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On homogeneous integral equations of Fredholm type in the spaces of Bohr almost periodic functions

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Abstract. In the present work we consider a new type of homogenous Fredholm integral equations (or limit integral equations). We study the question on transference of homogenous case of Fredholm theory to the Bohr spaces. As in the ordinary case, we establish that for every characteristic number corresponding subspace of solutions has finite dimension. For symmetric kernel we establish expansion of the kernel.

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1 Introduction

In the work [4] a new kind of integral equations of Fredholm type are considered in Bohr spaces. There the analog of the first theorem of Fredholm is obtained. We, using the main characteristic property of Bohr almost periodic function on representation of almost periodic function as a limit periodic function, succeed in leading of the question to investigation of some family of "ordinary" Fredholm type equations and obtain needed results. The same approach is applicable in the case of homogeneous equations (see [1,2,5,6,8]).

Let us consider a limit integral equation of a view:

$$\varphi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\varphi(\xi)d\xi \tag{1.1}$$

In ordinary case, this equation is called as a second kind Fredholm equation. When $f(x) = 0$ we call this equation homogenous. It is clear that the homogenous equation has a trivial solution $\psi(x) = 0, x \in \mathbb{R}$. Since the operator on right hand side of the equation (1.1) is a linear operator, then the set of solutions of the equation is a linear subspace in the

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Bohr space, i. e. every linear combination of solutions with real coefficients will be solution again. As trivial solutions are not substantive, we shall consider two basic questions for homogenous equations: has the equation (1.1) nonzero solutions and if yes, how we can find them? Moreover, we shall study the properties of solutions, to solve the question completely.

2 Some properties for analogs of Fredholm functions

In [4], following functions are introduced:

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{b_n \lambda^n}{n!},$$

with

$$b_n = (-1)^n \lim_{T\to\infty} \frac{1}{T^n} \times$$

$$\times \int_0^T \cdots \int_0^T \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) & \cdots & K(\xi_1,\xi_n) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) & \cdots & K(\xi_2,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n,\xi_1) & K(\xi_n,\xi_2) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_n; \tag{2.1}$$

also

$$D(x,y;\lambda) = \lambda K(x,y) + \sum_{n=1}^{\infty} (-1)^n \frac{Q_n(x,y)\lambda^{n+1}}{n!}; x, y \in R,$$

where

$$Q_n(x,\xi) = \lim_{T\to\infty} \frac{1}{T^n} \int_0^T \cdots \int_0^T \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & \cdots & K(x,\xi_n) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n,\xi) & K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n. \tag{2.2}$$

These functions are analogs of classical Fredholm functions (see p. 304 of the book [9]). The relation of following Lemma is an analog of the first fundamental relation of Fredholm.

Lemma 2.1. For real *x, y* and λ following equality holds true:

$$D(x,y;\lambda) = \lambda D(\lambda)K(x,y) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(u,y)D(x,u;\lambda)\, du; \quad x, y \in R.$$

Proof. For the proof of the lemma 1, we have to find the relation between coefficients of expansions of the functions $D(\lambda)$ and $D(x,y;\lambda)$ into power series with respect to the parameter λ. In the right-hand side of the equality (2.2), take the expansion of the determinant with respect to entries of the first column. Then we get

$$\int_0^T \cdots \int_0^T \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & \cdots & K(x,\xi_n) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n,\xi) & K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n$$

$$= \int_0^T \cdots \int_0^T K(x,\xi) \begin{vmatrix} K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \ddots & \vdots \\ K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n$$

$$+\sum_{i=1}^{n}(-1)^i\int_0^T\cdots\int_0^T K(\xi_i,\xi)\begin{vmatrix} K(x,\xi_1) & \cdots & K(x,\xi_n)\\ K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n)\\ \cdots & \cdots & \cdots\\ K(\xi_{i-1},\xi_1) & \cdots & K(\xi_{i-1},\xi_n)\\ K(\xi_{i+1},\xi_1) & \cdots & K(\xi_{i+1},\xi_n)\\ \cdots & \cdots & \cdots\\ K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n)\end{vmatrix} d\xi_1\cdots d\xi_n.$$

