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Some embeddings into the total Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$

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Abstract. *On the $\mathbb{R}^d$ the Dunkl operators $\{D_{k,j}\}_{j=1}^d$ are the differential-difference operators associated with the reflection group $\mathbb{Z}_2^d$ on $\mathbb{R}^d$. We study some embeddings into the total Morrey space (D_k-total Morrey space) $L_{p,\lambda,\mu}(\mu_k)$, $0 \le \lambda, \mu < d + 2\gamma_k$ associated with the Dunkl operator on $\mathbb{R}^d$. These spaces generalize the Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$ (D_k-Morrey space) so that $L_{p,\lambda}(\mu_k) \equiv L_{p,\lambda,\lambda}(\mu_k)$ and the modified Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$ (modified D_k-Morrey space) so that $\widetilde{L}_{p,\lambda}(\mu_k) \equiv L_{p,\lambda,0}(\mu_k)$.*

Keywords. Dunkl operator, generalized translation operator, total D_k-Morrey space.

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1 Introduction

Dunkl operators are differential reflection operators associated with finite reflection groups which generalize the usual partial derivatives as well as the invariant differential operators of Riemannian symmetric spaces. They play an important role in harmonic analysis and the study of special functions of several variables. These operators are associated with the differential-difference Dunkl operators on $\mathbb{R}^d$. Rosler in [10] shows that the Dunkl kernel verify a product formula. This allows us to define the Dunkl translations τ_x, $x \in \mathbb{R}^d$.

Morrey spaces, introduced by C. B. Morrey [7], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In [3] Guliyev introduce a variant of Morrey spaces called total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, see also, [9].

On the $\mathbb{R}^d$ the Dunkl operators $\{D_{k,j}\}_{j=1}^d$ are the differential-difference operators associated with the reflection group $\mathbb{Z}_2^d$ on $\mathbb{R}^d$. In the present work, we give basic properties of the total Morrey spaces (D_k-total Morrey space) $L_{p,\lambda,\mu}(\mu_k)$, $0 \le \lambda, \mu < d + 2\gamma_k$ associated with the Dunkl operator on $\mathbb{R}^d$ and study some embeddings into the total Morrey space $L_{p,\lambda,\mu}(\mu_k)$. These spaces generalize the Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$ (D_k-Morrey space) so that $L_{p,\lambda}(\mu_k) \equiv L_{p,\lambda,\lambda}(\mu_k)$ and the modified Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$ (modified D_k-Morrey space) so that $\widetilde{L}_{p,\lambda}(\mu_k) \equiv L_{p,\lambda,0}(\mu_k)$.

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The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In section 3, we give some embeddings into the total D_k-Morrey spaces.

Finally, we make some conventions on notation. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle \cdot , \cdot \rangle$ and its associated norm $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathbb{R}^d$. For any $v \in \mathbb{R}^d \setminus \{0\}$ let σ_v be the reflection in the hyperplane $H_v \subset \mathbb{R}^d$ orthogonal to v:

$$\sigma_v(x) := x - \Big(\frac{2\langle x, v\rangle}{\|v\|^2}\Big)v, \ \ x \in \mathbb{R}^d.$$

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a *root system*, if $\sigma_v R = R$ for all $v \in R$. We assume that it is normalized by $\|v\|^2 = 2$ for all $v \in R$.

The finite group G generated by the reflections $\{\sigma_v\}_{v\in R}$ is called the *reflection group* (or the *Coxeter-Weyl group*) of the root system. Then, we fix a G-invariant function $k : R \to \mathbb{C}$ called the *multiplicity function of the root system* and we consider the family of commuting operators $D_{k,j}$ defined for any $f \in C^1(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$ by

$$D_{k,j} f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{v\in R_+} k_v \frac{f(x) - f(\sigma_v(x))}{\langle x, v\rangle} \langle v, e_j\rangle, \ \ 1 \le j \le d,$$

where $C^1(\mathbb{R}^d)$ denotes the set of all functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\{\frac{\partial f}{\partial x_j}\}_{j=1}^d$ are continuous on $\mathbb{R}^d$, $\{e_i\}_{i=1}^d$ are the standard unit vectors of $\mathbb{R}^d$ and R_+ is a positive subsystem. These operators, defined by Dunkl [1], are independent of the choice of the positive subsystem R_+ and are of fundamental importance in various areas of mathematics and mathematical physics.

