delay term

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Abstract. *In this work, we consider a transmission problem for an elastic–thermoelastic bar with the elastic part being surrounded by two thermoelastic parts in the presence of an infinite distributed delay term. The heat flux of the system is governed by Cattaneo's law. Under suitable assumption on the weight of the delay, we establish the polynomial stability of the solution by introducing a suitable Lyapunov functional.*

Keywords. Transmission problem · second sound · distributed delay · polynomial stability.

Mathematics Subject Classification (2010): 35B35, 74F05, 74H55, 93D15, 93D20

1 Introduction

In this article, we study the transmission problem with second sound and a distributed delay term,

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1 u_t(x,t) & \\ + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s)ds + \gamma \theta_x(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ \theta_t(x,t) + k q_x(x,t) + \gamma u_{xt}(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ \tau q_t(x,t) + q(x,t) + k\theta_x(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2) \times (0,+\infty), \end{cases} \tag{1.1}$$

under the boundary and transmission conditions

$$\begin{cases} u(0,t) = u(L_3,t) = q(0,t) = \theta(0,t) = \theta(L_3,t) = 0, & t>0, \\ u(L_i,t) = v(L_i,t), & i=1,2, \ t>0, \\ q(L_i,t) = \theta(L_i,t) = 0, & i=1,2, \ t>0, \\ a u_x(L_i,t) = b v_x(L_i,t), & i=1,2, \ t>0, \end{cases} \tag{1.2}$$

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and the initial conditions

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ \theta(x,0) = \theta_0(x), \ q(x,0) = q_0(x), & x \in \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in (L_1, L_2), \\ u_t(x,-t) = f_0(x,-t), & x \in \Omega, t \in (0,\tau_2). \end{cases} \tag{1.3}$$

Where $0 < L_1 < L_2 < L_3$, $\Omega = (0, L_1) \cup (L_2, L_3)$, $a, \mu_1, \gamma, k, \tau, b$ are positive constants, and the initial data $(u_0, u_1, v_0, v_1, \theta_0, q_0)$ belongs to suitable space. Moreover, $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real number satisfying $0 \leq \tau_1 < \tau_2$.

Here, we will prove the stability results for system (1.1)-(1.3), under the assumption

$$\mu_1 > \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds. \tag{1.4}$$

In the classical thermoelasticity, the heat flux is given by Fourier's law. As a result, this theory predicts an infinite speed of heat propagation (predicts the physical paradox of in nite speed of heat propagation). In other words, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. To overcome this physical paradox but still keeping the essentials of heat conduction process, many theories have merged. One of which is the advent of the second sound effects observed experimentally in materials at low temperature. This theory suggests replacing the classic Fourier's law by a modied law of heat conduction called Cattaneo's law. For results concerning existence, blow up, and asymptotic behavior of smooth, as well as weak solutions in heat conduction with second sound, we refer the reader to ([1,3,4,9,11,12]). Liu et al. [9] considered the following transmission problem in infinite memory-type thermoelasticity with second sound

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \int_0^\infty g(s) \, u_{xx}(x,t-s) \, ds & \\ +\gamma \theta_x(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ \theta_t(x,t) + k q_x(x,t) + m u_{xt}(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ \tau q_t(x,t) + q(x,t) + k\theta_x(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2) \times (0,+\infty), \end{cases}$$

the main diffculties in handing this problem are that the system does not have any frictional damping term, and that the dissipative effects of heat conduction induced by Cattaneo's law are usually weaker than those induced by Fourier's law. Under appropriate hypothesis on the relaxation function, the authors established a general decay result.

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the stability of evolution systems with time delay effects has become an active area of research (e.g. [1,2, 5,6,8,13–15]). Kafini et al. [6] considered Timoshenko-type system of thermoelasticity of type III with distributive delay, they proved an exponential decay in the case of equal wave speeds and a polynomial decay result in the case of nonequal wave speeds with smooth initial data. Liu [8] considered the following transmission problem in a bounded domain with a distributed delay in the first equation

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1 u_t(x,t) & \\ + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s) ds = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2) \times (0,+\infty), \end{cases}$$

under suitable assumptions on the weight of the delay, the author established the well-posedness result and proved that the system is exponentially stable by introducing a suitable Lyapunov functional. In [2], the effect of the delay term $u_t(x, t-\tau)$ in the transmission system has been investigated by Benseghir. Recently, the well posedness and the decay of solution for a transmission problem in a bounded domain with a viscoelastic term and a delay term $u_t(x, t-\tau)$ have been studied in [7,17].

