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Regularity of solutions to nonlinear elliptic equations in generalized Morrey spaces

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Abstract. *In this paper, we prove the generalized Morrey estimates for the gradient of weak solutions to a class of nonlinear elliptic equations in a very general irregular domain. The nonlinearity* $\mathbf{a}(x,\xi)$ *is assumed to be measurable in* x *for almost every* ξ *and belongs to the small* BMO *class.*

Keywords. Nonlinear elliptic equations; Reifenberg flat domain; fractional maximal operator; generalized Morrey spaces, small BMO.

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1 Introduction

The aim of this study is to investigate the regularity properties of solutions to the following Dirichlet problem

$$\begin{cases} \operatorname{div} \mathbf{a}(x, Du) = \operatorname{div} f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and the nonlinearity $\mathbf{a}(x,\xi) = (a^1, ..., a^n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in x for almost every ξ, differentiable in ξ for almost every x and satisfies the conditions given below. There exist constants $\Lambda_1, \gamma > 0$ so that

$$|\mathbf{a}(x,\xi)| + |\xi| \cdot |D_\xi \mathbf{a}(x,\xi)| \leq \Lambda_1 |\xi|^{p-1} \tag{1.2}$$

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and

$$D_\xi \mathbf{a}(x,\xi)\eta \cdot \eta \geq \gamma |\xi|^{p-2}|\eta|^2 \tag{1.3}$$

for almost every $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$ and almost every $x \in \mathbb{R}^n$ with $p \in (1,\infty)$. Note that the primary structure conditions (1.2) and (1.3) imply the following inequality

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta))(\xi - \eta) \geq \Lambda_2 |\xi - \eta|^p \tag{1.4}$$

and $\mathbf{a}(x,0) = 0$ for $x \in \mathbb{R}^n$ while the universal constant Λ_2 depends only on Λ_1, γ, n.

Gradient estimates for nonlinear elliptic equations with discontinuous coefficients have been studied extensively by several authors [2,4,5,7,8,10,13,14,19,12,21–25,27,28,33, 37]. Some of these developments are mainly based on the perturbation method developed by Caffarelli and Peral [4] which allows us to investigate the discontinuous coefficient and highly nonlinear structure of equation (1.1). For the elliptic equations where $\mathbf{a}$ is Lipschitz continuous in both x and ξ variables, the interior $C^{1,\alpha}$ regularity of locally bounded weak solutions to corresponding homogeneous equation was established by DiBenedetto [6] and Tolksdorf [35] extending the celebrated $C^{1,\alpha}$ estimates by Uraltseva [36], Uhlenbeck [37], Evans [9], and Lewis [26] for the homogeneous p-Laplace equation. When $\mathbf{a}$ is not necessarily continuous in x but has sufficiently small BMO oscillation, it was established by Nguyen and Phan [32] if $|f|^{\frac{1}{p-1}} \in L_{q,\mathrm{loc}}$, then $\nabla u \in L_{q,\mathrm{loc}}$ for any $q > p$.

Let $B_r(x) = B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ is an open ball in $\mathbb{R}^n$ with center x and radius $r > 0$, $B_r = B_r(x_0) = \{y \in \mathbb{R}^n : |x_0 - y| < r\}$, $B_r^c(x) = \mathbb{R}^n \setminus B_r(x)$, $2B_r(x) = B(x,2r)$, $\Omega_r(x) = \Omega \cap B_r(x)$, $x \in \Omega$, $B_r^+(x_0) \equiv B^+(x_0,r) = B(x_0,r) \cap \{x_n > 0\}$, where $x_0 = (x',0)$.

In connection with elliptic partial differential equations, Morrey proposed a weak condition for the solution to be continuous enough in [30]. Later on, his condition became a family of normed spaces and they are called Morrey spaces $L_{p,\lambda}$. Although the notion is originally from the partial differential equations, the space turned out to be important in many branches of mathematics. Moreover, various Morrey spaces have been defined during the studies, for example, generalized Morrey spaces $M_{p,\varphi}$ were introduced in different ways by the first author, Mizuhara and Nakai [16,29,31] (see, also [17,34]). The generalized Morrey spaces given by Guliyev are defined as:

Let $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. A function $f \in L_p(\Omega)$, $1 < p < \infty$, belongs to the generalized Morrey space $M_{p,\varphi}(\Omega)$ if the following norm is finite

$$\|f\|_{M_{p,\varphi}(\Omega)} = \sup_{x\in\Omega,\, r>0} \frac{1}{\varphi(x,r)} \Big(\frac{1}{r^n}\int_{\Omega_r(x)} |f(y)|^p dy\Big)^{1/p} < \infty.$$

If $\varphi(x,r) \equiv r^{(\lambda-n-2)/p}$, then $M_{p,\varphi}(\Omega)$ coincides with the classical Morrey space $L_{p,\lambda}(\Omega)$, $\lambda \in (0,n)$ (see [18]).