Throughout this paper, we assume that $k_v \ge 0$ for all $v \in R$ and we denote by h_k the weight function on $\mathbb{R}^d$ given by

$$h_k(x) := \prod_{v\in R_+} |\langle x, v\rangle|^{k_v}, \quad x \in \mathbb{R}^d.$$

The function h_k is G-invariant and homogeneous of degree γ_k, where $\gamma_k := \sum_{v\in R_+} k_v$.

Closely related to them is the so-called intertwining operator V_κ (the subscript means that the operator depends on the parameters κ_i, except in the rank-one case where the subscript is then a single parameter). The *intertwining operator* V_κ is the unique linear isomorphism of $\oplus_{n\ge 0} P_n$ such that

$$V(P_n) = P_n, V_k(1) = 1, D_i V_k = V_k \frac{\partial}{\partial x_i} \text{ for any } i \in \{1, ..., d\}$$

with P_n being the subspace of homogeneous polynomials of degree n in d variables. The explicit formula of V_k is not known in general (see [11]). For the group $G := \mathbb{Z}_2^d$ and $h_k(x) := \prod_{i=1}^d |x_i|^{k_i}$ for all $x \in \mathbb{R}^d$, it is an integral transform

$$V_k f(x) := b_k \int\limits_{[-1,1]^d} f\big(x_1 t_1, \cdots , x_d t_d\big) \prod_{i=1}^d (1 + t_i)\Big(1 - t_i^2\Big)^{k_i - 1} dt, \ \ x \in \mathbb{R}^d. \qquad (2.1)$$

We denote by μ_k the measure on $\mathbb{R}^d$ given by $d\mu_k(x) := h_k(x)dx$, and we introduce the Mehta-type constant c_k, by

$$c_k^{-1} := \int_{\mathbb{R}^d} e^{-|x|^2/2} d\mu_k(x).$$

For $y \in \mathbb{R}^d$, the initial problem $D_{k,j}u(\cdot, y)(x) = y_j u(x, y)$, $j = 1, \dots, d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel (see e.g., [1,5]). This kernel has the Laplace-type representation [11]:

$$E_k(x, z) = \int_{\mathbb{R}^d} e^{<y,z>} d\Gamma_x(y), \ x \in \mathbb{R}^d, \ z \in \mathbb{C}^d, \tag{2.2}$$

where $< y, z >:= \sum_{i=1}^d y_i z_i$ and Γ_x is a probability measure on $\mathbb{R}^d$, such that

$$\text{supp}(\Gamma_x) \subset \{y \in \mathbb{R}^d : |y| \leq |x|\}.$$

This kernel possesses the following properties: for $x, y \in \mathbb{R}^d$, we have

$$E_k(x, y) = E_k(y, x), E_k(x, 0) = 1, \ E_k(-ix, y) = \overline{E_k(ix, y)}, \ |E_k(\pm ix, y)| \leq 1. \tag{2.3}$$

Let $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$ denote the ball in $\mathbb{R}^d$ that centered in $x \in \mathbb{R}^d$ and having radius $r > 0$, $B_r = B(0, r)$. Then having

$$\mu_k(B_r) = \int_{B_r} d\mu_k(x) = b_k \, r^{d+2\gamma_k}, \tag{2.4}$$

where

$$b_k = \left(\frac{a_k}{d + 2\gamma_k}\right) \text{ and } a_k := \left(\int_{S^{d-1}} h_k^2(x) \, d\sigma(x)\right)^{-1},$$

S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$.

We denote by $L_p(\mu_k) \equiv L_p(\mathbb{R}^d, d\mu_k)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L_p(\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(x)|^p d\mu_k(x)\Big)^{1/p} < \infty, \ 1 \leq p < \infty,$$
$$\|f\|_{L_\infty(\mathbb{R}^d)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < \infty.$$

For $f \in L_1(\mu_k)$ the Dunkl transform is defined (see [2]) by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y) d\mu_k(x), \ x \in \mathbb{R}^d.$$

The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L_2(\mu_k)$ onto itself. In particular,

$$\|\mathcal{F}_k f\|_{L_2(\mu_k)} = \|f\|_{L_2(\mu_k)}. \tag{2.5}$$

The Dunkl transform allows us to define a generalized translation operator on $L_2(\mu_k)$ by setting $\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y)\mathcal{F}_k(f)(y)$, $y \in \mathbb{R}^d$. It is the definition of Thangavelu and Xu given in [13]. It plays the role of the ordinary translation $\tau_x f(\cdot) = f(x + \cdot)$ in $\mathbb{R}^d$, since the Euclidean Fourier transform satisfies $\mathcal{F}(\tau_x f)(y) = e^{ixy}\mathcal{F}(f)(y)$.