The aim of this article is to study the asymptotic stability of system (1.1)–(1.3) provided that (1.4) is satis ed. The paper is organized as follows. In Section 2, we present some assumptions and preliminary works. In Section 3, by introducing the extra second-order energy, we prove the polynomial decay result.

2 Preliminaries

As in [15], we introduce the new variable

$$z(x,\rho,t,s) = u_t(x, t-\rho s),\ x \in \Omega,\ \rho \in (0,1),\ t>0,\ s \in (\tau_1,\tau_2). \tag{2.1}$$

Then, we obtain

$$s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0,\ x \in \Omega,\ \rho \in (0,1),\ t>0,\ s \in (\tau_1,\tau_2).$$

Hence, system (1.1)–(1.3) is equivalent to

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1 u_t(x,t) & \\ + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)ds + \gamma\theta_x(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ \theta_t(x,t) + k q_x(x,t) + \gamma u_{xt}(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ \tau q_t(x,t) + q(x,t) + k\theta_x(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2) \times (0,+\infty), \\ s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, & (x,\rho,t,s) \in \Omega \times (0,1) \times (0,+\infty) \times (\tau_1,\tau_2), \\ z(x,0,t,s) = u_t(x,t), & (x,t,s) \in \Omega \times (0,+\infty) \times (\tau_1,\tau_2), \\ u(0,t) = u(L_3,t) = q(0,t) = \theta(L_3,t) = 0, & t \in (0,+\infty), \\ u(L_i,t) = v(L_i,t), & i=1,2,\ \ t>0, \\ q(L_i,t) = \theta(L_i,t) = 0, & i=1,2,\ \ t>0, \\ a u_x(L_i,t) = b v_x(L_i,t), & i=1,2,\ \ t>0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x), & x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in (L_1,L_2), \\ z(x,\rho,0,s) = f_0(x,\rho s), & (x,\rho,s) \in \Omega \times (0,1) \times (0,\tau_2). \end{cases} \tag{2.2}$$

Defining $U = (u, v, \varphi, \theta, q, \psi, z)^T$, where $\varphi = u_t$ and $\psi = v_t$, we formally get that U satisfies

$$\begin{cases} U'(t) = AU(t), \quad t>0, \\ U(0) = U_0 = (u_0, v_0, u_1, \theta_0, q_0, v_1, f_0)^T, \end{cases} \tag{2.3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} \varphi \\ \psi \\ a u_{xx} - \mu_1 u_t - \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)ds - \gamma\theta_x \\ -k q_x - \gamma u_{xt} \\ -\dfrac{1}{\tau} q - \dfrac{k}{\tau}\theta_x \\ b v_{xx} \\ -\dfrac{1}{s} z_\rho(x,\rho,t,s) \end{pmatrix}.$$

We consider the following spaces

$$X_* = \{(u,v) \in H^1(\Omega) \cap H^1(L_1,L_2);\ u(0,t) = u(L_3,t) = 0,\\ u(L_i,t) = v(L_i,t),\ au_x(L_i,t) = bv_x(L_i,t), i=1,2\}.$$

Let

$$\mathcal{H} = X_* \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(L_1,L_2) \times L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)),$$

be the Hilbert space equipped with the inner product

$$\left(U,\tilde{U}\right)_{\mathcal{H}} = \int_\Omega \left(\varphi\tilde{\varphi} + au_x\tilde{u}_x + \theta\tilde{\theta} + \tau q\tilde{q}\right)dx + \int_{L_1}^{L_2}\left(\psi\tilde{\psi} + bv_x\tilde{v}_x\right)dx \\ + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\ z(x,\rho,t,s)\tilde{z}(x,\rho,t,s)dsd\rho dx.$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{H}) = \left\{\begin{array}{c} (u,v,\varphi,\theta,q,\psi,z)^T \in \mathcal{H},\ u \in H^2(\Omega) \cap H^1(\Omega), \\ v \in H^2(L_1,L_2) \cap H^1(L_1,L_2), \varphi \in H^1(\Omega), \theta \in \tilde{H}^1_*(\Omega), \\ q \in \bar{H}^1_*(\Omega), \psi \in H^1(L_1,L_2),\ z, z_\rho \in L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)) \end{array}\right\},$$

where

$$\tilde{H}^1_*(\Omega) = \{\theta \in H^1(\Omega);\ \theta(0,t) = 0,\ \theta(L_i,t) = 0, i=1,2,3\},\\ \bar{H}^1_*(\Omega) = \{q \in H^1(\Omega);\ q(0,t) = 0,\ q(L_i,t) = 0, i=1,2,\}.$$

Using the semigroup approach, the Hille-Yosida theorem and the procedure similar to that of ([8,16,17]), we can obtain the following well-posedness of system (2.2):