In every determinant of the sum over i we make $i-1$ replacements of the i-th column, consequently with every column before it, while this column does not stand in the position of the first column. Then we get the sum of following expressions:

$$(-1)^{2i-1}\int_0^T\cdots\int_0^T K(\xi_i,\xi)\begin{vmatrix} K(x,\xi_i) & K(x,\xi_1) & \cdots & K(x,\xi_n)\\ K(\xi_1,\xi_i) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n)\\ \cdots & \cdots & \cdots & \cdots\\ K(\xi_{i-1},\xi_i) & K(\xi_{i-1},\xi_1) & \cdots & K(\xi_{i-1},\xi_n)\\ K(\xi_{i+1},\xi_i) & K(\xi_{i+1},\xi_1) & \cdots & K(\xi_{i+1},\xi_n)\\ \cdots & \cdots & \cdots & \cdots\\ K(\xi_n,\xi_i) & K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n)\end{vmatrix} d\xi_1\cdots d\xi_n.$$

Now under the integral we replace the variables $\xi_i,\xi_{i+1},...,\xi_n$, correspondingly by variables $y,\xi_i,...,\xi_{n-1}$. Then we get n identical expressions of form (due to symmetricity of the kernel):

$$-\int_0^T\cdots\int_0^T K(y,\xi)\begin{vmatrix} K(x,y) & K(x,\xi_1) & \cdots & K(x,\xi_{n-1})\\ K(\xi_1,y) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_{n-1})\\ \cdots & \cdots & \cdots & \cdots\\ K(\xi_{i-1},y) & K(\xi_{i-1},\xi_1) & \cdots & K(\xi_{i-1},\xi_{n-1})\\ K(\xi_i,y) & K(\xi_i,\xi_1) & \cdots & K(\xi_i,\xi_{n-1})\\ \cdots & \cdots & \cdots & \cdots\\ K(\xi_{n-1},y) & K(\xi_{n-1},\xi_1) & \cdots & K(\xi_{n-1},\xi_{n-1})\end{vmatrix} dyd\xi_1\cdots d\xi_{n-1},$$

being equal to the corresponding multiple integral in the expression of $-Q_{n-1}(x,y)$, in consent with (2.2). So,

$$Q_n(x,y)=(-1)^n b_n K(x,y)-n\lim_{T\to\infty}\frac{1}{T}\int_0^T Q_{n-1}(x,t)K(t,y)\,dt. \tag{2.3}$$

Substituting this expression in the series for $D(x,y;\lambda)$, and using the relation (2.1), we obtain

$$D(x,y;\lambda)=\lambda K(x,y)+\sum_{n=1}^{\infty}(-1)^n\frac{Q_n(x,y)\lambda^{n+1}}{n!}$$

$$=\lambda K(x,y)+\sum_{n=1}^{\infty}(-1)^n\frac{\lambda^{n+1}}{n!}(b_nK(x,y)-n\lim_{T\to\infty}\frac{1}{T}\int_0^T Q_{n-1}(x,t)K(t,y)\,dt)$$

$$=\lambda D(\lambda)K(x,y)+\lambda\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{n+1}}{n!}\lim_{T\to\infty}\frac{1}{T}\int_0^T Q_n(x,t)K(t,y)\,dt$$

$$=\lambda D(\lambda)K(x,y)+\lambda\lim_{T\to\infty}\frac{1}{T}\int_0^T K(t,y)\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{n+1}Q_n(x,t)}{n!}dt$$

$$= \lambda D(\lambda)K(x,y) + \lambda \lim_{T\to\infty} \frac{1}{T}\int_0^T D(x,t,\lambda)K(t,y)\,dt. \quad (2.4)$$

Proof of Lemma 2.1 is finished.

Denoting

$$k(x,y,\lambda) = \frac{D(x,y,\lambda)}{\lambda D(\lambda)},$$

we can rewrite the previous equation as follows

$$k(x,y;\lambda) - K(x,y) = \lambda \lim_{T\to\infty} \frac{1}{T}\int_0^T K(u,y)\,k(x,u;\lambda)\,du; \quad x,y \in R. \quad (2.5)$$

The last equality shows that the function $k(x,u;\lambda)$ is a solution of the equation (1.2) of the work [4], with the function $K(x,y)$ at the right-hand side, instead of $f(x)$.