In this paper we study the general nonlinear problem (1.1) in generalized Morrey space $M_{p,\varphi}(\Omega)$ when the principal part also depends on the variable $\xi = \nabla u$. Specifically, we find the conditions on the nonlinearity of equation and the most general geometric requirement to the boundary $\partial\Omega$ to obtain the following global $W^1_{p,\varphi}(\Omega)$ estimate

$$\|Du\|_{M_{q,\varphi_2}(\Omega)} \leq C_1 \, \||f|^{\frac{1}{p-1}}\|_{M_{q_1,\varphi}(\Omega)}. \tag{1.5}$$

The method used in [13] is based on the boundedness of the maximal operator in generalized Morrey spaces and the Calderón-Zygmund decomposition. In this study, unlike in [13], we used the boundedness of the fractional maximal operator instead of the maximal operator.

The generalized Sobolev-Morrey space $W^1_{p,\varphi}(\Omega)$ consists of all functions $u \in W^1_p(\Omega)$ with distributional derivatives $D^s u \in M_{p,\varphi}(\Omega)$ endowed with the norm

$$\|u\|_{W^1_{p,\varphi}(\Omega)} = \sum_{0\le|s|\le 1} \|D^s u\|_{M_{p,\varphi}(\Omega)}.$$

The space $W^1_{p,\varphi}(\Omega) \cap \mathring{W}^1_p(\Omega)$ consists of all functions $u \in \mathring{W}^1_p(\Omega)$ with $D^s u \in M_{p,\varphi}(\Omega)$, $0 \le |s| \le 1$ and is endowed by the same norm. Recall that $\mathring{W}^1_p(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W^1_p(\Omega)$.

A function $u \in \mathring{W}^1_p(\Omega)$ is said to be a weak solution to (1.1) if the following holds

$$\int_\Omega \mathbf{a}(x, Du) \cdot D\varphi dx = \int_\Omega f D\varphi dx \tag{1.6}$$

for all $\phi \in C^\infty(\Omega)$ vanishing in a neighborhood of $\partial\Omega$. Assume that the nonlinearity $\mathbf{a}$ satisfy (1.2) and (1.3). We set

$$\Theta(\mathbf{a}, \Omega_r(y))(x) = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\mathbf{a}(x,\xi) - \bar{\mathbf{a}}_{\Omega_r(y)}(\xi)|}{|\xi|^{p-1}},$$

where

$$\bar{\mathbf{a}}_{\Omega_r(y)}(\xi) \equiv \fint_{\Omega_r(y)} \mathbf{a}(x,\xi)dx = \frac{1}{|\Omega_r(y)|}\int_{\Omega_r(y)} \mathbf{a}(x,\xi)dx.$$

The nonlinearity $\mathbf{a}$ is said to be satisfy the small $(\delta, R_0) - BMO$ condition if

$$[\mathbf{a}]_{p,R_0} = \sup_{0<r\le R_0} \sup_{y\in\mathbb{R}^n} \fint_{\Omega_r(y)} |\Theta(\mathbf{a}, \Omega_r(y))(x)|^p dx \le \delta^p, \ \delta > 0. \tag{1.7}$$

From the assumptions (1.2), (1.3) and the small $(\delta, R_0) - BMO$ condition, it is easy to see that for any $\gamma \in [1, \infty)$ there exists $\varepsilon > 0$ such that

$$[\mathbf{a}]_{\gamma,R_0} = \sup_{y\in\mathbb{R}^n} \sup_{0<r\le R_0} \fint_{\Omega_r(y)} |\Theta(\mathbf{a}, \Omega_r(y))(x)|^\gamma dx \le \delta^\varepsilon.$$

Note that $\partial\Omega$ satisfies the following a more general geometric requirement. We say that Ω is (δ, R_0) Reifenberg flat if for every $x \in \partial\Omega$ and every $\rho \in (0, R_0)$ there exists a coordinate system $\{y_1, ..., y_n\}$ which depends on ρ and x such that $x = 0$ in this coordinate system and

$$B_\rho(0) \cap \{y_n > \rho\delta\} \subset B_\rho(0) \cap \Omega \subset B_\rho(0) \cap \{y_n > -\rho\delta\}.$$

In the above definitions we assume δ is to be a small positive constant while one can assume $R = 1$ or any other constant by a scaling. If Ω is (δ, R_0) Reifenberg flat, then there is the following measure density condition

$$\frac{|B_r(x)|}{|B_r(x) \cap \Omega|} \le \Big(\frac{2}{1-\delta}\Big)^n$$

which can be found in [11].