Theorem 2.1 *Under the assumption (1.4), for any $U_0 \in \mathcal{H}$, there exists a unique weak solution $U \in C(\mathbb{R}^+,\mathcal{H})$ of problem (2.3). Moreover, if $U_0 \in D(\mathcal{H})$, then*

$$U \in C\left(\mathbb{R}^+, D(\mathcal{H})\right) \cap C\left(\mathbb{R}^+,\mathcal{H}\right).$$

3 Polynomial stability

In this section, we prove the polynomial decay for system (2.2). It will be achieved by using the perturbed energy method. we define the following first-order and the second-order energy:

$$E_1(t) = \frac{1}{2}\int_\Omega \left[u_t^2(x,t) + au_x^2(x,t) + \theta^2(x,t) + \tau q^2(x,t)\right]dx \\ + \frac{1}{2}\int_{L_1}^{L_2}\left[v_t^2(x,t) + bv_x^2(x,t)\right]dx + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\ z^2(x,\rho,t,s)dsd\rho dx, \tag{3.1}$$

$$E_2(t) = \frac{1}{2}\int_\Omega \left[u_{tt}^2(x,t) + au_{xt}^2(x,t) + \theta_t^2(x,t) + \tau q_t^2(x,t)\right]dx \\ + \frac{1}{2}\int_{L_1}^{L_2}\left[v_{tt}^2(x,t) + bv_{xt}^2(x,t)\right]dx + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\ z_t^2(x,\rho,t,s)dsd\rho dx, \tag{3.2}$$

and

$$E(t) = E_1(t) + E_2(t).$$

The stability result reads as follows.

Theorem 3.1 *Let (u, v, θ, q, z) be the solution of (2.2). Assume that (1.4) and*

$$\frac{a}{b} < \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)}. \tag{3.3}$$

There exist positive constant k_0, such that

$$E_1(t) \leq \frac{k_0}{t},\ \forall t \geq 0. \tag{3.4}$$

The proof will be established through the following Lemmas.

Lemma 3.1 *Let (u, v, θ, q, z) be the solution of (2.2) and assume (1.4) holds. Then we have the inequality*

$$E_1'(t) \leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_\Omega u_t^2\,(x,t)\,dx - \int_\Omega q^2\,(x,t)\,dx. \tag{3.5}$$

Proof. Multiplying $(2.2)_1$, $(2.2)_2$ and $(2.2)_3$ by u_t, θ and q, respectively, integrating over Ω and using integration by parts and the boundary conditions, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2\,(x,t)\,dx + \frac{a}{2}\frac{d}{dt}\int_\Omega u_x^2\,(x,t)\,dx - [a u_t u_x]_{\partial\Omega} \\ &= -\mu_1 \int_\Omega u_t^2\,(x,t)\,dx - \int_\Omega u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\,ds\,dx + \gamma \int_\Omega u_{xt}\theta\, dx,\end{aligned} \tag{3.6}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \theta^2\,(x,t)\,dx - k\int_\Omega \theta_x q\,dx + \gamma\int_\Omega \theta u_{xt}\,dx = 0, \tag{3.7}$$

and

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega q^2\,(x,t)\,dx + \int_\Omega q^2\,(x,t)\,dx + k\int_\Omega \theta_x q\,(x,t)\,dx = 0. \tag{3.8}$$

Multiplying $(2.2)_4$ by v_t and integrating by parts over (L_1, L_2), we get

$$\frac{1}{2}\frac{d}{dt}\int_{L_1}^{L_2} v_t^2\,(x,t)\,dx + \frac{b}{2}\frac{d}{dt}\int_{L_1}^{L_2} v_x^2\,(x,t)\,dx - [b v_t v_x]_{L_1}^{L_2} = 0. \tag{3.9}$$

Multiplying $(2.2)_5$ by $|\mu_2(s)|\, z$, integrating the product over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, and recall that $z(x,0,t,s) = u_t(x,t)$, yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, z^2(x,\rho,t,s)\,ds\,d\rho\,dx \\ &= -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\,ds\,dx + \frac{1}{2}\int_\Omega u_t^2 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds\,dx.\end{aligned} \tag{3.10}$$

Adding Eqs. (3.6)–(3.10) up and using (1.2) gives

$$\begin{aligned}E_1'(t) = &-\left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds\right)\int_\Omega u_t^2\,(x,t)\,dx \\ &- \int_\Omega q^2\,(x,t)\,dx - \int_\Omega u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\,ds\,dx \\ &- \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\,ds\,dx\end{aligned} \tag{3.11}$$

Meanwhile, using Young's inequality, we have

$$
\begin{aligned}
&-\int_{\Omega} u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) ds dx \\
&\leq \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_{\Omega} u_t^2 dx + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s) ds dx. \qquad (3.12)
\end{aligned}
$$

Simple substitution of (3.12) into (3.11) and using (1.4) give (3.5). The proof is complete.

