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Existence and uniqueness of a generalized solution to a boundary value problem for a multidimensional parabolic equation with an integral boundary condition

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Abstract. *In this paper, we consider a nonlocal boundary value problem for a multidimensional linear parabolic equation containing the integral of the desired solution. The coefficients of the considered parabolic equation are discontinious functions. The integral boundary condition represents the relationship that binds the value of the derivative of the desired solution with respect to the spatial variables at the boundary points and the value of the solution in the internal area. By using Galerkin method the existence of the generalized solution from* $V_2^{1,0}(Q_T)$ *is proved. The energy inequality is obtained and the uniqueness of the generalized solution is proved. It is proved that for many strong assumptions about the data of the problem, the generalized solution from* $V_2^{1,0}(Q_T)$ *belongs to space* $W_2^{1,1}(Q_T)$.

Keywords. parabolic equation · integral boundary condition · generalized solution.

Mathematics Subject Classification (2010): 35K20, 49J20

1 Introduction

In the mathematical modeling of numerous practice processes, boundary value problems with nonlocal boundary conditions for partial differential equations arise. Nonlocal boundary conditions represent relations connecting the values of the desired solution at the boundary and interior points of the domain. Integral conditions hold a special place among nonlocal boundary conditions. Nonlocal boundary value problems for parabolic equations with integral conditions were studied in [1-4, 6, 9] and others. Note that such boundary value problems in the classes of generalized solutions are studied least of all.

2 Statement of the problem

Suppose that, Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary $S = S' \bigcup S''$, $Q_T = \Omega \times (0,T)$ is a cylinder, $T > 0$ is a given number, $S_T = S \times (0,T)$ is the lateral

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surface of the cylinder Q_T, $S'_T = S' \times (0,T)$, $S''_T = S'' \times (0,T)$. The designations of functional spaces and their norms used in this work correspond to [5].

Consider the linear parabolic equation in the cylinder Q_T

$$u_t - \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\, u_{x_j}\right)_{x_i} + a(x,t)\, u = f(x,t), \quad (x,t) \in Q_T. \tag{2.1}$$

For equation (2.1), we pose the following boundary value problem: it is required to find a solution subject to the initial condition in the domain Q_T

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{2.2}$$

with boundary condition

$$u|_{S'_T} = 0, \tag{2.3}$$

and the nonlocal condition

$$\left.\frac{\partial u}{\partial N}\right|_{(x,t)\in S''_T} = \int_\Omega K(x,y,t)\, u(y,t)\, dy|_{(x,t)\in S''_T}, \tag{2.4}$$

where $\frac{\partial u}{\partial N} = \sum_{i,j=1}^{n} a_{ij}(x,t)\, u_{x_j} \cos(\nu, x_i)$ is the derivative corresponding to the conormal, ν is the outer normal to the boundary S'', $a_{ij}(x,t)$, $i,j = \overline{1,n}$, $a(x,t)$, $f(x,t)$, $\varphi(x)$, $K(x,y,t)$ are the given measurable functions satisfying the conditions

$$a_{ij}(x,t) = a_{ji}(x,t), \quad i,j = \overline{1,n},$$

$$\nu\xi^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\, \xi_i \xi_j \le \mu\xi^2, \quad \xi = (\xi_1, \dots, \xi_n), \quad \xi^2 = \sum_{i=1}^{n} \xi_i^2,$$

$$|a(x,t)| \le \mu \quad al.e.\ Q_T,$$

$$|K(x,y,t)| \le \mu_1 \quad al.e. \quad S'' \times \Omega \times (0,T),\ \nu,\ \mu,\ \mu_1 = const > 0, \tag{2.5}$$

$$\varphi \in L_2(\Omega), \quad f \in L_{2,1}(Q_T). \tag{2.6}$$

We define a generalized solution $u = u(x,t)$ for the problem (2.1)-(2.4) from $V_2^{1,0}(Q_T)$ as an element of $V_{2,0}^{1,0}(Q_T) = \left\{u : u \in V_2^{1,0}(Q_T), \quad u|_{S'_T} = 0\right\}$ satisfying the integral identity

$$\int_{Q_T} \left(-u\eta_t + \sum_{i,j=1}^{n} a_{ij}(x,t)\, u_{x_j} \eta_{x_i} + a(x,t) u \eta\right) dxdt$$
$$- \int_{S''_T} \left[\int_\Omega K(s,y,t)\, u(y,t)\, dy\right] \eta(s,t)\, dsdt$$
$$= \int_\Omega \varphi(x)\, \eta(x,0)\, dx + \int_{Q_T} f(x,t)\, \eta dxdt \tag{2.7}$$

for any function $\eta = \eta(x,t) \in \hat{W}_{2,0}^{1,1}(Q_T) = \Big\{\eta : \eta \in W_2^{1,1}(Q_T), \eta\Big|_{S'_T} = 0,$

$\eta(x,T) = 0,\ x \in \Omega\ \}$.

3 Existence and uniqueness of a generalized solution for the problem (2.1) - (2.4)

Theorem 3.1 *Let the conditions (2.5), (2.6) be satisfied. Then problem (2.1) - (2.4) is uniquely solvable in the class* $V_2^{1,0}(Q_T)$ *and satisfies the estimate*

$$|u|_{Q_T} \equiv \|u\|_{V_2^{1,0}(Q_T)} \leq M_1 \left(\|\varphi\|_{2,\Omega} + 2\|f\|_{2,1,Q_T} \right), \tag{3.1}$$

here constant M_1 *does not depend on* φ *and* f.

Proof. For the proof, we use the Galerkin method. Let $\{\psi_m(x)\}$ be some fundamental system of functions from $W_{2,0}^1(\Omega) = \left\{\psi : \psi = \psi(x) \in W_2^1(\Omega),\ \psi(x) = 0,\ x \in S'\right\}$ and orthonormal in $L_2(\Omega)$. Approximate solutions $u^N(x,t)$ of problem (2.1) - (2.4) will be sought in the form

$$u^N(x,t) = \sum_{m=1}^{N} c_m^N(t)\,\psi_m(x),$$

here $c_m^N(t) = \left(u^N, \psi_m\right)_{2,\Omega}$ are to be determined from the conditions

$$\int_\Omega u_t^N(x,t), \psi_m(x)\,dx + \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,t)\,u_{x_j}^N(x,t)\,\psi_{mx_i}(x)\,dx$$

$$+\int_\Omega a(x,t)\,u^N(x,t)\,\psi_m(x)\,dx - \int_{S''}\left[\int_\Omega K(s,y,t)\,u^N(y,t)\,dy\right]\psi_m(s)\,ds$$

$$= \int_\Omega f(x,t)\,\psi_m(x)\,dx, \quad m = \overline{1,N}, \tag{3.2}$$

$$c_m^N(0) = \int_\Omega \varphi(x)\,\psi_m(x)\,dx, \quad m = \overline{1,N}. \tag{3.3}$$

Conditions (3.2) represent a system of linear ordinary differential equations in the form

$$\frac{dc_m^N(t)}{dt} + \sum_{k=1}^{N} A_{mk}(t)\,c_k^N(t) + F_m(t) = 0, \quad m = \overline{1,N}, \tag{3.4}$$

here

$$A_{mk}(t) = \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(x,t)\,\psi_{kx_j}(x)\,\psi_{mx_i}(x) + a(x,t)\,\psi_k(x)\,\psi_m(x)\right] dx$$

$$-\int_{S''}\left[\int_\Omega K(s,y,t)\,\psi_k(y)\,dy\right]\psi_m(s)\,ds,\ m,k = \overline{1,N},$$

$$F_m(t) = -\int_\Omega f(x,t)\,\psi_m(x)\,dx, \quad m = \overline{1,N}.$$

It follows from conditions (2.5), (2.6) that the coefficients $A_{mk}(t)$ for $c_k^N(t)$ are bounded with respect to t, and the free terms are summable on $(0,T)$ functions. Then, by a well-known theorem from [8, p. 27], we conclude that systems (3.4) with initial conditions (3.3) have a unique absolutely continuous solution on $[0,T]$. Consequently, the functions $u^N(x,t)$ are uniquely determined for any N.

Let us show that the sequence $\{u^N\}$ satisfies the estimate

$$\left|u^N\right|_{Q_T} \leq M_2, \quad N = 1, 2, \ldots, \tag{3.5}$$

where the constant $M_2 > 0$ depends only on the input data and does not depend on N.

We multiply each of the equations (3.2) by its own $c_m^N(t)$, add the resulting equations over m from 1 to N and integrate the result over t from 0 to $t \leq T$. Then we obtain the equality

$$\frac{1}{2}\left\|u^N(x,t)\right\|_{2,\Omega}^2 + \int_{Q_t}\left[\sum_{i,j=1}^n a_{ij}u_{x_j}^N u_{x_i}^N + a\left(u^N\right)^2\right]dxdt$$
$$= \frac{1}{2}\left\|u^N(x,0)\right\|_{2,\Omega}^2 + \int_{Q_t} f u^N dxdt$$
$$+ \int_{S_t''}\left[\int_\Omega K(s,y,t)\, u^N(y,t)\, dy\right] u^N(s,t)\, dsdt, \tag{3.6}$$

here $Q_t = \Omega \times (0,t)\,, \quad S_t'' = S'' \times (0,t)$.

Note that any function $u \in W_2^{1,0}(Q_T)$ satisfies the inequality [5, p. 77]

$$\int_{S''} u^2(s,t)\, ds \leq \int_\Omega \left[\varepsilon u_x^2(x,t) + c_\varepsilon u^2(x,t)\right] dx \tag{3.7}$$

for almost all $t \in (0,T)$, here $\varepsilon > 0$ is any number, $c_\varepsilon = c\left(\frac{c}{4\varepsilon} + 1\right),\ c > 0$ is the constant in inequality (6.23) from [5, p. 77].

Then using conditions (2.5), (2.6), an obvious inequality $ab \leq \left(a^2 + b^2\right)/2$ and (3.7) for $\varepsilon = \nu\,|\Omega|^{-1}$ from (3.6) we obtain the inequalities

$$\frac{1}{2}\left\|u^N(x,t)\right\|_{2,\Omega}^2 + \nu\left\|u_x^N\right\|_{2,Q_t}^2 \leq \frac{1}{2}\left\|u^N(x,0)\right\|_{2,\Omega}^2 + \frac{\nu}{2}\left\|u_x^N\right\|_{2,Q_t}^2$$
$$+\frac{1}{2}\left(\mu_1^2\,|S''| + c_\varepsilon\,|\Omega| + 2\mu\right)\left\|u^N\right\|_{2,Q_t}^2 + \|f\|_{2,1,Q_t}\max_{0\leq\tau\leq t}\left\|u^N(x,\tau)\right\|_{2,\Omega}, \tag{3.8}$$

here $|\Omega| = mes\Omega, \quad |S''| = mesS''$.

We use the notation $y^N(t) = \max\limits_{0\leq\tau\leq t}\left\|u^N(x,\tau)\right\|_{2,\Omega}$. It is obvious that

$$\left\|u^N\right\|_{2,Q_t}^2 \leq t\left(y^N(t)\right)^2, \left\|u^N(x,0)\right\|_{2,\Omega} \leq \|\varphi\|_{2,\Omega}\,,$$
$$\left\|u^N(x,0)\right\|_{2,\Omega}^2 \leq y^N(t)\left\|u^N(x,0)\right\|_{2,\Omega}. \tag{3.9}$$

Multiplying both sides of inequality (3.8) by 2 and taking into (3.9), we have

$$\left\|u^N(x,t)\right\|_{2,\Omega}^2 + \nu\left\|u_x^N\right\|_{2,Q_t}^2$$
$$\leq y^N(t)\,\|\varphi\|_{2,\Omega} + dt\left(y^N(t)\right)^2 + 2\,\|f\|_{2,1,Q_t}\, y^N(t)\,, \tag{3.10}$$

here $d = \mu_1^2\,|S''| + c_\varepsilon\,|\Omega| + 2\mu$.

Then from (3.10), as was proved in [5, p. 167], the inequality follows

$$\left|u^N\right|_{Q_t} \leq M_3\left[\|\varphi\|_{2,\Omega} + 2\,\|f\|_{2,1,Q_T}\right], \tag{3.11}$$

in which the constant $M_3 > 0$ does not depend on N, φ and f. Therefore, estimate (3.5) is valid.

By virtue of (3.5), a subsequence $\left\{u^{N_m}\right\}$ can be distinguished from the sequence $\left\{u^N\right\}$ which converges to some function $u \in W_{2,0}^{1,0}(Q_T)$ weakly in $L_2(Q_T)$ together with the derivatives $\left\{u_x^{N_m}\right\}$ and converges to u weakly in $L_2(\Omega)$ uniformly with respect to $t \in [0,T]$. Without loss of generality, we will assume that the whole sequence $\left\{u^N\right\}$ converges to u this way. By virtue of the well-known property of weak convergence, inequality (3.11) remains for the limit function u and, therefore, the function u satisfies estimate (3.1) and it is an element of $V_{2,0}^{1,0}(Q_T)$. Let us show that the limit function $u = u(x,t)$ satisfies identity (2.7), that is, it is a generalized solution of the problem (2.1) - (2.4). For this, we take arbitrary absolutely continuous functions $d_m(t)\,,\ m = \overline{1,N}$ with $d'_m(t)$ from $L_2(0,T)$ and equal to zero at $t = T$. We multiply each equation (3.2) for $u^N = u^{N_m}$ by its function $d_m(t)$, add the obtained equalities over all m from 0 to N and integrate the result over t from 1 to T. This gives the identity

$$\int_{Q_T}\left(-u^{N_m}\Phi_t^N + \sum_{i,j=1}^{n} a_{ij}(x,t)\, u_{x_j}^N \Phi_{x_i}^N + a(x,t)\, u^{N_m}\Phi^N\right) dxdt$$
$$-\int_{S''_T}\left(\int_{\Omega} K(s,y,t)\, u^{N_m}(y,t)dy\right)\Phi^N(s,t)\, dsdt$$
$$= \int_{\Omega} u^{N_m}(x,0)\,\Phi^N(x,0)\, dx + \int_{Q_T} f\Phi^N dxdt, \tag{3.12}$$

where $\Phi^N(x,t) = \sum\limits_{m=1}^{N} d_m(t)\,\psi_m(x)$.

Passing to the limit along the chosen subsequence $\left\{u^{N_m}\right\}$ in equality (3.12), we obtain the integral identity

$$\int_{Q_T}\left(-u\Phi_t^N + \sum_{i,j=1}^{n} a_{ij}(x,t)\, u_{x_j}\Phi_{x_i}^N + a(x,t)\, u\Phi^N\right) dxdt$$
$$-\int_{S''_T}\left(\int_{\Omega} K(s,y,t)\, u(y,t)dy\right)\Phi^N(s,t)\, dsdt$$
$$= \int_{\Omega} \varphi(x)\,\Phi^N(x,0)\, dx + \int_{Q_T} f\Phi^N dxdt. \tag{3.13}$$

Let us denote the set of all functions $\Phi^N(x,t) = \sum\limits_{m=1}^{N} d_m(t)\,\psi_m(x)$ by P_N. The family of functions $\bigcup\limits_{p=1}^{\infty} P_p$ is everywhere dense in $\hat{W}_{2,0}^{1,1}(Q_T)$. Let the function $\eta = \eta(x,t)$ be the limit of a sequence $\left\{\Phi^N\right\}$ from $\bigcup\limits_{p=1}^{\infty} \mathrm{P}_p$ in the norm of $W_2^{1,1}(Q_T)$. Let us show that the equality

$$\lim_{N\to\infty}\int_{S''_T}\left(\int_{\Omega} K(s,y,t)\, u(y,t)dy\right)\Phi^N(s,t)\, dsdt$$

$$= \int_{S_T''} \left(\int_{\Omega} K(s, y, t) u(y, t) dy \right) \eta(s, t) \, dsdt. \tag{3.14}$$

According to the theorem about traces [7, p. 140], the following inequality holds:

$$\left\| \Phi^N(s, t) - \eta(s, t) \right\|_{2, S_T''} \le M_4 \left\| \Phi^N(x, t) - \eta(x, t) \right\|_{2, Q_T}^{(1,1)}, \tag{3.15}$$

where the constant $M_4 > 0$ does not depend on N.

Then using the Cauchy-Bunyakovsky inequality, condition (2.5) for the function $K(x, y, t)$, estimate (3.1), and inequality (3.15), we obtain

$$\left| \int_{S_T''} \left(\int_{\Omega} K(s, y, t) u(y, t) dy \right) \left[\Phi^N(s, t) - \eta(s, t) \right] dsdt \right|$$

$$\le \left\{ \int_{S_T''} \left(\int_{\Omega} K(s, y, t) u(y, t) dy \right)^2 dsdt \right\}^{1/2} \left\{ \int_{S_T''} \left| \Phi^N(s, t) - \eta(s, t) \right|^2 dsdt \right\}^{1/2}$$

$$\le \mu_1 \sqrt{|\Omega|} \, \|u\|_{2, Q_T} \cdot M_4 \left\| \Phi^N - \eta \right\|_{2, Q_T}^{(1,1)}$$

$$\le \mu_1 M_1 M_4 \sqrt{|\Omega|} \left[\|\varphi\|_{2, \Omega} + 2 \|f\|_{2,1, Q_T} \right] \left\| \Phi^N - \eta \right\|_{2, Q_T}^{(1,1)} \to 0,$$

for $N \to \infty$.

Consequently, equality (3.14) is true.

Then, when passing to the limit for $N \to \infty$ in equality (3.13), we obtain that it holds for any $\eta \in \hat{W}_{2,0}^{(1,1)}(Q_T)$. Therefore, the function $u(x, t)$ satisfies the integral identity (2.7) from the definition of a generalized solution of the problem (2.1)-(2.4). Therefore, a generalized solution of problem (2.1)-(2.4) exists.

Let us show that problem (2.1)-(2.4) cannot have two different generalized solutions from the class $V_{2,0}^{1,0}(Q_T)$. Indeed, if it had two such solutions as u_1 and u_2 then their difference would be a generalized solution of problem (2.1)-(2.4) from the class $V_{2,0}^{1,0}(Q_T)$ corresponding to the functions $\varphi = 0, \ f = 0$. Then, according to (3.1), for the function u we have the estimate: $|u|_{Q_T} \le 0$, which means the coincidence of the solutions u_1 and u_2. Theorem 1 is proved.

4 Smoothness of the generalized solution

Let us show that under several stronger assumptions about the data of problem (2.1)-(2.4), generalized solutions from $V_{2,0}^{1,0}(Q_T)$ belong to $W_{2,0}^{1,1}(Q_T) = \{u : u \in W_2^{1,1}(Q_T), u|_{S_T'} = 0\}$. Let, in addition to conditions (2.5) and (2.6), the following conditions be fulfilled:

$$|a_{ijt}(x, t)| \le \mu_2, i, j = \overline{1, n}, \ al.e. \ Q_T, \ |K_t(x, y, t)| \le \mu_3 \ al.e. \ S'' \times \Omega \times (0, T), \tag{4.1}$$

$$\varphi \in W_{2,0}^1(\Omega), \ f \in L_2(Q_T), \tag{4.2}$$

where $\mu_2, \mu_3 > 0$ are some constants.

Theorem 4.1 *Let conditions (2.5), (2.6), (4.1), (4.2) be satisfied. Then problem (2.1)-(2.4) is uniquely solvable in the class* $W_{2,0}^{1,1}(Q_T)$ *and satisfies the estimate*

$$\max_{0\le t\le T}\|u_x(x,t)\|_{2,\Omega}^2+\|u_t\|_{2,Q_T}^2\le M_5\left[\left(\|\varphi\|_{2,\Omega}^{(1)}\right)^2+\|f\|_{2,Q_T}^2\right], \tag{4.3}$$

where the constant $M_5>0$ *does not depend on* φ *and* f.

Proof. For the proof, we again use the Galerkin method. We multiply each of the equations (3.2) by its own $\frac{dc_m^N(t)}{dt}$, add all the obtained equalities over m from 1 to N and integrate over t from 0 to t. This will give the relationship

$$\int_{Q_t}\left[\left(u_t^N\right)^2+\sum_{i,j=1}^n a_{ij}(x,t)\,u_{x_j}^N u_{tx_i}^N+au^N u_t^N\right]dxdt$$
$$-\int_{S_t''}\left(\int_\Omega K(s,y,t)\,u^N(y,t)dy\right)u_t^N(s,t)\,dsdt=\int_{Q_t}fu_t^N dxdt. \tag{4.4}$$

Using the formulas for integration by parts, we obtain the following equalities:

$$\int_{Q_t}\sum_{i,j=1}^n a_{ij}(x,t)\,u_{x_j}^N u_{tx_i}^N dxdt=\frac{1}{2}\int_\Omega\sum_{i,j=1}^n a_{ij}u_{x_j}^N u_{x_i}^N\,dx\Big|_{t=0}^{t=t}$$
$$-\frac{1}{2}\int_{Q_t}\sum_{i,j=1}^n a_{ijt}u_{x_j}^N u_{x_i}^N dxdt,$$

$$\int_{S_t''}\left(\int_\Omega K(s,y,t)\,u^N(y,t)dy\right)u_t^N(s,t)\,dsdt$$
$$=\int_{S''}\left(\int_\Omega K(s,y,t)\,u^N(y,t)dy\right)u^N(s,t)\,ds\Bigg|_{t=0}^{t=t}$$
$$-\int_{S_t''}\left(\int_\Omega\left(K_t(s,y,t)\,u^N(y,t)+K(s,y,t)\,u_t^N(y,t)\right)dy\right)u^N(s,t)\,dsdt.$$

Substituting these equalities into (4.4), we obtain the relationship

$$\frac{1}{2}\int_\Omega\sum_{i,j=1}^n a_{ij}u_{x_j}^N u_{x_i}^N dx+\int_{Q_t}\left(u_t^N\right)^2dxdt=\frac{1}{2}\int_\Omega\sum_{i,j=1}^n a_{ij}(x,0)\,u_{x_j}^N(x,0)\,u_{x_i}^N(x,0)\,dx$$
$$+\frac{1}{2}\int_{Q_t}\sum_{i,j=1}^n a_{ijt}u_{x_j}^N u_{x_i}^N dxdt-\int_{Q_t}au^N u_t^N dxdt$$
$$+\int_{S''}\left(\int_\Omega K(s,y,t)\,u^N(y,t)dy\right)u^N(s,t)\,ds$$
$$-\int_{S''}\left(\int_\Omega K(s,y,0)\,u^N(y,0)dy\right)u^N(s,0)\,ds$$
$$-\int_{S_t''}\left(\int_\Omega\left(K_t(s,y,t)\,u^N(y,t)+K(s,y,t)\,u_t^N(y,t)\right)dy\right)u^N(s,t)\,dsdt+\int_{Q_t}fu_t^N dxdt.$$

Hence, by virtue of conditions (2.5), (4.1) and the “Cauchy inequality with ε” [5, p. 33], we derive the following inequality

$$\frac{\nu}{2}\left\|u_x^N(x,t)\right\|_{2,\Omega}^2+\left\|u_t^N\right\|_{2,Q_t}^2\le\frac{\mu}{2}\left\|u_x^N(x,0)\right\|_{2,\Omega}^2+\frac{\mu_2}{2}\left\|u_x^N\right\|_{2,Q_t}^2+\frac{\varepsilon_1}{2}\|f\|_{2,Q_t}^2$$
$$+\frac{1}{2\varepsilon_1}\left\|u_t^N\right\|_{2,Q_t}^2+\mu\left[\frac{\varepsilon_2}{2}\left\|u^N\right\|_{2,Q_t}^2+\frac{1}{2\varepsilon_2}\left\|u_t^N\right\|_{2,Q_t}^2\right]$$
$$+\mu_1\left[\frac{\varepsilon_3}{2}\left\|u^N(x,t)\right\|_{2,\Omega}^2+\frac{|\Omega|}{2\varepsilon_3}\int_{S''}\left(u^N(s,t)\right)^2ds\right]$$
$$+\mu_1\left[\frac{1}{2}\left\|u^N(x,0)\right\|_{2,\Omega}^2+\frac{|\Omega|}{2}\int_{S''}\left(u^N(s,0)\right)^2ds\right]$$
$$+\mu_1\left[\frac{\varepsilon_4}{2}\left\|u_t^N\right\|_{2,Q_t}^2+\frac{|\Omega|}{2\varepsilon_4}\int_{S_t''}\left(u^N(s,t)\right)^2dsdt\right]$$
$$+\mu_3\left[\frac{1}{2}\left\|u^N\right\|_{2,Q_t}^2+\frac{|\Omega|}{2}\int_{S_t''}\left(u^N(s,t)\right)^2dsdt\right], \tag{4.5}$$

where $\varepsilon_i>0,\quad i=\overline{1,4}$ are arbitrary constants.

Further, to estimate $\left(u^N(s,t)\right)^2$, we use inequality (3.7) and present similar terms. Then from (4.5) we derive the inequality

$$\left(\frac{\nu}{2}-\frac{\mu_1\varepsilon|\Omega|}{2\varepsilon_3}\right)\left\|u_x^N(x,t)\right\|_{2,\Omega}^2+\left(1-\frac{\mu}{2\varepsilon_2}-\frac{1}{2\varepsilon_1}-\frac{\mu_1\varepsilon_4}{2}\right)\left\|u_t^N\right\|_{2,Q_t}^2$$
$$\le\left(\frac{\mu_1}{2}+\frac{\mu_1|\Omega|c_\varepsilon}{2}\right)\left\|u^N(x,0)\right\|_{2,\Omega}^2+\left(\frac{\mu}{2}+\frac{\mu_1\varepsilon|\Omega|}{2}\right)\left\|u_x^N(x,0)\right\|_{2,\Omega}^2$$
$$+\left(\frac{\mu\varepsilon_2}{2}+\frac{\mu_1c_\varepsilon|\Omega|}{2\varepsilon_4}+\frac{\mu_3}{2}+\frac{\mu_3c_\varepsilon|\Omega|}{2}\right)\left\|u^N\right\|_{2,Q_t}^2$$
$$+\left(\frac{\mu_1|\Omega|c_\varepsilon}{2\varepsilon_3}+\frac{\mu_1\varepsilon_3}{2}\right)\left\|u^N(x,t)\right\|_{2,\Omega}^2+\left(\frac{\mu_2}{2}+\frac{\mu_1|\Omega|\varepsilon}{2\varepsilon_4}+\frac{\mu_3\varepsilon|\Omega|}{2}\right)\left\|u_x^N\right\|_{2,Q_t}^2. \tag{4.6}$$

Let’s take $\varepsilon=\frac{\nu}{4},\quad\varepsilon_1=4,\quad\varepsilon_2=4\mu,\quad\varepsilon_3=\frac{\mu_1|\Omega|}{2},\quad\varepsilon_4=\frac{1}{\mu_1}$. Then (4.6) transforms into the following inequality:

$$\nu\left\|u_x^N(x,t)\right\|_{2,\Omega}^2+\left\|u_t^N\right\|_{2,Q_t}^2\le2\mu_1\left(1+|\Omega|c_\varepsilon\right)\left\|u^N(x,0)\right\|_{2,\Omega}^2$$
$$+\left(2\mu+\frac{\mu_1\nu|\Omega|}{2}\right)\left\|u_x^N(x,0)\right\|_{2,\Omega}^2+2\left(4\mu^2+\mu_1^2c_\varepsilon|\Omega|+\mu_3+\mu_3c_\varepsilon|\Omega|\right)\left\|u^N\right\|_{2,Q_t}^2$$
$$+\left(2\mu_2+\frac{\mu_1^2\nu|\Omega|}{2}+\frac{\mu_3\nu|\Omega|}{2}\right)\left\|u_x^N\right\|_{2,Q_t}^2$$
$$+\left(4c_\varepsilon+\mu_1^2|\Omega|\right)\left\|u^N(x,t)\right\|_{2,\Omega}^2+8\|f\|_{2,Q_t}^2. \tag{4.7}$$

Then, using estimates (3.11) from (4.7), we derive the estimate

$$\max_{0\le t\le T}\left\|u_x^N(x,t)\right\|_{2,\Omega}^2+\left\|u_t^N\right\|_{2,Q_T}^2\le M_3\left[\left(\|\varphi\|_{2,\Omega}^{(1)}\right)^2+\|f\|_{2,Q_T}^2\right]. \tag{4.8}$$

in which the constant $M_6 > 0$ does not depend on N, φ and f.

Thus, estimate (4.8) with a constant M_6 independent of the number of approximations is valid for the Galerkin approximations u^N. By virtue of (3.5) and (4.8), a subsequence $\{u^{N_m}\}$ can be selected from the sequence $\{u^N\}$ converging to some function $u \in W_{2,0}^{1,1}(Q_T)$ weakly in $L_2(Q_T)$ together with the derivatives $\{u_x^{N_m}\}$, $\left\{u_t^{N_m}\right\}$, and subsequences $\{u^{N_m}\}$, $\{u_x^{N_m}\}$, converging to $u,\ u_x$ weakly in$L_2(\Omega)$ uniformly with respect to $t \in [0, T]$. By virtue of the well-known property of weak convergence, inequality (4.8) remains valid for the limit function and, therefore, estimate (4.3) is valid.

Theorem 2 is proved.

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