Note that from (2.3) and (2.5), the definition makes sense and

$$\|\tau_x f\|_{L_2(\mu_k)} \le \|f\|_{L_2(\mu_k)}.$$

Rösler [12] introduced the Dunkl translation operators for radial functions. If f are radial functions, $f(x) = F(|x|)$, then

$$\tau_x f(y) = \int_{\mathbb{R}^d} F(\sqrt{|z|^2 + |y|^2 + 2 < y, z >})\, d\Gamma_x(y),\ x \in \mathbb{R}^d,$$

where Γ_x is the representing measure given by (2.2).

This formula allows us to establish the following result, see [13, 14].

For all $1 \le p \le 2$ and for all $x \in \mathbb{R}^d$, the Dunkl translation $\tau_x : L_p^{\mathrm{rad}}(\mu_k) \to L_p(\mu_k)$ is a bounded operator, and for $f \in L_p^{\mathrm{rad}}(\mu_k)$,

$$\|\tau_x f\|_{L_p(\mu_k)} \le \|f\|_{L_p(\mu_k)}. \tag{2.6}$$

If $G = \mathbb{Z}_2^d$, then for all $1 \le p \le \infty$ and for all $x \in \mathbb{R}^d$, the Dunkl translation $\tau_x : L_p(\mu_k) \to L_p(\mu_k)$ is a bounded operator, and for $f \in L_p(\mu_k)$,

$$\|\tau_x f\|_{L_p(\mu_k)} \le C_0 \|f\|_{L_p(\mu_k)}. \tag{2.7}$$

In the analysis of this generalized translation a particular role is played by the space (cf. [11–13, 15])

$$A_k(\mathbb{R}^d) = \{f \in L_1(\mu_k) : \mathcal{F}_k f \in L_1(\mu_k)\}.$$

The operator τ_x satisfies the following properties:

Proposition 2.1 *Assume that $f \in A_k(\mathbb{R}^d)$ and $g \in L_1(\mu_k)$, then*

$$(i)\ \int_{\mathbb{R}^d} \tau_x f(y)\, g(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y)\, \tau_{-x} g(y) d\mu_k(y);$$
$$(i)\ \tau_x f(y) = \tau_{-y} f(-x).$$

3 Some embeddings into the total D_k-Morrey spaces

Definition 3.1 *[6] Let $1 \le p < \infty$, $0 \le \lambda \le d + 2\gamma_k$ and $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda}(\mu_k)$ Morrey space ($\equiv D_k$-Morrey space), by $\widetilde{L}_{p,\lambda}(\mu_k)$ the modified Morrey space ($\equiv$ modified D_k-Morrey space), associated with the Dunkl operator [6, 8] and by $L_{p,\lambda,\mu}(\mu_k)$ Morrey space ($\equiv$ total D_k-Morrey space) as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^d$, with the finite norms*

$$\|f\|_{L_{p,\lambda}(\mu_k)} := \sup_{x \in \mathbb{R}^d,\, t>0} \left(t^{-\lambda} \int_{B_t} \tau_x |f|^p(y)\, d\mu_k(y) \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)} := \sup_{x \in \mathbb{R}^d,\, t>0} \left([t]_1^{-\lambda} \int_{B_t} \tau_x |f|^p(y)\, d\mu_k(y) \right)^{1/p},$$

$$\|f\|_{L_{p,\lambda,\mu}(\mu_k)} := \sup_{x \in \mathbb{R}^d,\, t>0} \left([t]_1^{-\lambda}\, [1/t]_1^{\mu} \int_{B_t} \tau_x |f|^p(y)\, d\mu_k(y) \right)^{1/p},$$

respectively.

If $\min\{\lambda,\mu\}<0$ or $\max\{\lambda,\mu\}>d+2\gamma_k$, then $L_{p,\lambda,\mu}(\mu_k)=\Theta(\mathbb{R}^d)$, where $\Theta(\mathbb{R}^d)$ is the set of all functions equivalent to 0 on $\mathbb{R}^d$.