Lemma 3.2 *Let (u, v, θ, q, z) be the solution of (2.2) and assume (1.4) holds. Then we have the inequality*

$$
\begin{aligned}
E_2'(t) &\leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_{\Omega} u_{tt}^2 (x,t)\, dx - \int_{\Omega} q_t^2 (x,t)\, dx \\
&\leq -\int_{\Omega} q_t^2 (x,t)\, dx \qquad (3.13)
\end{aligned}
$$

Proof. *Differentiating Eqs. $(2.2)_1$–$(2.2)_5$ with respect to t and using the same procedure as in the proof of Lemma 3.1, we arrive at our conclusion.*

Lemma 3.3 *Let (u, v, θ, q, z) be the solution of (2.2). Then the functional*

$$
I_1(t) = \int_{\Omega} u u_t dx + \frac{\mu_1}{2} \int_{\Omega} u^2 dx + \int_{L_1}^{L_2} v v_t dx,
$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$
\begin{aligned}
I_1'(t) \leq &-\left(a - \varepsilon_1 c_0^2\right) \int_{\Omega} u_x^2 dx - b \int_{L_1}^{L_2} v_x^2 dx + \int_{\Omega} u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx \\
&+ \frac{1}{2\varepsilon_1} \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\, ds dx + \frac{\gamma^2}{2\varepsilon_1} \int_{\Omega} \theta_x^2 dx, \qquad (3.14)
\end{aligned}
$$

where $c_0 = \max\{L_1, L_3 - L_2\}$.

Proof. Taking the derivative of $I_1(t)$ with respect to t, using $(2.2)_1$ and (1.2), we obtain

$$
\begin{aligned}
I_1'(t) = &-a \int_{\Omega} u_x^2 dx - b \int_{L_1}^{L_2} v_x^2 dx + \int_{\Omega} u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx \\
&- \int_{\Omega} u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) ds dx - \gamma \int_{\Omega} u \theta_x dx. \qquad (3.15)
\end{aligned}
$$

Using the boundary condition (1.2), we obtain

$$
u^2(x,t) = \left(\int_0^x u_x(x,t)\, dx\right)^2 \leq L_1 \int_0^{L_1} u_x^2(x,t)\, dx,\ x \in [0, L_1],
$$

$$
u^2(x,t) \leq (L_3 - L_2) \int_{L_2}^{L_3} u_x^2(x,t)\, dx,\ x \in [L_2, L_3],
$$

which imply the following Poincaré's inequality

$$
\int_{\Omega} u^2(x,t)\, dx \leq c_0^2 \int_{\Omega} u_x^2(x,t)\, dx,\ x \in \Omega, \qquad (3.16)
$$

where $c_0 = \max\{L_1, L_3 - L_2\}$ is the Poincaré's constant.

Using Young's inequality and (3.16), we obtain that for any $\varepsilon_1 > 0$,

$$\gamma \int_\Omega u\theta_x dx \leq \frac{\varepsilon_1}{2} c_0^2 \int_\Omega u_x^2 dx + \frac{\gamma^2}{2\varepsilon_1} \int_\Omega \theta_x^2 dx, \tag{3.17}$$

$$\begin{aligned} &- \int_\Omega u \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,t,s)\, ds dx \\ &\leq \frac{\varepsilon_1}{2} c_0^2 \int_\Omega u_x^2 dx + \frac{1}{2\varepsilon_1} \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\, ds dx, \end{aligned} \tag{3.18}$$

Inserting the estimates (3.17) and (3.18) into (3.15), then (3.14) is fulfilled. The proof is complete.

Now, as in [10], we introduce the function

$$p(x) = \begin{cases} x - \dfrac{L_1}{2}, & x \in [0, L_1], \\ \dfrac{L_1}{2} - \dfrac{L_1 + L_3 - L_2}{2(L_2 - L_1)}(x - L_1), & x \in (L_1, L_2), \\ x - \dfrac{L_2 + L_3}{2} & x \in [L_2, L_3]. \end{cases} \tag{3.19}$$

It is clear to see that $p(x)$ is bounded, that is $|p(x)| \leq M$,where $M = \max\left\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\right\}$ is a positive constant.

We define the two functionals

$$I_2(t) = -\int_\Omega p(x) u_x u_t dx, \quad I_3(t) = -\int_{L_1}^{L_2} p(x) v_x v_t\, dx.$$

Then, we have the following estimates.