Definition 2.1. The function $k(x,u;\lambda)$ we call a function relative to $-K(x,y)$; also the function

$$r(x,u;\lambda) = \lambda k(x,u;\lambda) = \frac{D(x,y,\lambda)}{D(\lambda)}$$

we call a resolvent of the equation (1.1) (see [7]).

If λ is not a root of the function $D(\lambda)$ then the relative function is unique, in accordance with Theorem 2.2 of the work [3]. We can formulate the analog of the first theorem of Fredholm from [4].

Lemma 2.2. Let λ be a real number such that $D(\lambda) \neq 0$. Then the equation (1.1) has unique almost periodic solution given by the equality

$$\varphi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\xi)k(x,u;\lambda)d\xi. \quad (2.6)$$

Theorem 2.1. Let a real number λ be distinct from any roots of the function $D(\lambda)$, and the function $K(x,y)$ be some symmetric almost periodic function, and $k(x,u;\lambda)$ be relative to it. If the function $\varphi(x)$ is a solution of the equation

$$\varphi(x) - f(x) = \lambda \lim_{T\to\infty} \frac{1}{T}\int_0^T K(x,\xi)\varphi(\xi)d\xi$$

then the function $f(x)$ is a solution of the equation

$$f(\xi) - \varphi(\xi) = \lambda \lim_{T\to\infty} \frac{1}{T}\int_0^T k(\xi,u;\lambda)f(u)du.$$

Proof. Multiplying the first equation in Theorem 2.1 by the function $k(x,u;\lambda)$, let us integrate it with respect to the variable x. Then we get

$$\lambda \lim_{T\to\infty} \frac{1}{T}\int_0^T k(x,u;\lambda)f(x)dx = \lambda \lim_{T\to\infty} \frac{1}{T}\int_0^T k(\xi,x;\lambda)\varphi(x)dx$$

$$-\lambda^2 \lim_{T\to\infty} \frac{1}{T}\int_0^T k(x,u;\lambda)dx \lim_{T\to\infty} \frac{1}{T}\int_0^T K(x,\xi)\varphi(\xi)d\xi.$$

Using boundedness and continuity, we can change the order of integration and limiting processes. So, we can rewrite the second integral in the right-hand side of the last equation as follows

$$= \lambda^2 \lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(\xi)d\xi \lim_{T\to\infty} \frac{1}{T}\int_0^T K(x,\xi)k(x,u;\lambda)dx.$$

Inner mean value in the right hand-side of the last equation can be substitute by the left hand-side of the equation (2.5). Now, using conditions of the theorem, represent the last relation as below

$$\lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(\xi)(k(u,\xi;\lambda) - K(u,\xi))d\xi$$

$$= \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(\xi)(k(u,\xi;\lambda))d\xi - (\varphi(u) - f(u)).$$

So, we have

$$\lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T k(x,u;\lambda) f(x)dx = f(u) - \varphi(u),$$

as stated in the theorem. Theorem 1 is proven.

Definition 2.2. Roots of the equation $D(\lambda) = 0$ are called characteristic numbers of the equation or the kernel $K(x,y)$.

As the function $D(\lambda)$ is an entire function then it has no more than denumerable set of complex roots with finite multiplicities. They have not any limit points in a compact subset of complex plane. Defined by Theorem 2.2 of the work [4] formula, cannot be applicable when the number λ is a characteristic number.

Theorem 2.2. Every characteristic number of integral equation is a pole of the resolvent function $r(x,u;\lambda)$.

Proof. Taking into account the expression (2.5) and letting *x*=*y*, we find

$$D(x,x;\lambda) = \lambda D(\lambda) K(x,x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(u,x) D(x,u;\lambda)\, du; \quad x \in R.$$

Let us derive from the equality (2.3) another relation between the coefficients of the series $D(\lambda)$ and $D(x,u;\lambda)$. If to put $x = y$ in the formula for $Q_n(x,y)$, and take the mean value, we obtain $(-1)^{n+1}\, b_{n+1}$:

$$(-1)^{n+1}\, b_{n+1} = \lim_{T\to\infty} \frac{1}{T} \int_0^T Q_n(x,x)\, dx. \tag{2.7}$$

In consent with this formula every coefficient of the series $D(\lambda)$ and $D(x,u;\lambda)$ could be calculated by the recurrent relations (2.3) and (2.7), using initial values $b_0 = 1,\ Q_0(x,y) = K(x,y)$. Using these relations, we shall find some formulae from which we shall derive our theorem. Note firstly that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T D(x,x,\ \lambda)\, dx = \lim_{T\to\infty} \frac{1}{T} \int_0^T (\sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+1} Q_n(x,t)}{n!})\, dx$$

$$= -\sum_{n=0}^{\infty} \frac{b_{n+1}}{n!} \lambda^{n+1} = -\lambda \sum_{n=0}^{\infty} \frac{b_{n+1}}{n!} \lambda^n.$$

Differentiating $D(\lambda)$ we write:

$$\lambda \frac{dD(\ \lambda)}{d\,\lambda} = \sum_{n=0}^{\infty} \frac{b_{n+1}\lambda^n}{n!} = -\lim_{T\to\infty} \frac{1}{T} \int_0^T D(x,x,\ \lambda)\, dx. \tag{2.8}$$

Dividing every part of the last equality by $D(\lambda)$, we get a formula for the logarithmic derivative of the function $D(\lambda)$ in terms of mean values for the functions $k(x,x,\lambda)$ and $r(x,x,\lambda)$.

$$\frac{d\ln D(\ \lambda)}{d\ \lambda} = -\lim_{T\to\infty}\frac{1}{T}\int_0^T k(x,x,\ \lambda)\,dx = -\lambda\lim_{T\to\infty}\frac{1}{T}\int_0^T r(x,x,\lambda)\,dx.$$

To prove our theorem, suppose that λ is not a pole for the resolvent. Then λ must be a root of the nominator of the resolvent, moreover, the order p of multiplicity of this root is not less than the order q for multiplicity of λ , as a root of $D(\lambda)$, that is $p \geq q$. Differentiating the equality (2.8) term by term, we find out:

$$\frac{d^{h+1}D(\ \lambda)}{d\ \lambda^{h+1}} = -\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{d^h}{d\ \lambda^h}D(x,x,\ \lambda)\,dx.$$

When $h = 1,\ 2,\ldots,p-1$, we have

$$\frac{d^h}{d\ \lambda^h}D(x,x,\ \lambda) = 0,\ x\in R.$$

Then,

$$\frac{d^{h+1}D(\ \lambda)}{d\ \lambda^{h+1}} = 0,$$

which shows that $q \geq p+1$. The obtained contradiction proves Theorem 2.2.

3 Solution of homogeneous equations

In [4] it was studied the Fredholm type equation in the case of non-homogenous equations with values of the parameter does not satisfying the equation $D(\lambda) = 0$. If we take homogenous equations, then from the results of the works [5,9] one can deduce, as in the case of ordinary Fredholm equations, the equation has trivial solutions only, when $D(\lambda) \neq 0$.

In this section we consider homogenous equations and find their solutions. It is clear that the equation (1.1) has a trivial solution $\varphi(x) = 0$ which we can count as an almost periodic function. But one needs in nontrivial solutions of this equation. If $\varphi_1(x),\ldots,\varphi_k(x)$ are almost periodic solutions, then every linear combinations of these functions also will be a solution of the equation. So, the set of solutions sets up a linear space. We shall show that this is a finite dimensional linear space and find its basis.

Theorem 3.1. If the number λ is a characteristic number of the kernel $K(u,x)$, then the equation (1.1) has nonzero solutions.

Proof. Taking the equality

$$D(x,y;\lambda) = \lambda D(\lambda)K(x,y) + \lambda\lim_{T\to\infty}\frac{1}{T}\int_0^T K(u,y)D(x,u;\lambda)\,du;\quad x,y\in R$$

of the Lemma 1, we note that when λ is a characteristic number then

$$D(x,y;\lambda) = \lambda\lim_{T\to\infty}\frac{1}{T}\int_0^T K(u,y)D(x,u;\lambda)\,du,$$

that is, the function $D(x,y;\lambda)$ is a solution of the equation (1.1). But this function can be equal to zero identically. Let $D(\lambda_0) = 0$ for some real λ_0. Since $D(\lambda)$ is an entire function, then it can be expanded into power series

$$D(\lambda) = d_s(\lambda-\lambda_0)^s + d_{s+1}(\lambda-\lambda_0)^{s+1} + \cdots\ ;$$

here s is a natural number. The number λ_0 can also be a root of the function $D(x, y; \lambda)$ which also is an entire function, and, therefore, can be expanded into power series:

$$D(x,\ y, \lambda) = a_r(x, y)(\lambda - \lambda_0)^r + a_{r+1}(x, y)(\lambda - \lambda_0)^{r+1} + \cdots ; r \geq 0.$$

Let's denote

$$c_m = \lim_{T\to\infty} \frac{1}{T} \int_0^T a_m(u, u)\ du; m = 0,\ 1,\ \ldots .$$

Then, using (2.8) we get

$$-\frac{dD(\ \lambda)}{d\,\lambda} = c_r(\lambda - \lambda_0)^r + c_{r+1}(\lambda - \lambda_0)^{r+1} + \cdots .$$

The multiplicity of the root λ_0 in the left hand-side is equal to $s-1$ (we suppose that $\lambda \neq 0$, otherwise the integral equation has a trivial solution). Since the right hand-side has the zero λ_0 of multiplicity r, then $s - 1 \geq r$.

Substituting expansions of functions $D(\lambda)$ and $D(x, y; \lambda)$ in the relation of Lemma 2.1, we can write

$$\begin{aligned} a_r(x, y)(\lambda - \lambda_0)^r + a_{r+1}(x, y)(\lambda - \lambda_0)^{r+1} + \cdots \\ = \lambda K(x, y)(d_s(\lambda - \lambda_0)^s + d_{s+1}(\lambda - \lambda_0)^{s+1} + \cdots) \\ + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(u, y)(a_r(x, y)(\lambda - \lambda_0)^r + a_{r+1}(x, y)(\lambda - \lambda_0)^{r+1} + \cdots)\ du. \end{aligned}$$

Dividing by $(\lambda - \lambda_0)^r$, and passing to the limit as $\lambda \to \lambda_0$, we find

$$a_r(x, y) = \ \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(u, y) a_r(x, y) du\ .$$

Moreover, the function $a_r(x, x)$ at some x is distinct from zero. So, $a_r(x, y)$ is not identically zero for some y=y_0. The proof of the Theorem 3.1 is finished.

Consequence. The equation (1.1) has not any solutions, or it has infinitely many solutions, when $\lambda = \lambda_0$.

Note that, if $D'(\lambda_0) \neq 0$, then from the said above one deduces that the function $D(x, y;\ \lambda_0)$ is not identically zero for some y=y_0. Then, the function $D(x, y;\ \lambda)$ is a solution of the homogenous equation (1.1), when $\lambda = \lambda_0$.

The Theorem 3.1 answers the question on existence of the solutions of the homogenous equation. Moreover, the set of solutions sets up a linear space. The main question is consisted in defining of dimension of this space and its basis.

This question is successfully solvable for equations with symmetric kernel.

4 Orthogonal system of almost periodic functions

Definition 4.1. We say that almost periodic functions $f_1(x)$ and $f_2(x)$ are orthogonal, if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f_1(x)\ f_2(x)\ dx = 0.$$

System of almost periodic functions $f_1(x)$, $f_2(x)$,..., $f_k(x)$ is called orthogonal, if fore every pair of different indices $1 \leq i < j \leq k$ following relation is valid:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f_i(x)\ f_j(x)\ dx = 0.$$

We call the expression

$$\|f\| = \left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T |f(x)|^2 \, dx \right\}^{1/2}$$

a norm of the almost periodic function f. If the norm of almost periodic function is equal to 1 then this function is called normalized.

From existence of mean values for almost periodic functions and Parseval formula ([1–3,5,6,8]) one deduces that the norm of the almost periodic function is equal to zero if and only if the function is identically zero.

Let us consider homogenous equation

$$f(x) = \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) f(\xi) d\xi.$$

Let $\lambda = \lambda_0$ be some characteristic number for which some system of orthogonal solutions $f_1(x)$, $f_2(x)$,..., $f_k(x)$ is given. We shall call this system of solutions a system of characteristic functions. Note that such a system of functions could well constructed by standard Gram-Shmidt process. We may also suppose all of these functions be normalized.

Theorem 4.1. The number n of various normalized characteristic functions related to given characteristic number λ_0 satisfies the inequality:

$$n \le \lambda_0^2 \lim_{T\to\infty} \frac{1}{T} \int_0^T \lim_{T\to\infty} \frac{1}{T} \int_0^T (K(x,y))^2 dx dy.$$

Proof. Consider the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \sum_{m=1}^k f_m(y) \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) f_m(\xi) d\xi \right|^2 dy$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{m=1}^k f_m(y)$$

$$\times \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) f_m(\xi) d\xi \sum_{n=1}^k f_n(y) \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\theta) f_n(\theta) d\theta dy.$$

Using orthogonality and normality (with respect to y) one can rewrite the last expression as follows

$$\sum_{m=1}^k \left| \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) f_m(\xi) d\xi \right|^2$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,y) \sum_{n=1}^k f_m(y) \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\theta) f_n(\theta) d\theta dy = \lim_{T\to\infty} \frac{1}{T} \int_0^T KX dy,$$

where

$$K = K(x,y), \; X = \sum_{n=1}^k f_m(y) \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\theta) f_n(\theta) d\theta.$$

Using orthogonality again, we can write

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X^2 dy = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^k f_n(y) \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\theta) f_n(\theta) d\theta$$

$$\times \sum_{m=1}^{k} f_m(y) \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\theta) f_m(\eta) d\eta dy = \sum_{m=1}^{k} \left| \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) f_m(\xi) d\xi \right|^2.$$

Comparing both sides of last two equalities we get:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T KX dy = \lim_{T\to\infty} \frac{1}{T} \int_0^T X^2 dy.$$

Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T K^2 dy - \lim_{T\to\infty} \frac{1}{T} \int_0^T (K-X)^2 dy$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T (2KX - X^2) dy = \lim_{T\to\infty} \frac{1}{T} \int_0^T KX dy.$$

So,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T KX dy \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T K^2 dy,$$

because we omitted the non-positive addend. So, using the fact that the taken functions are solution of homogenous equation, we get:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \sum_{m=1}^{k} f_m(x) \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) f_m(\xi) d\xi \right|^2 dx$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \sum_{m=1}^{k} \frac{f_m(y) f_m(x)}{\lambda_0} \right|^2 dy \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T (K(x,y))^2 dx.$$

Using orthonormality, we find

$$\sum_{m=1}^{k} \left(\frac{f_m(x)}{\lambda_0} \right)^2 \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T (K(x,y))^2 dx.$$

Taking mean value with respect to x, and using orthonormality, we get, finally:

$$n \leq \lambda_0^2 \lim_{T\to\infty} \frac{1}{T} \int_0^T \lim_{T\to\infty} \frac{1}{T} \int_0^T (K(x,y))^2 dx dy.$$

Theorem 4.1 is proven.

Theorem 4.2. Characteristic functions related to various characteristic numbers are orthogonal.

Proof. Let functions $f_1(x)$, $f_2(x)$ be characteristic functions relative to characteristic numbers λ_1, λ_2, correspondingly. Then we have

$$f_1(x) f_2(x) = \lambda_1 \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) f_1(\xi) f_2(x) d\xi.$$

Taking mean value, we write:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f_1(x) f_2(x) dx = \lambda_1 \lim_{T\to\infty} \frac{1}{T} \int_0^T \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi) f_1(\xi) f_2(x) dx d\xi.$$

Analogically,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f_1(x)f_2(x)dx = \lambda_2 \lim_{T\to\infty}\frac{1}{T}\int_0^T \lim_{T\to\infty}\frac{1}{T}\int_0^T K(x,\xi)f_1(x)f_2(\xi)dxd\xi.$$

Subtracting, we get

$$\lambda_1(\lambda_1-\lambda_2)\lim_{T\to\infty}\frac{1}{T}\int_0^T f_1(\xi)f_2(\xi)d\xi = 0.$$

Since first two multipliers are distinct from zero, then we have established the demanded result. Theorem 4.2 is proven.

Theorem 4.3. If the kernel of the equation is symmetric then all characteristic numbers are real.

Proof. Consider the equation

$$f(x) = \lambda \lim_{T\to\infty}\frac{1}{T}\int_0^T K(x,\xi)f(\xi)d\xi,$$

with real symmetric kernel. Taking complex conjugate, we get

$$\overline{f(x)} = \overline{\lambda}\lim_{T\to\infty}\frac{1}{T}\int_0^T K(x,\xi)\overline{f(\xi)}d\xi;$$

so, the characteristic function relative to the characteristic number $\overline{\lambda}$ is complex conjugate to $f(x)$. Supposing that λ is not real, we see that $\lambda \neq \overline{\lambda}$. By Theorem 4.1,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \overline{f(x)}f(x)dx = \lim_{T\to\infty}\frac{1}{T}\int_0^T \lceil f(x)\rceil^2 dx = 0,$$

which is not true. The got contradiction completes the proof of Theorem 4.3.

As in the case of ordinary integral equations, in the case of limit integral equations with symmetric kernel, the equation has only real characteristic numbers. The set of characteristic functions is a linear subspace with finite dimension. Above we obtained an estimate for that dimension. The characteristic functions relative to different characteristic numbers are orthogonal. Since Fredholm functions are entire functions, then the set of characteristic numbers is a countable set. Therefore, the set of all characteristic functions is a countable orthogonal system of functions.

Let the sequence $f_1(x)$, $f_2(x)$,... be a complete system of orthogonal functions satisfying following homogenous limit integral equation

$$f(x) = \lambda \lim_{T\to\infty}\frac{1}{T}\int_0^T K(x,\xi)f(\xi)d\xi,$$

and $\lambda_1, \lambda_2, \ldots$ be sequence of all characteristic numbers. Suppose that the series

$$\sum_{m=1}^{\infty}\frac{f_m(y)f_m(x)}{\lambda_m}$$

uniformly converges in the product $R\times R$; here characteristic numbers are taken with their multiplicities.

Theorem 4.4. For the kernel of the equation (1.1) the relation below is satisfied:

$$K(x,y) = \sum_{m=1}^{\infty}\frac{f_m(y)f_m(x)}{\lambda_m}$$

Proof. Consider a new kernel

$$L(x,y)=K(x,y)-\sum_{m=1}^{\infty}\frac{f_m(y)f_m(x)}{\lambda_m}.$$

This is a symmetric kernel. If this function is not equal to zero, then it has non-zero characteristic number μ. Take any characteristic function

$$f(x)=\mu\lim_{T\to\infty}\frac{1}{T}\int_0^T L(x,\xi)f(\xi)d\xi.$$

Let us take the mean value

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f_m(y)f(y)dy$$
$$=\mu\lim_{T\to\infty}\frac{1}{T}\int_0^T\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{K(x,y)-\sum_{n=1}^{\infty}\frac{f_n(y)f_n(x)}{\lambda_n}\right\}f_m(y)f(y)dydx.$$

Performing term by term integration, we get the following expression in the right hand-side:

$$\frac{\mu}{\lambda_m}\lim_{T\to\infty}\frac{1}{T}\int_0^T f_m(\xi)f(\xi)d\xi-\frac{\mu}{\lambda_m}\lim_{T\to\infty}\frac{1}{T}\int_0^T f_m(\xi)f(\xi)d\xi=0.$$

Then, from previous relation it follows that the function $f(x)$ is orthogonal to every function from the set of characteristic functions. So,

$$f(x)=\mu\lim_{T\to\infty}\frac{1}{T}\int_0^T L(x,\xi)f(\xi)d\xi$$
$$=\mu\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{K(x,y)-\sum_{n=1}^{\infty}\frac{f_n(y)f_n(x)}{\lambda_n}\right\}f(\xi)d\xi$$
$$=\mu\lim_{T\to\infty}\frac{1}{T}\int_0^T K(x,\xi)f(\xi)d\xi.$$

These equalities show that μ is a characteristic number and the characteristic function $f(x)$ must be a linear combination of several number of characteristic functions, relative to this characteristic number:

$$f(x)=\sum_{j=1}^{S}c_jf_j(x).$$

The function $f(x)$ is orthogonal to all functions. Then, mean value for absolute value of this function is equal to zero, because:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\xi)f_j(x)d\xi=c_j=0,$$

for every j. Since the function $f(x)$ is almost periodic, then this function is identically equal to zero, which is a contradiction. Therefore, the kernel constructed above, that is, the function

$$L(x,y)=K(x,y)-\sum_{m=1}^{\infty}\frac{f_m(y)f_m(x)}{\lambda_m}$$

identically equals to zero. Proof of Theorem 4.4 is completed.

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