Note that

$$L_{p,0,0}(\mu_k)=\widetilde{L}_{p,0}(\mu_k)=L_{p,0}(\mu_k)=L_p(\mu_k),$$
$$L_{p,\lambda,\lambda}(\mu_k)=L_{p,\lambda}(\mu_k),\ L_{p,\lambda,0}(\mu_k)=\widetilde{L}_{p,\lambda}(\mu_k)$$

$$L_{p,\lambda,\mu}(\mu_k)\subset_{\succ} L_{p,\lambda}(\mu_k) \quad \text{and} \quad \|f\|_{L_{p,\lambda}(\mu_k)}\le \|f\|_{L_{p,\lambda,\mu}(\mu_k)}, \tag{3.1}$$

$$L_{p,\lambda,\mu}(\mu_k)\subset_{\succ} L_{p,\mu}(\mu_k) \quad \text{and} \quad \|f\|_{L_{p,\mu}(\mu_k)}\le \|f\|_{L_{p,\lambda,\mu}(\mu_k)}. \tag{3.2}$$

Definition 3.2 *[4] Let $1\le p<\infty$, $0\le\lambda\le d+2\gamma_k$. We denote by $WL_{p,\lambda}(\mu_k)$ weak D_k-Morrey space, by $W\widetilde{L}_{p,\lambda}(\mu_k)$ the modified weak D_k-Morrey space [6,8] and by $WL_{p,\lambda,\mu}(\mu_k)$ weak total D_k-Morrey space as the set of locally integrable functions $f(x)$, $x\in\mathbb{R}^d$ with finite norms*

$$\|f\|_{WL_{p,\lambda}(\mu_k)}:=\sup_{r>0} r \sup_{x\in\mathbb{R}^d,\,t>0}\Big(t^{-\lambda}\mu_k\left\{y\in B_t:\ \tau_x|f|(y)>r\right\}\Big)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mu_k)}:=\sup_{r>0} r \sup_{x\in\mathbb{R}^d,\,t>0}\Big([t]_1^{-\lambda}\mu_k\left\{y\in B_t:\ \tau_x|f|(y)>r\right\}\Big)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mu_k)}:=\sup_{r>0} r \sup_{x\in\mathbb{R}^d,\,t>0}\Big([t]_1^{-\lambda}\,[1/t]_1^{\mu}\,\mu_k\left\{y\in B_t:\ \tau_x|f|(y)>r\right\}\Big)^{1/p},$$

respectively.

We note that

$$L_{p,\lambda,\mu}(\mu_k)\subset WL_{p,\lambda,\mu}(\mu_k) \ \text{ and } \ \|f\|_{WL_{p,\lambda,\mu}(\mu_k)}\le\|f\|_{L_{p,\lambda,\mu}(\mu_k)}.$$

Lemma 3.1 *If $0<p<\infty$, $0\le\lambda\le d+2\gamma_k$ and $0\le\mu\le d+2\gamma_k$, then*

$$L_{p,d+2\gamma_k,\mu}(\mu_k)\subset_{\succ} L_\infty(\mathbb{R}^d)\subset_{\succ} L_{p,\lambda,|d|}(\mu_k)$$

and

$$\|f\|_{L_{p,\lambda,d+2\gamma_k}(\mu_k)}\le b_k^{1/p}\|f\|_{L_\infty(\mathbb{R}^d)}\le\|f\|_{L_{p,d+2\gamma_k,\mu}(\mu_k)}.$$

Proof. Let $f\in L_\infty(\mathbb{R}^d)$. Then for all $x\in\mathbb{R}^d$ and $0<t\le1$

$$\Big(t^{-\lambda}\int_{B_t}\tau_x|f(y)|^p\,d\mu_k(y)\Big)^{1/p}\le b_k^{1/p}\,\|f\|_{L_\infty},\quad 0\le\lambda\le d+2\gamma_k$$

and for all $x\in\mathbb{R}^d$ and $t\ge1$

$$\Big(t^{-d-2\gamma_k}\int_{B_t}\tau_x|f(y)|^p\,d\mu_k(y)\Big)^{1/p}\le b_k^{1/p}\,\|f\|_{L_\infty}.$$

Therefore $f\in L_{p,\lambda,d+2\gamma_k}(\mu_k)$ and

$$\|f\|_{L_{p,\lambda,d+2\gamma_k}(\mu_k)}\le b_k^{1/p}\|f\|_{L_\infty}.$$

Let $f\in L_{p,d+2\gamma_k,\mu}(\mu_k)$. By the Lebesgue's differentiation theorem we have (see [8, Section 2, Corolarry 2.2])

$$\lim_{t\to0}\mu_k(B_t)^{-1}\int_{B_t}\tau_x|f(y)|^p\,d\mu_k(y)=|f(x)|^p\quad a.e.\ x\in\mathbb{R}^d.$$

Then

$$|f(x)| = \left(\lim_{t\to 0} \mu_k(B_t)^{-1} \int_{B_t} \tau_x |f(y)|^p \, d\mu_k(y)\right)^{1/p}$$
$$\leq b_k^{1/p} \left(t^{-d-2\gamma_k} \int_{B_t} \tau_x |f(y)|^p \, d\mu_k(y)\right)^{1/p} \leq b_k^{1/p} \, \|f\|_{L_{p,d+2\gamma_k,\mu}(\mu_k)}.$$

Therefore $f \in L_\infty(\mathbb{R}^d)$ and

$$\|f\|_{L_\infty(\mathbb{R}^d)} \leq b_k^{1/p} \, \|f\|_{L_{p,d+2\gamma_k,\mu}(\mu_k)}.$$

Corollary 3.1 *If* $0 < p < \infty$*, then*

$$L_{p,d+2\gamma_k}(\mu_k) = \widetilde{L}_{p,d+2\gamma_k}(\mu_k) = L_\infty(\mathbb{R}^d)$$

and

$$\|f\|_{L_{p,d+2\gamma_k}(\mu_k)} = \|f\|_{\widetilde{L}_{p,d+2\gamma_k}(\mu_k)} = b_k^{1/p} \|f\|_{L_\infty}.$$

Lemma 3.2 *Let* $1 \leq p < \infty$, $0 \leq \lambda \leq d + 2\gamma_k$ *and* $0 \leq \mu \leq d + 2\gamma_k$. *Then*

$$L_{p,\lambda,\mu}(\mu_k) = L_{p,\lambda}(\mu_k) \cap L_{p,\mu}(\mu_k)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mu_k)} = \max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_{p,\mu}(\mu_k)}\right\}.$$

Proof. Let $f \in L_{p,\lambda,\mu}(\mu_k)$. Then by (3.1) and (3.2) we have

$$L_{p,\lambda,\mu}(\mu_k) \subset\!\succ L_{p,\lambda}(\mu_k) \cap L_{p,\mu}(\mu_k)$$

and

$$\max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_{p,\mu}(\mu_k)}\right\} \leq \|f\|_{L_{p,\lambda,\mu}(\mu_k)}.$$

Let $f \in L_{p,\lambda}(\mu_k) \cap L_{p,\mu}(\mu_k)$. Then by Proposition 2.1 we have

$$\|f\|_{L_{p,\lambda,\mu}(\mu_k)} = \sup_{x\in\mathbb{R}^d, t>0} \left([t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{B_t} \tau_x |f|^p(y) \, d\mu_k(y)\right)^{1/p}$$
$$= \max\left\{ \sup_{x\in\mathbb{R}^d, 0<t\leq 1} \left(t^{-\lambda} \int_{B_t} \tau_x |f|^p(y) \, d\mu_k(y)\right)^{1/p},\right.$$
$$\left.\sup_{x\in\mathbb{R}^d, t>1} \left(t^{-\mu} \int_{B_t} \tau_x |f|^p(y) \, d\mu_k(y)\right)^{1/p}\right\}$$
$$\leq \max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_{p,\mu}(\mu_k)}\right\}.$$

Therefore, $f \in L_{p,\lambda,\mu}(\mu_k)$ and the embedding $L_{p,\lambda}(\mu_k) \cap L_{p,\mu}(\mu_k) \subset\!\succ L_{p,\lambda,\mu}(\mu_k)$ is valid.

Thus $L_{p,\lambda,\mu}(\mu_k) = L_{p,\lambda}(\mu_k) \cap L_{p,\mu}(\mu_k)$ and

$$\|f\|_{L_{p,\lambda,\mu}(\mu_k)} = \max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_{p,\mu}(\mu_k)}\right\}.$$

Corollary 3.2 *[8, Lemma 3.2] If $0 < p < \infty$, $0 \le \lambda \le d + 2\gamma_k$, then*

$$\widetilde{L}_{p,\lambda}(\mu_k) = L_{p,\lambda}(\mu_k) \cap L_p(\mu_k)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)} = \max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_p(\mu_k)}\right\}.$$

From Lemmas 3.1 and 3.2 for $1 \le p < \infty$ we have

$$\widetilde{L}_{p,d+2\gamma_k}(\mu_k) = L_\infty(\mathbb{R}^d) \cap L_p(\mu_k). \tag{3.3}$$

Lemma 3.3 *Let $0 < p < \infty$, $0 \le \lambda \le d + 2\gamma_k$ and $0 \le \mu \le d + 2\gamma_k$. Then*

$$WL_{p,\lambda,\mu}(\mu_k) = WL_{p,\lambda}(\mu_k) \cap WL_{p,\mu}(\mu_k)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mu_k)} = \max\left\{\|f\|_{WL_{p,\lambda}(\mu_k)}, \|f\|_{WL_{p,\mu}(\mu_k)}\right\}.$$

Corollary 3.3 *If $0 < p < \infty$, $0 \le \lambda \le d + 2\gamma_k$, then*

$$W\widetilde{L}_{p,\lambda}(\mu_k) = WL_{p,\lambda}(\mu_k) \cap WL_p(\mu_k)$$

and

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mu_k)} = \max\left\{\|f\|_{WL_{p,\lambda}(\mu_k)}, \|f\|_{WL_p(\mu_k)}\right\}.$$

Remark 3.1 If $0 < p < \infty$, and $\min\{\lambda, \mu\} < 0$ or $\max\{\lambda, \mu\} > |d|$, then

$$L_{p,\lambda,\mu}(\mu_k) = WL_{p,\lambda,\mu}(\mu_k) = \Theta(\mathbb{R}^d).$$

Lemma 3.4 *If $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le d + 2\gamma_k$ and $0 \le \mu_1 \le \mu_2 \le d + 2\gamma_k$, then*

$$L_{p,\lambda_1,\mu_1}(\mu_k) \subset\!\!\succ L_{p,\lambda_2,\mu_2}(\mu_k)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mu_k)} \le \|f\|_{L_{p,\lambda_1,\mu_1}(\mu_k)}.$$

Proof. Let $f \in L_{p,\lambda,\mu}(\mu_k)$, $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le d+2\gamma_k$, $0 \le \mu_1 \le \mu_2 \le d+2\gamma_k$. Then

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mu_k)} = \max\Bigg\{\sup_{x\in\mathbb{R}^d,\, 0<t\le 1}\left(t^{\lambda_1-\lambda_2}t^{-\lambda_1}\int_{B_t}\tau_x|f|^p(y)\,d\mu_k(y)\right)^{1/p},$$
$$\sup_{x\in\mathbb{R}^d,\, t\ge 1}\left(t^{\mu_1-\mu_2}t^{-\mu_1}\int_{B_t}\tau_x|f|^p(y)\,d\mu_k(y)\right)^{1/p}\Bigg\} \le \|f\|_{L^d_{p,\lambda_1,\mu_1}}.$$

On the total D_k-Morrey spaces the following embedding is valid.

Lemma 3.5 *Let $0 \le \lambda < d + 2\gamma_k$, $0 \le \mu < d + 2\gamma_k$, $0 \le \alpha < d + 2\gamma_k - \lambda$ and $0 \le \beta < d + 2\gamma_k - \mu$. Then for $\frac{d+2\gamma_k-\lambda}{\alpha} \le p \le \frac{d+2\gamma_k-\mu}{\beta}$*

$$L_{p,\lambda,\mu}(\mu_k) \subset\!\!\succ L_{1,d+2\gamma_k-\alpha,d+2\gamma_k-\beta}(\mu_k)$$

and for $f \in L_{p,\lambda,\mu}(\mu_k)$ the following inequality

$$\|f\|_{L_{1,d+2\gamma_k-\alpha,d+2\gamma_k-\beta}(\mu_k)} \le b_k^{\frac{1}{p'}}\,\|f\|_{L_{p,\lambda,\mu}(\mu_k)}$$

is valid.

Proof. Let $0 < \lambda < d+2\gamma_k$, $0 < \alpha < d+2\gamma_k-\lambda$, $0 < \beta < d+2\gamma_k-\mu$, $f \in L_{p,\lambda,\mu}(\mu_k)$ and $\frac{d+2\gamma_k-\lambda}{\beta} \le p \le \frac{d+2\gamma_k}{\beta}$. By the Hölder's inequality we have

$$\|f\|_{L_{1,d+2\gamma_k-\alpha,d+2\gamma_k-\beta}(\mu_k)} = \sup_{x\in\mathbb{R}^d,\, t>0} [t]_1^{\alpha-d-2\gamma_k}\,[1/t]_1^{-\beta+d+2\gamma_k} \int_{B_t} \tau_x|f|(y)\,d\mu_k(y)$$
$$\le b_k^{\frac{1}{p'}} \sup_{x\in\mathbb{R}^d,\, t>0} \Big([t]_1\, t^{-1}\Big)^{-(d+2\gamma_k)/p'} [t]_1^{\alpha-\frac{d+2\gamma_k-\lambda}{p}} \Big([t]_1^{-\lambda}\,[1/t]_1^{\mu} \int_{B_t} \tau_x|f|^p(y)\,d\mu_k(y)\Big)^{1/p}$$
$$\le b_k^{\frac{1}{p'}}\, \|f\|_{L_{p,\lambda,\mu}(\mu_k)} \sup_{t>0} \big([t]_1\, t^{-1}\big)^{\frac{d+2\gamma_k-\mu}{p}-\beta} [t]_1^{\alpha-\frac{d+2\gamma_k-\lambda}{p}}.$$

Note that

$$\sup_{t>0} \big([t]_1\, t^{-1}\big)^{\frac{d+2\gamma_k-\mu}{p}-\beta} [t]_1^{\alpha-\frac{d+2\gamma_k-\lambda}{p}} = \max\{\sup_{0<t\le 1} t^{\alpha-\frac{d+2\gamma_k-\lambda}{p}}, \sup_{t>1} t^{\beta-\frac{d+2\gamma_k-\mu}{p}}\} < \infty$$

$$\text{if and only if} \quad \frac{d+2\gamma_k-\lambda}{\alpha} \le p \le \frac{d+2\gamma_k-\mu}{\beta}.$$

Therefore $f \in L_{1,d+2\gamma_k-\alpha,d+2\gamma_k-\beta}(\mu_k)$ and

$$\|f\|_{L_{1,d+2\gamma_k-\alpha,d+2\gamma_k-\beta}(\mu_k)} \le b_k^{\frac{1}{p'}}\, \|f\|_{L_{p,\lambda,\mu}(\mu_k)}.$$

Corollary 3.4 *[8, Lemma 3.4] Let* $0 \le \lambda < d+2\gamma_k$ *and* $0 \le \beta < d+2\gamma_k-\lambda$. *Then for* $p = \frac{d+2\gamma_k-\lambda}{\beta}$

$$L_{p,\lambda}(\mu_k) \subset\!\!\succ L_{1,d+2\gamma_k-\beta}(\mu_k)$$

and for $f \in L_{p,\lambda}(\mu_k)$ *the following inequality*

$$\|f\|_{L_{1,d+2\gamma_k-\beta}(\mu_k)} \le b_k^{\frac{1}{p'}}\, \|f\|_{L_{p,\lambda}(\mu_k)}$$

is valid.

Corollary 3.5 *[8, Lemma 3.5] Let* $G = \mathbb{Z}_2^d$, $0 \le \lambda < d+2\gamma_k$ *and* $0 \le \beta < d+2\gamma_k-\lambda$. *Then for* $\frac{d+2\gamma_k-\lambda}{\beta} \le p \le \frac{d+2\gamma_k-\mu}{\beta}$

$$\widetilde{L}_{p,\lambda}(\mu_k) \subset\!\!\succ \widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k)$$

and for $f \in \widetilde{L}_{p,\lambda}(\mu_k)$ *the following inequality*

$$\|f\|_{\widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k)} \le b_k^{\frac{1}{p'}}\, \|f\|_{L_{p,\lambda}(\mu_k)}$$

is valid.

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