Lemma 3.4 *For any $\varepsilon_2 > 0$, the functionals $I_2(t)$ and $I_3(t)$ satisfy*

$$\begin{aligned} I_2'(t) \leq{}& \left(\frac{a}{2} + \varepsilon_2\right) \int_\Omega u_x^2 dx + C_1(\varepsilon_2) \int_\Omega u_t^2 dx + C_2(\varepsilon_2) \int_\Omega \theta_x^2 dx \\ &+ C_3(\varepsilon_2) \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s) ds dx \\ &- \frac{a}{4}\left[L_1 u_x^2(L_1,t) + (L_3 - L_2)\, u_x^2(L_2,t)\right] \\ &- \frac{1}{4}\left[L_1 u_t^2(L_1,t) + (L_3 - L_2)\, u_t^2(L_2,t)\right]. \end{aligned} \tag{3.20}$$

where

$$C_1(\varepsilon_2) = \frac{1}{2} + \frac{\mu_1^2 M^2}{4\varepsilon_2},\ C_2(\varepsilon_2) = \frac{\gamma^2 M^2}{4\varepsilon_2},\ C_3(\varepsilon_2) = \frac{M^2}{2\varepsilon_2},$$

and

$$\begin{aligned} I_3'(t) ={}& -\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} \left(b \int_{L_1}^{L_2} v_x^2 dx + \int_{L_1}^{L_2} v_t^2 dx\right) + \frac{L_1}{4} v_t^2(L_1,t) \\ &+ \frac{L_3 - L_2}{4} v_t^2(L_2,t) + \frac{b}{4}\left[(L_3 - L_2)\, v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\right]. \end{aligned} \tag{3.21}$$

Proof. Taking the derivative of $I_2(t)$ with respect to t, using $(2.2)_1$, we get

$$\begin{aligned} I_2'(t) &= -\int_\Omega p(x)u_x u_{tt}dx - \int_\Omega p(x)u_{xt}u_t dx \\ &= -a\int_\Omega p(x)u_x u_{xx}dx + \int_\Omega p(x)u_x \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,t,s)dsdx \\ &\quad +\mu_1\int_\Omega p(x)u_x u_t dx + \gamma\int_\Omega p(x)u_x\theta_x dx - \int_\Omega p(x)u_{xt}u_t dx. \end{aligned} \tag{3.22}$$

Integrating by parts, and noticing (3.19), equation (3.22) becomes

$$\begin{aligned} I_2'(t) &= \frac{a}{2}\int_\Omega u_x^2 dx - \frac{1}{2}\left[ap(x)u_x^2\right]_{\partial\Omega} + \frac{1}{2}\int_\Omega u_t^2 dx - \frac{1}{2}\left[p(x)u_t^2\right]_{\partial\Omega} \\ &\quad +\mu_1\int_\Omega p(x)u_x u_t dx + \int_\Omega p(x)u_x\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)dsdx \\ &\quad +\gamma\int_\Omega p(x)u_x\theta_x dx. \end{aligned} \tag{3.23}$$

By Young's inequality, for any $\varepsilon_2 > 0$, we obtain

$$\begin{aligned} I_2'(t) &\le \left(\frac{a}{2}+\varepsilon_2\right)\int_\Omega u_x^2 dx - \frac{1}{2}\left[ap(x)u_x^2\right]_{\partial\Omega} - \frac{1}{2}\left[p(x)u_t^2\right]_{\partial\Omega} \\ &\quad + \left(\frac{1}{2}+\frac{\mu_1^2 M^2}{4\varepsilon_2}\right)\int_\Omega u_t^2 dx + \frac{\gamma^2 M^2}{4\varepsilon_2}\int_\Omega \theta_x^2 dx \\ &\quad + \frac{M^2}{2\varepsilon_2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,t,s)dsdx. \end{aligned}$$

On the other hand, by using the boundary conditions (1.2), we have

$$\begin{aligned} \frac{1}{2}\left[ap(x)u_x^2\right]_{\partial\Omega} &= \frac{a}{4}\left[L_1 u_x^2(L_1,t) + (L_3-L_2)u_x^2(L_2,t)\right], \\ \frac{1}{2}\left[p(x)u_t^2\right]_{\partial\Omega} &= \frac{1}{4}\left[L_1 u_t^2(L_1,t) + (L_3-L_2)u_t^2(L_2,t)\right] \ge 0. \end{aligned}$$

Estimate (3.20) follows by inserting the above two equalities into (3.23).

By the same method, taking the derivative of $I_3(t)$ with respect to t, and integrating by parts and noticing (3.19), we obtain

$$\begin{aligned} I_3'(t) &= -\int_{L_1}^{L_2} p(x)v_x v_{tt}\,dx - \int_{L_1}^{L_2} p(x)v_{tx}v_t\,dx \\ &= -\frac{L_1+L_3-L_2}{4(L_2-L_1)}\left(b\int_{L_1}^{L_2} v_x^2 dx + \int_{L_1}^{L_2} v_t^2 dx\right) + \frac{L_1}{4}v_t^2(L_1,t) \\ &\quad + \frac{L_3-L_2}{4}v_t^2(L_2,t) + \frac{b}{4}\left[(L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\right], \end{aligned}$$

which is exactly (3.21).

Lemma 3.5 *Let (u,v,θ,q,z) be the solution of (2.2). Then the functional*

$$I_4(t) = -\tau\int_\Omega q\left(\int_0^x \theta(y,t)\,dy\right)dx,$$

satisfies the estimate

$$I_4'(t) \le -\frac{k}{2}\int_\Omega \theta^2 dx + \left(2\tau k + \frac{1}{2k}\right)\int_\Omega q^2 dx + \frac{\tau\gamma^2}{4k}\int_\Omega u_t^2 dx. \tag{3.24}$$

Proof. Differentiating $I_4(t)$ with respect to t, using the second and the third equations in (2.2), and integration by parts, we obtain

$$\begin{aligned} I_4'(t) &= -\int_\Omega (-q - k\theta_x)\left(\int_0^x \theta(y,t)\,dy\right)dx - \tau\int_\Omega q\left(\int_0^x (-kq_y - \gamma u_{yt})\,dy\right)dx \\ &= -k\int_\Omega \theta^2 dx + \tau k\int_\Omega q^2 dx + \tau\gamma\int_\Omega q u_t dx + \int_\Omega q\left(\int_0^x \theta(y,t)\,dy\right)dx. \end{aligned} \tag{3.25}$$

Now, we estimate the terms in the right hand side of (3.25) using Young's Poincaré's inequalities

$$\begin{aligned} \int_\Omega q\left(\int_0^x \theta(y,t)\,dy\right)dx &\le \frac{k}{2}\int_\Omega \left(\int_0^x \theta(y,t)\,dy\right)^2 dx + \frac{1}{2k}\int_\Omega q^2 dx \\ &\le \frac{k}{2}\int_\Omega \theta^2 dx + \frac{1}{2k}\int_\Omega q^2 dx, \end{aligned} \tag{3.26}$$

$$\tau\gamma\int_\Omega q u_t dx \le \tau k\int_\Omega q^2 dx + \frac{\tau\gamma^2}{4k}\int_\Omega u_t^2 dx. \tag{3.27}$$

Estimate (3.24) follows by substituting (3.26) and (3.27) into (3.25).

Lemma 3.6 *Let (u, v, θ, q, z) be the solution of (2.2). Then the functional*

$$I_5(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,s,t)\, ds d\rho dx,$$

satisfies, for some positive constant n_1, the estimate

$$\begin{aligned} I_5'(t) \le &-n_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,t,s)\, ds dx + \mu_1 \int_\Omega u_t^2 dx \\ &-n_1 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|\, z^2(x,\rho,t,s)\, ds d\rho dx. \end{aligned} \tag{3.28}$$

Proof. By differentiating $I_5(t)$ with respect to t, and using the equation $(2.2)_5$, we obtain

$$\begin{aligned} I_5'(t) &= -2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)|\, z(x,\rho,t,s)\, z_\rho(x,\rho,t,s)\, ds d\rho dx \\ &= -\frac{d}{d\rho}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,t,s)\, ds d\rho dx \\ &\quad - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,t,s)\, ds d\rho dx \\ &= -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left[e^{-s} z^2(x,1,t,s) - z^2(x,0,t,s)\right] ds dx \\ &\quad - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)|\, z^2(x,\rho,t,s)\, ds d\rho dx. \end{aligned}$$

Using the fact that $z(x,0,t,s) = u_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$I_5'(t) \leq -\int_\Omega \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x,1,t,s)\, ds dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_\Omega u_t^2 dx$$
$$-\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s} |\mu_2(s)| z^2(x,\rho,t,s)\, ds d\rho dx.$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$.
Finally, setting $n_1 = e^{-\tau_2}$ and recalling (1.4), we obtain (3.28).

Now, we turn to prove our main result in this section.

Proof. **(of Theorem** 3.1)
We define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E.

$$\mathcal{L}(t) = NE(t) + N_1 I_1(t) + I_2(t) + N_3 I_3(t) + I_4(t) + N_5 I_5(t), \tag{3.29}$$

where N, N_1, N_3 and N_5 are positive constants to be selected later.
Taking the derivative of (3.29) with respect to t and making the use of the above lemmas, we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) N - N_1 - C_1(\varepsilon_2) - \frac{\tau\gamma^2}{4k} - \mu_1 N_5\right] \int_\Omega u_t^2(x,t) dx \\
& -\left[\left(a - \varepsilon_1 c_0^2\right) N_1 - \left(\frac{a}{2} + \varepsilon_2\right)\right] \int_\Omega u_x^2(x,t) dx \\
& -\frac{k}{2} \int_\Omega \theta^2(x,t) dx + \left[\frac{\gamma^2}{2\varepsilon_1} N_1 + C_2(\varepsilon_2)\right] \int_\Omega \theta_x^2(x,t) dx \\
& -\left[N - \left(2\tau k + \frac{1}{2k}\right)\right] \int_\Omega q^2(x,t) dx - N \int_\Omega q_t^2(x,t)\, dx \\
& -\left[bN_1 + \frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} bN_3\right] \int_{L_1}^{L_2} v_x^2(x,t) dx \\
& -\left[\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} N_3 - N_1\right] \int_{L_1}^{L_2} v_t^2(x,t) dx \\
& -\left(n_1 N_5 - \frac{1}{2\varepsilon_1} N_1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds - C_3(\varepsilon_2) N_2 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \\
& \times \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds dx \\
& -n_1 N_5 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx \\
& -\frac{1}{4}\left(1 - \frac{a}{b} N_3\right) \left[L_1 a u_x^2(L_1,t) + (L_3 - L_2) a u_x^2(L_2,t)\right] \\
& -\frac{1}{4}(1 - N_3) \left[L_1 u_t^2(L_1,t) + (L_3 - L_2) u_t^2(L_2,t)\right].
\end{aligned} \tag{3.30}$$

Minding that $\int_{\Omega}\theta_x^2(x,t)dx$ term is positive, we need to use $\int_{\Omega}q^2(x,t)dx$ and $\int_{\Omega}q_t^2(x,t)dx$ to control it the third equation of (2.2) implies that

$$\left[\frac{\gamma^2}{2\varepsilon_1}N_1+C_2(\varepsilon_2)\right]\int_{\Omega}\theta_x^2(x,t)\,dx\leq\frac{1}{k^2}\left[\frac{\gamma^2}{2\varepsilon_1}N_1+C_2(\varepsilon_2)\right]\int_{\Omega}q^2(x,t)\,dx$$
$$+\frac{\tau^2}{k^2}\left[\frac{\gamma^2}{2\varepsilon_1}N_1+C_2(\varepsilon_2)\right]\int_{\Omega}q_t^2(x,t)\,dx.$$

So we get

$$\begin{aligned}
\mathcal{L}'(t)\leq&-\left[\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)N-N_1-C_1(\varepsilon_2)-\frac{\tau\gamma^2}{4k}-\mu_1N_5\right]\int_{\Omega}u_t^2(x,t)dx\\
&-\left[\left(a-\varepsilon_1c_0^2\right)N_1-\left(\frac{a}{2}+\varepsilon_2\right)\right]\int_{\Omega}u_x^2(x,t)dx-\frac{k}{2}\int_{\Omega}\theta^2(x,t)dx\\
&-\left[N-\frac{\gamma^2}{2k^2\varepsilon_1}N_1-\frac{C_2(\varepsilon_2)}{k^2}-\left(2\tau k+\frac{1}{2k}\right)\right]\int_{\Omega}q^2(x,t)dx\\
&-\left[N-\left(\frac{\tau^2\gamma^2}{2k^2\varepsilon_1}N_1+\frac{\tau^2C_2(\varepsilon_2)}{k^2}\right)\right]\int_{\Omega}q_t^2(x,t)\,dx\\
&-\left[bN_1+\frac{L_1+L_3-L_2}{4(L_2-L_1)}bN_3\right]\int_{L_1}^{L_2}v_x^2(x,t)dx\\
&-\left[\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_3-N_1\right]\int_{L_1}^{L_2}v_t^2(x,t)dx\\
&-\left(n_1N_5-\frac{1}{2\varepsilon_1}N_1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds-C_3(\varepsilon_2)N_2\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\\
&\times\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,z^2(x,1,t,s)\,dsdx\\
&-n_1N_5\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}s\,|\mu_2(s)|\,z^2(x,\rho,s,t)\,dsd\rho dx\\
&-\frac{1}{4}\left(1-\frac{a}{b}N_3\right)\left[L_1au_x^2(L_1,t)+(L_3-L_2)\,au_x^2(L_2,t)\right]\\
&-\frac{1}{4}(1-N_3)\left[L_1u_t^2(L_1,t)+(L_3-L_2)\,u_t^2(L_2,t)\right].
\end{aligned}\tag{3.31}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. In fact, it follows from the assumption (3.3) that we can always choose N_1 and N_3 such that

$$1-N_3>0,\ 1-\frac{a}{b}N_3>0,\ \frac{L_1+L_3-L_2}{4(L_2-L_1)}N_3-N_1>0.$$

Once the above constants are fixed, we may choose ε_1 and ε_2 small enough such that

$$\varepsilon_1c_0^2N_1+\varepsilon_2<a\left(N_1-\frac{1}{2}\right).$$

Then we can take N_5 su ciently large such that

$$n_1N_5-\frac{1}{2\varepsilon_1}N_1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds-C_3(\varepsilon_2)N_2\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds>0.$$

Finally, noticing the assumption (1.4), we can always choose N sufficiently large such that

$$\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) N - N_1 - C_1(\varepsilon_2) - \frac{\tau\gamma^2}{4k} - \mu_1 N_5 > 0,$$

$$N - \frac{\gamma^2}{2k^2\varepsilon_1} N_1 - \frac{C_2(\varepsilon_2)}{k^2} - \left(2\tau k + \frac{1}{2k}\right) > 0,$$

$$N - \left(\frac{\tau^2\gamma^2}{2k^2\varepsilon_1} N_1 + \frac{\tau^2 C_2(\varepsilon_2)}{k^2}\right) > 0.$$

Consequently, from the above, we deduce that there exist a positive constant δ_1 such that (3.31) becomes

$$\mathcal{L}'(t) \le -\delta_1 E_1(t). \tag{3.32}$$

On the other hand, if we let

$$\mathcal{G}(t) = N_1 I_1(t) + I_2(t) + N_3 I_3(t) + I_4(t) + N_5 I_5(t),$$

we obtain

$$\begin{aligned} |\mathcal{G}(t)| \le{}& N_1 \int_\Omega |u u_t|\, dx + \frac{\mu_1}{2} N_1 \int_\Omega u^2 dx + N_1 \int_{L_1}^{L_2} |v v_t|\, dx \\ &+ \int_\Omega p(x) u_x u_t dx + N_3 \int_{L_1}^{L_2} p(x) v_x v_t\, dx \\ &+\tau \int_\Omega \left| q\left(\int_0^x \theta(y,t)\, dy\right)\right| dx \\ &+ N_5 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s \left|\mu_2(s) e^{-s\rho}\right| z^2(x,\rho,s,t)\, ds d\rho dx \end{aligned}$$

Exploiting Young's, Poincaré, Cauchy-Schwarz inequalities, (3.1), and the fact that $e^{-s\rho} \le 1$ for all $\rho \in [0,1]$, we obtain

$$\begin{aligned} |\mathcal{G}(t)| \le{}& \delta \int_\Omega \left[u_t^2(x,t) + u_x^2(x,t) + \theta^2(x,t) + q^2(x,t)\right] dx \\ &+ \delta \int_{L_1}^{L_2} \left[v_t^2(x,t) + v_x^2(x,t)\right] dx + \delta \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\, |\mu_2(s)|\, z^2(x,\rho,t,s) ds d\rho dx \\ \le{}& \delta E_1(t), \end{aligned}$$

for some $\delta > 0$.

Consequently, $|\mathcal{L}(t) - N E(t)| \le \delta E_1(t)$,that is,

$$(N-\delta) E_1(t) + N E_2(t) \le \mathcal{L}(t) \le (N+\delta) E_1(t) + N E_2(t),$$

for N large enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1 (E_1(t) + E_2(t)) \le \mathcal{L}(t) \le \beta_2 (E_1(t) + E_2(t)), \ \ \forall t \ge 0.$$

A simple integration of (3.32), we get

$$\begin{aligned} t E_1(t) \le{}& \int_0^t E_1(y)\, dy \le \frac{1}{\delta_1} (\mathcal{L}(0) - \mathcal{L}(t)) \le \frac{1}{\delta_1} \mathcal{L}(0) \\ \le{}& \frac{\beta_2}{\delta_1} (E_1(0) + E_2(0)), \end{aligned}$$

which gives us,

$$E_1(t) \leq \frac{k_0}{t}, \quad t \geq 0,$$

where $k_0 = \dfrac{\beta_2}{\delta_1}\,(E_1(0) + E_2(0))$.

This completes the proof.

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