It is well known that when f is regular, for example if $f \in L_2(\Omega)$, then according to classical theory in [24] there exists a unique weak solution to problem (1.1).

We state our main theorem as follows.

Theorem 1.1 *Let $p \in (1,\infty)$, $q_1 \in (p-1, n(p-1))$, $\frac{n}{q_1} - \frac{n}{q} = \frac{1}{p-1}$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<s<\infty} s^{-\frac{n}{q}} \operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi_1(\Omega_\sigma(y))\, \sigma^{\frac{n}{q_1}} \le C\, \varphi_2(\Omega_r(y)), \tag{1.8}$$

where C does not depend on y and r. Assume that (1.2) *and* (1.3) *hold and u is a weak solution to problem* (1.1). *There is a small $\delta(n, \Lambda_1, \gamma) > 0$ such that if Ω is (δ, R_0) - Reifenberg flat and the nonlinearity* $\mathbf{a}$ *satisfies the small $(\delta, R_0) - BMO$ condition for some $R_0 > 0$ and $|f|^{\frac{1}{p-1}} \in M_{q_1,\varphi_1}(\Omega)$, then $Du \in M_{q,\varphi_2}(\Omega)$ with the estimate*

$$\|Du\|_{M_{q,\varphi_2}(\Omega)} \le C_1\, \||f|^{\frac{1}{p-1}}\|_{M_{q_1,\varphi}(\Omega)}, \tag{1.9}$$

where the constant C_1 depends on n, q, Λ_1,γ, R_0, $|\Omega|$.

If we take $\varphi_1(x,r) = \varphi(r)$, $\varphi_2(x,r) = r^{\frac{1}{p-1}}\varphi(r)$, then from Theorem 1.1 we get the following corollary.

Corollary 1.1 *Let $p \in (1,\infty)$, $q_1 \in (p-1, n(p-1))$, $\frac{n}{q_1} - \frac{n}{q} = \frac{1}{p-1}$ and φ satisfy the conditions* (2.3) *and* (2.4). *Assume that* (1.2) *and* (1.3) *hold and u is a weak solution to problem* (1.1). *There is a small $\delta(n, \Lambda_1, \gamma) > 0$ such that if Ω is (δ, R_0) - Reifenberg flat and the nonlinearity* $\mathbf{a}$ *satisfies the small $(\delta, R_0) - BMO$ condition for some $R_0 > 0$ and $|f|^{\frac{1}{p-1}} \in M_{q_1,\varphi}(\Omega)$, then $Du \in M_{q, r^{\frac{1}{p-1}}\varphi(r)}(\Omega)$ with the estimate*

$$\|Du\|_{M_{q, r^{\frac{1}{p-1}}\varphi(r)}}(\Omega) \le C_1\, \||f|^{\frac{1}{p-1}}\|_{M_{q_1,\varphi}(\Omega)}, \tag{1.10}$$

where the constant C_1 depends on n, q, Λ_1,γ, R_0, $|\Omega|$.

This paper is organized as follows. In section 2 we give the boundedness of the fractional maximal operator in generalized Morrey spaces. We also give a version of Vitali covering lemma and use some standart arguments of measure theory. In sections 3 and 4 we study interior and boundary estimates of the regularized problem, respectively. In Section 5 we give Calderón-Zygmund type estimates for weak solutions of a class of nonlinear elliptic equations and prove our main result Theorem 1.1.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Boundedness of the fractional maximal operator in generalized Morrey spaces

Firstly we introduce some notations. For a measurable function f on a measurable subset E of $\mathbb{R}^n$ we define

$$\bar{f}_E = \fint_E f dx = \frac{1}{|E|}\int_E f dx.$$

The fractional maximal function $M_\alpha f$ of a locally integrable function f defined in $\mathbb{R}^n$ is a function (see [1]) such that

$$M_\alpha f(y) = \sup_{\rho>0} |B_\rho(y)|^{\frac{\alpha}{n}} \fint_{B_\rho(y)} |f(x)| dx, \quad \alpha \in [0,n).$$

In particular, we get $M_0 f(y) \equiv Mf(y)$ as the usual Hardy-Littlewood maximal function when $\alpha = 0$. If f is defined on a bounded subset E of $\mathbb{R}^n$, then we define the restricted maximal function $M_E f$ by $M_E(f) = M(f\chi_E)$, where χ_E is the standard characteristic function on E. Moreover, when $\alpha = 1$, $M_1 f(y)$ can be defined as

$$M_1 f(y) = \sup_{\rho>0} \frac{\rho \, \|f\|_{L_1(B_\rho(y))}}{|B_\rho(y)|}.$$

The boundedness results of maximal operators in generalized Morrey spaces is obtained in [18].

Lemma 2.1 *[18] Assume that* $1 < q_1 < \infty$ *and the condition*

$$\sup_{r<s<\infty} s^{-\frac{n}{q_1}} \operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi(\Omega_\sigma(x)) \, \sigma^{\frac{n}{q_1}} \le C \, \varphi(\Omega_r(x)), \tag{2.1}$$

holds, where C *does not depend on* x *and* r. *Then there is a constant* $C_{q_1} > 0$ *such that*

$$\|f\|_{M_{q_1,\varphi}(\mathbb{R}^n)} \le \|Mf\|_{M_{q_1,\varphi}(\mathbb{R}^n)} \le C_q \|f\|_{M_{q_1,\varphi}(\mathbb{R}^n)}, \; f \in M_{q_1,\varphi}(\mathbb{R}^n).$$

In the following we give the boundedness of the fractional maximal operator in generalized Morrey spaces.

Theorem 2.1 *[20, Theorem 4.3] Let* $1 \le q_1 < \infty$, $0 \le \alpha < \frac{n}{q_1}$, $\frac{n}{q_1} - \frac{n}{q} = \alpha$ *and* (φ_1, φ_2) *satisfy the condition*

$$\sup_{r<s<\infty} s^{-\frac{n}{q}} \operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi_1(\Omega_\sigma(y)) \, \sigma^{\frac{n}{q_1}} \le C \, \varphi_2(\Omega_r(y)), \tag{2.2}$$

where C *does not depend on* y *and* r.

Then for $q_1 > 1$, M_α *is bounded from* $M_{q_1,\varphi_1}(\Omega)$ *to* $M_{q,\varphi_2}(\Omega)$ *and*

$$\|M_\alpha f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{q_1,\varphi_1}}$$

and for $q_1 = 1$, M_α *is bounded from* $M_{1,\varphi_1}(\Omega)$ *to* $WM_{q,\varphi_2}(\Omega)$ *and*

$$\|M_\alpha f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Corollary 2.1 *Let* $p \in (1, \infty)$, $q_1 \in (p-1, n(p-1))$, $\frac{n}{q_1} - \frac{n}{q} = \frac{1}{p-1}$ *and* (φ_1, φ_2) *satisfy the condition* (2.2). *Then*

$$\|(M_1 f)^{\frac{1}{p-1}}\|_{M_{q,\varphi_2}(\Omega)} \lesssim \||f|^{\frac{1}{p-1}}\|_{M_{q_1,\varphi_1}(\Omega)}.$$

Corollary 2.2 *Let* $1 \le q_1 < \infty$, $0 \le \alpha < \frac{n}{q_1}$, $\frac{n}{q_1} - \frac{n}{q} = \alpha$ *and* φ *satisfy the condition*

$$\sup_{r<s<\infty} s^{-\frac{n}{q}} \operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi(\Omega_\sigma(y)) \, \sigma^{\frac{n}{q_1}} \le C \, r^\alpha \, \varphi(\Omega_r(y)),$$

where C *does not depend on* y *and* r.

Then for $q_1 > 1$, M_α *is bounded from* $M_{q_1,\varphi}(\Omega)$ *to* $M_{q,r^\alpha\varphi(r)}(\Omega)$ *and*

$$\|M_\alpha f\|_{M_{q,r^\alpha\varphi(r)}(\Omega)} \lesssim \|f\|_{M_{q_1,\varphi}(\Omega)}$$

and for $q_1 = 1$ M_α *is bounded from* $M_{1,\varphi}(\Omega)$ *to* $WM_{q,r^\alpha\varphi(r)}(\Omega)$ *and*

$$\|M_\alpha f\|_{WM_{q,r^\alpha\varphi(r)}(\Omega)} \lesssim \|f\|_{M_{1,\varphi}(\Omega)}.$$

Corollary 2.3 *Let $p \in (1,\infty)$, $q_1 \in (p-1, n(p-1))$, $\frac{n}{q_1} - \frac{n}{q} = \frac{1}{p-1}$ and φ satisfy the condition*

$$\sup_{r<s<\infty} s^{-\frac{n}{q}} \operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi(\Omega_\sigma(y))\, \sigma^{\frac{n}{q_1}} \leq C\, r^{\frac{1}{p-1}}\, \varphi(\Omega_r(y)), \tag{2.3}$$

where C does not depend on y and r. Then

$$\|(M_1 f)^{\frac{1}{p-1}}\|_{M_{q, r^{\frac{1}{p-1}}\varphi(r)}(\Omega)} \lesssim \||f|^{\frac{1}{p-1}}\|_{M_{q_1,\varphi}(\Omega)}.$$

Fix $y_0 \in \Omega$ and denote $K_r = K_r(y_0)$.

Lemma 2.2 *Assume that Ω is $(\delta, 1)$- Reifenberg flat. Let $E \subset F \subset K_r$ be measurable subsets of K_r satisfying the following conditions: there exists $\varepsilon \in (0,1)$ such that for each $y \in E$*

$$|E \cap \Omega_1(y)| < \varepsilon\, |\Omega_1(y)|$$

and for all $y \in E$ and for every $\rho \in (0,1]$ with $|E \bigcap \Omega_\rho(y)| \geq \varepsilon |\Omega_\rho(y)|$, $\Omega_\rho(y) \cap K_r \subset F$. Then we have $|E| \leq \left[\frac{10}{1-\delta}\right]^n \varepsilon\, |F|$.

We also use the following arguments of measure theory.

Lemma 2.3 *Let $f \in L_1(\Omega)$ be a non-negative and measurable function in $\mathbb{R}^n$, φ satisfy the conditions* (2.1) *and*

$$\varphi(\Omega_r(y))^{q_1}\, r^n \leq \varphi(\Omega_s(z))^{q_1}\, s^n \ \text{ for all } \ \Omega_r(y) \subset \Omega_s(z), \tag{2.4}$$

where $q_1 \in (1,\infty)$, and $\theta > 0$, $\lambda > 1$ are constants. Then $f \in M_{q_1,\varphi}(\Omega)$ if and only if

$$S = \sup_{y\in\Omega,\, r>0} \sum_{k\geq 1} \frac{\lambda^{kq_1} |\{x \in \Omega_r(y) : f(x) > \theta\lambda^k\}|}{\varphi(\Omega_r(y))^{q_1}\, r^n} < \infty.$$

Moreover, we have

$$\frac{1}{C} S \leq \|f\|^{q_1}_{M_{q_1,\varphi}(\Omega)} \leq C(1+S),$$

where $C > 0$ is a constant depending only on θ, λ, φ and q_1.

Now we give normalization invariance property of problem (1.1).

Lemma 2.4 *Assume the nonlinearity $\mathbf{a}(x,\xi)$ satisfies the conditions* (1.2), (1.3) *and $u \in \mathring{W}^1_p(\Omega)$ is the weak solution to problem* (1.1). *Let $\mathbf{a}(x,\xi)$ satisfy the small (δ, R_0) - BMO condition. For each $\lambda \geq 1$ if we define*

$$\mathbf{a}_\lambda(x,\xi) = \frac{\mathbf{a}(x,\lambda\xi)}{\lambda}, \ \ u_\lambda(x) = \frac{u(x)}{\lambda}, \ \ f_\lambda(x) = \frac{f(x)}{\lambda},$$

then $\mathbf{a}_\lambda(x,\xi)$ satisfies the conditions (1.2) *and* (1.3) *with the same constants Λ_1, γ, and the small (δ, R_0)-BMO condition. Moreover, $u_\lambda \in \mathring{W}^1_p(\Omega)$ is the weak solution to*

$$\begin{cases} \operatorname{div} \mathbf{a}_\lambda(x, Du_\lambda) = \operatorname{div} f_\lambda, & \text{in } \ \Omega, \\ u_\lambda = 0, & \text{on } \ \partial\Omega. \end{cases}$$

3 Interior estimates

Together with problem (1.1), we consider the following problem

$$\begin{cases} \operatorname{div} \mathbf{a}(x, w) = 0, & \text{in } \ \Omega_{2R}, \\ w = u, & \text{on } \ \partial\Omega_{2R}, \end{cases} \tag{3.1}$$

where $0 < R < \frac{R_0}{2}$, $B_{2R} = B_{2R}(x_0) \subset \Omega$ and $\Omega_{2R} = \Omega_{2R}(x_0) = B_{2R} \times \Omega$ for $x_0 \in \Omega$. Let u be a weak solution to problem (1.1) and $w \in W_2^1(B_{2R})$ be a weak solution to problem (3.1). Due to the lack of regularity with respect to the time variable, the weak solutions u and w could not be chosen as a test function. However, in order to overcome this trouble, we can make use of the Steklov averages. Then we have the following estimate.

Lemma 3.1 *([18]) