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Commutators of Marcinkiewicz integral with rough kernels on generalized weighted Morrey spaces

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Abstract. *We study the boundedness of the commutators of Marcinkiewicz operators $\mu_{\Omega,b}$ with rough kernels $\Omega \in L_s(S^{n-1})$ for some $s \in (1,\infty]$ and BMO function b on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. In the case of $b \in BMO(\mathbb{R}^n)$ we find the sufficient conditions on the pair (φ_1,φ_2) with $s' < p < \infty$ and $w \in A_{p/s'}$ or $1 < p < s$ and $w^{1-p'} \in A_{p'/s'}$ which ensures the boundedness of the operators $\mu_{\Omega,b}$ from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$.*

Keywords. Marcinkiewicz operator; rough kernel; generalized weighted Morrey spaces; commutator; $A_p(\mathbb{R}^n)$ weights.

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1 Introduction

The classical Morrey spaces were originally introduced by Morrey in [16] to study the local behavior of solutions to second order elliptic partial differential equations. Guliyev, Mizuhara and Nakai [7,15,18] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [8,19]). Recently, Komori and Shirai [14] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [9] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [9] Guliyev also studied the boundedness of the classical operators and their commutators in these spaces $M_{p,\varphi}(w)$, see also [1,6,10,11].

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Suppose that Ω satisfies the following conditions.

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(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \quad \text{for all} \quad t > 0 \ \text{ and } \ x \in \mathbb{R}^n. \tag{1.1}$$

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the com- mutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b} f(x) = \left(\int_0^\infty |F^b_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F^b_{\Omega,t}(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left[b(x) - b(y)\right] f(y)dy.$$

In the case of $b \in BMO(\mathbb{R}^n)$ we find the sufficient conditions on the pair (φ_1, φ_2) with $s' < p < \infty$ and $w \in A_{p/s'}$ or $1 < p < s$ and $w^{1-p'} \in A_{p'/s'}$ which ensures the boundedness of the operators $\mu_{\Omega,b}$ from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

We recall that a weight function w is in the Muckenhoupt's class $A_p(\mathbb{R}^n)$ [17], $1 < p < \infty$, if

$$\begin{aligned} [w]_{A_p} :&= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'}dx \right)^{p-1}, \end{aligned} \tag{2.1}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B by Hölder's inequality

$$[w]^{1/p}_{A_p(B)} = |B|^{-1} \|w\|^{1/p}_{L_1(B)} \, \|w^{-1/p}\|_{L_{p'}(B)} \geq 1. \tag{2.2}$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ $A_\infty(\mathbb{R}^n) = \bigcup_{1\leq p<\infty} A_p(\mathbb{R}^n)$ and $[w]_{A_\infty} = \inf\limits_{1\leq p<\infty} [w]_{A_p}$.

Remark 2.1 It is known that

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]^{q'/p'}_{A_{p'/q'}(B)} = |B|^{-1} \|w^{1-p'}\|^{q'/p'}_{L_1(B)} \, \|w^{q'/p}\|_{L_{(p'/q')'}(B)}.$$

Moreover, we can write $w^{1-p'} \in A_{p'/q'} \Rightarrow w^{1-p'} \in A_{p'}$ because of $w^{1-p'} \in A_{p'/q'} \subset A_{p'}$. Therefore, we get

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow w^{1-p'} \in A_{p'} \\ &\Rightarrow [w^{1-p'}]^{1/p'}_{A_{p'}(B)} = |B|^{-1} \|w^{1-p'}\|^{1/p'}_{L_1(B)} \, \|w^{1/p}\|_{L_p(B)}. \end{aligned} \tag{2.3}$$

But the opposite is not true.

Remark 2.2 Let's write $w^{1-p'} \in A_{p'/q'}$ and used the definitions A_p classes we get the following

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow [w^{1-p'}]^{\frac{q(p-1)}{p(q-1)}}_{A_{p'/q'}} = |B|^{-1} \|w^{1-p'}\|^{\frac{q(p-1)}{p(q-1)}}_{L_1(B)} \, \|w^{q'/p}\|_{L_{(p'/q')'}(B)} \\ &\Rightarrow [w^{1-p'}]^{1/p'}_{A_{p'/q'}} = |B|^{-\frac{q-1}{q}} \|w^{1-p'}\|^{1/p'}_{L_1(B)} \, \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B)}, \end{aligned} \tag{2.4}$$

where the following equalities are provided.

$$1 - p' = -\frac{p'}{p}, \;\; \frac{q'}{p} = \frac{q}{p(q-1)}, \;\; \frac{q'}{p'} = \frac{q(p-1)}{p(q-1)}, \;\; \left(\frac{q}{p}\right)' = \frac{q}{q-p}, \;\; \left(\frac{p'}{q'}\right)' = \frac{p(q-1)}{q-p}.$$

Then from eq.(2.3) and eq.(2.4) we have

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow [w^{1-p'}]^{1/p'}_{A_{p'/q'}} \\ &= |B|^{\frac{1}{q}} [w^{1-p'}]^{1/p'}_{A_{p'}(B)} \, \|w^{1/p}\|^{-1}_{L_p(B)} \, \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B)}. \end{aligned} \tag{2.5}$$

We define the generalized weighed Morrey spaces as follows, see [9].

Definition 2.1 *Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, w(B(x,r))^{-\frac{1}{p}} \, \|f\|_{L_{p,w}(B(x,r))}.$$

The operator μ_Ω was first defined by Stein [20]. And Stein proved that if is continuous and satisfies a $\mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \le 1)$ condition, then μ_Ω is an operator of type (p,p) $(1 < p \le 2)$ and of weak type $(1,1)$. In [2], Benedek, Calderón and Panzone proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The L_p boundedness of μ_Ω has been studied extensively. See [2,13,20,21], among others. Ding, Fan and Pan [3] proved the weighted $L_p(\mathbb{R}^n)$ boundedness with A_p weighs for a class of rough Marcinkiewicz integrals. Recently, Ding, Fan and Pan [4] improved the results mentioned above and showed that if $\Omega \in H^1(S^{n-1})$, the Hardy space on the unit sphere, then μ_Ω is still a bounded operator on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. In [22], Xu, Chen and Ying proved the same result as [4] using a different method.

Theorem 2.1 *[5] Suppose that Ω be satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L_s(S^{n-1})$ *for some* $s \in (1, \infty]$. *Let also* $b \in BMO(\mathbb{R}^n)$. *Then for every* $s' < p < \infty$ *and* $w \in A_{\frac{p}{s'}}(\mathbb{R}^n)$ *or* $1 < p < s$ *and* $w^{1-p'} \in A_{\frac{p'}{s'}}(\mathbb{R}^n)$, *there is a constant* $C > 0$ *independent of* f *such that*

$$\left\|\mu_{\Omega,b} f\right\|_{L_{p,w}} \leq C \|f\|_{L_{p,w}}.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty \left(1 + \frac{s}{t}\right) g(s)\, w(s)\, ds, \qquad 0 < t < \infty,$$

where w is a weight. The following theorem was proved in [9].

Theorem 2.2 *[9] Let* v_1, v_2 *and* w *be weights on* $(0, \infty)$ *and* $v_1(t)$ *be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w^* g(t) \leq C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0, \infty)$ *if and only if*

$$B := \sup_{t>0} v_2(t) \int_t^\infty \left(1 + \frac{s}{t}\right) \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3 Commutators of Marcinkiewicz operator with rough kernels $\mu_{\Omega,b}$ in the spaces $M_{p,\varphi}(w)$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 3.1 *Suppose that* $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, *and let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Remark 3.1 Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \leq C \|b\|_* \ln \frac{t}{r} \ \text{ for } \ 0 < 2r < t, \tag{3.1}$$

where C is independent of b, x, r and t.

The following Guliyev type local estimates are valid, see [7–9].

Lemma 3.1 *Suppose that Ω be satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$. *Let* $b \in BMO(\mathbb{R}^n)$.

If $s' < p < \infty$ *and* $w \in A_{p/s'}$, *then the inequality*

$$\|\mu_{\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

If $1 < p < s$ *and* $w^{1-p'} \in A_{p'/s'}$, *then the inequality*

$$\|\mu_{\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|^{1/p}_{L_{\frac{s}{s-p}}(B(x_0,r))} \int_{2r}^{\infty} \Big(1+\ln \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))}\, \|w\|^{-1/p}_{L_{\frac{s}{s-p}}(B(x_0,t))} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\infty)$, $b \in BMO(\mathbb{R}^n)$ and Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$.

For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \;\; f_1(y) = f(y)\chi_{2B}(y), \;\; f_2(y) = f(y)\chi_{\complement_{(2B)}}(y) \tag{3.2}$$

and have

$$\|\mu_{\Omega,b} f\|_{L_{p,w}(B)} \le \|\mu_{\Omega,b} f_1\|_{L_{p,w}(B)} + \|\mu_{\Omega,b} f_2\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}$, $\mu_{\Omega,b} f_1 \in L_{p,w}$ and from the boundedness of $\mu_{\Omega,b}$ in $L_{p,w}$ for $s' < p < \infty$ and $w \in A_{p/s'}$ (see Theorem 2.1) it follows that

$$\begin{aligned}\|\mu_{\Omega,b} f_1\|_{L_{p,w}(B)} &\le \|\mu_{\Omega,b} f_1\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L_s(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{s'}}}\, \|b\|_*\, \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &= \|\Omega\|_{L_s(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{s'}}}\, \|b\|_*\, \|f\|_{L_{p,w}(2B)}.\end{aligned}$$

For $x \in B$ we have

$$\mu_{\Omega,b}(f_2(x)) \lesssim \int_{\complement_{(2B)}} |b(y)-b(x)||\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n} dy.$$

Then

$$\begin{aligned}\|\mu_{\Omega,b} f_2\|_{L_{p,w}(B)} &\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} |b(y)-b(x)||\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n}\, dy \right)^p w(x)dx \right)^{\frac{1}{p}} \\ &\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} |b(y)-b_{B,w}||\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n}\, dy \right)^p w(x)dx \right)^{\frac{1}{p}} \\ &\quad + \left(\int_B \left(\int_{\complement_{(2B)}} |b(x)-b_{B,w}||\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n}\, dy \right)^p w(x)dx \right)^{\frac{1}{p}} \\ &= I_1 + I_2.\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned} I_1 &= w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} |b(y)-b_{B,w}|\,|\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n}\,dy \\ &\approx w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} |b(y)-b_{B,w}|\,|\Omega(x-y)|\,|f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}\,dy \\ &\approx w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{2r\le |x_0-y|\le t} |b(y)-b_{B,w}|\,|\Omega(x-y)|\,|f(y)|\,dy \frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_{B,w}|\,|\Omega(x-y)|\,|f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned}$$

Set $m = p/q' > 1$. Since $w \in A_m$, from (2.3), we know $w^{1-m'} \in A_{m'}$. Applying Hölder's inequality and by (3.1), we get

$$\begin{aligned} I_1 &\lesssim \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} \, \||b(y)-b_{B,w}|f\|_{L_{s'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_s(S^{n-1})} \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|b-b_{B,w}\|_{L_{m's',w^{1-m'}}(B(x_0,t))} \, \|f\|_{L_{p,w}(B(x_0,t))} \, |B(x_0,t+|x-x_0|)|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_s(S^{n-1})} \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) (w^{1-m'}(B(x_0,t)))^{\frac{1}{m's'}} \, \|f\|_{L_{p,w}(B(x_0,t))} \, |B(x_0,t)| \frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_s(S^{n-1})} [w]_{A_{\frac{p}{s'}}}^{\frac{1}{p}} \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}. \end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \left(\int_B |b(x)-b_{B,w}|^p w(x)dx\right)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}\,dy.$$

By Fubini's theorem we have

$$\begin{aligned} &\int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy \approx \int_{\complement_{(2B)}} |\Omega(x-y)|\,|f(y)| \int_{|x-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &= \int_{2r}^{\infty} \int_{2r\le|x_0-y|<t} |\Omega(x-y)|\,|f(y)| dy \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)|\,|f(y)| dy \frac{dt}{t^{n+1}}\,. \end{aligned}$$

By applying Hölder's inequality for $s' < p < \infty$ and $w \in A_{p/s'}$, we get

$$\begin{aligned} &\int_{\complement_{(2B)}} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n} dy \lesssim \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} \, \|f\|_{L_{s'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_s(S^{n-1})} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \, \|w^{-s'/p}\|_{L_{(p/s')'}(B(x_0,t))}^{\frac{1}{s'}} \, |B(0,t+|x-x_0|)|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\lesssim \|\Omega\|_{L_s(S^{n-1})} \, [w]_{A_{\frac{p}{s'}}}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, |B(x_0,t)|^{\frac{1}{s'}} \, |B(0,t)|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\approx \|\Omega\|_{L_s(S^{n-1})} \, [w]_{A_{\frac{p}{s'}}}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}. \qquad (3.3) \end{aligned}$$

By (3.1) and (3.3), we get

$$I_2 \lesssim \|b\|_* \, w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}\, dy$$
$$\lesssim \|\Omega\|_{L_s(S^{n-1})}[w]^{\frac{1}{p}}_{A_{\frac{p}{s'}}} \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}.$$

Summing up I_1 and I_2, for all $p \in (1,\infty)$ we get

$$\|\mu_{\Omega,b} f_2\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_s(S^{n-1})}[w]^{\frac{1}{p}}_{A_{\frac{p}{s'}}} \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}.$$

Thus

$$\|\mu_{\Omega,b} f\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_s(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{s'}}} \, \|b\|_* \Big(\|f\|_{L_{p,w}(2B)}$$
$$+ \; w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}\Big). \tag{3.4}$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B|\, \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim w(B)^{\frac{1}{p}}\, \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \|w^{-1/p}\|_{L_{p'}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim [w]^{\frac{1}{p}}_{A_p}\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}. \tag{3.5}$$

Then, by (3.4) and (3.5) we get

$$\|\mu_{\Omega,b} f\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_s(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{s'}}} \, \|b\|_* \, w(B)^{\frac{1}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}.$$

Let also $1 < p < s$ and $w^{1-p'} \in A_{p'/s'}$. Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $\mu_{\Omega,b}\big(f_1\big) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of $\mu_{\Omega,b}$ in $L_{p,w}(\mathbb{R}^n)$ for $w^{1-p'} \in A_{p'/s'}$ and $1 < p < s$ (see Theorem 2.1) it follows that

$$\|\mu_{\Omega,b}\,(f_1)\,\|_{L_{p,w}(B)} \le \|\mu_{\Omega,b}\,(f_1)\,\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L_s(S^{n-1})}\, [w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{s'}}} \, \|b\|_* \, \|f_1\|_{L_{p,w}(\mathbb{R}^n)}$$
$$\approx \|\Omega\|_{L_s(S^{n-1})}\, [w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{s'}}} \, \|b\|_* \, \|f\|_{L_{p,w}(2B)}.$$

If $1 < p < s$ and $w^{1-p'} \in A_{p'/s'}$, then using Minkowski theorem and Hölder inequality,

$$
\begin{aligned}
I_1 &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B,w}| \, |\Omega(x-y)| \, |f(y)| dy \frac{dt}{t^{n+1}} \\
&\leq \left(\int_B \left(\int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B,w}| \, |\Omega(x-y)||f(y)| dy \, \frac{dt}{t^{n+1}} \right)^p \, w(x) dx \right)^{\frac{1}{p}} \\
&\leq \int_{2r}^{\infty} \int_{B(x_0,t)} \| |b(\cdot) - b_{B,w}| \, \Omega(\cdot - y) \|_{L_{p,w}(B)} |f(y)| \, dy \, \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_s(B)} \, \|b(\cdot) - b_{B,w}\|_{L_{(s/p)'}(B)}^{\frac{1}{p}} \, |f(y)| \, dy \, \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* \, \|\Omega\|_{L_s(S^{n-1})} \, \|w\|_{L_{(s/p)'}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left(1 + \ln \frac{t}{r}\right) |B(0, r + |x_0 - y|)|^{\frac{1}{s}} \, |f(y)| \, dy \, \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* \, \|\Omega\|_{L_s(S^{n-1})} \, \|w\|_{L_{(s/p)'}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_1(B(x_0,t))} \, |B(0, r+t)|^{\frac{1}{s}} \, \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* \, \|\Omega\|_{L_s(S^{n-1})} \, \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \, \|w^{-p'/p}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} \, |B(x_0, t)|^{\frac{1}{s}} \, \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* \, \|\Omega\|_{L_s(S^{n-1})} \, |B|^{\frac{1}{s}} \, \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \, \|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} \, |B(x_0, t)|^{\frac{1}{s}} \, \frac{dt}{t^{n+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \left(\int_B |b(x) - b_{B,w}|^p w(x) dx \right)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0 - y|^n} \, dy \\
&\leq \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)| \, |f(y)| dy \frac{dt}{t^{n+1}} \\
&\leq \|b\|_* \left(\int_B \left(\int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)||f(y)| dy \, \frac{dt}{t^{n+1}} \right)^p \, w(x) dx \right)^{\frac{1}{p}} \\
&\leq \|b\|_* \int_{2r}^{\infty} \int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_p(B)} \, |f(y)| dy \frac{dt}{t^{n+1}} \\
&\leq \|b\|_* \int_{2r}^{\infty} \int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_s(B)} \, \|w\|_{L_{(s/p)'}(B)}^{\frac{1}{p}} \, |f(y)| dy \frac{dt}{t^{n+1}} \\
&\leq \|b\|_* \, \|\Omega\|_{L_s(S^{n-1})} \, \|w\|_{L_{(s/p)'}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_s(B)} \, |B(0, r + |x_0 - y|)|^{\frac{1}{s}} \, |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* \, \|\Omega\|_{L_s(S^{n-1})} \, \|w\|_{L_{(s/p)'}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \, |B(0, r+t)|^{\frac{1}{s}} \, \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* \, \|\Omega\|_{L_s(S^{n-1})} \, \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \, \|w^{-p'/p}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} \, |B(x_0, t)|^{\frac{1}{s}} \, \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* \, \|\Omega\|_{L_s(S^{n-1})} \, |B|^{\frac{1}{s}} \, \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \, \|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} \, |B(x_0, t)|^{\frac{1}{s}} \, \frac{dt}{t^{n+1}}
\end{aligned}
$$

is obtained. By applying (2.3) for $\|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}}$ and (2.5) for $\|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}}$ we have the following inequality

$$\|\mu_{\Omega,b}(f_2)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_s(S^{n-1})}\,[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{s'}}}\,\|w\|_{L^{\frac{1}{p}}_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\,\|f\|_{L_{p,w}(B(x_0,t))}\,\|w\|^{-\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x_0,t))}\,\frac{dt}{t}$$

is valid. Thus

$$\|\mu_{\Omega,b}(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_s(S^{n-1})}\,[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{s'}}}\Big(\|f\|_{L_{p,w}(2B)}$$
$$+\|w\|_{L^{\frac{1}{p}}_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\,\|f\|_{L_{p,w}(B(x_0,t))}\,\|w\|^{-\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x_0,t))}\,\frac{dt}{t}\Big).$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B|\|f\|_{L_{p,w}(2B)}\int_{2r}^{\infty}\frac{dt}{t^{n+1}}$$
$$\lesssim |B|\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$= [w^{1-p'}]^{-\frac{1}{p'}}_{A_{p'}(B)}\,|B|^{\frac{1}{s}}\|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B)}\,\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,\omega}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim [w^{1-p'}]^{-\frac{1}{p'}}_{A_{p'}(B)}\,\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,\omega}(B(x_0,t))}\,|B(x_0,t)|^{\frac{1}{s}}\|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim \|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,\omega}(B(x_0,t))}\,\|w\|^{-\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x_0,t))}\,\frac{dt}{t}.$$

Thus

$$\|\mu_{\Omega,b}(f)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_s(S^{n-1})}\,[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{s'}}}\,\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\,\|f\|_{L_{p,w}(B(x_0,t))}\,\|w\|^{-\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x_0,t))}\,\frac{dt}{t}.$$

Thus we complete the proof of Lemma 3.1.

Theorem 3.1 *Suppose that $b \in BMO(\mathbb{R}^n)$, Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$.*

Let $s' < p < \infty$, $w \in A_{\frac{p}{s'}}(\mathbb{R}^n)$ and the pair (φ_1, φ_2) satisfy the condition

$$\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}}\,\frac{dt}{t} \le C\,\varphi_2(x,r),$$

where C does not depend on x and r. Let also, $1 < p < s$, $w^{1-p'} \in A_{p'/s'}$ and the pair (φ_1, φ_2) satisfy the condition

$$\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)\|w\|^{1/p}_{L_{\frac{s}{s-p}}(B(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{s}{s-p}}(B(x,t))}}\,\frac{dt}{t} \le C\,\varphi_2(x,r)\,\frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x,r))}},$$

where C does not depend on x and r.

Then the operator $\mu_{\Omega,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover

$$\|\mu_{\Omega,b}f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. By Lemma 3.1 and Theorem 2.2 with $\nu_1(r) = \varphi_1(x,r)^{-1}w(B(x,t))^{-\frac{1}{p}}$, $\nu_2(r) = \varphi_2(x,r)^{-1}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}}$ we have

$$\begin{aligned}
&\|\mu_{\Omega,b}f\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_{\Omega,b}f\|_{L_{p,w}(B(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-\frac{1}{p}}\,\frac{dt}{t} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}B(x,r)} \\
&= \|b\|_*\, \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}$$

Remark 3.2 Note that Lemma 3.1 and Theorem 3.1 in the case $s = \infty$ was proved in [12].

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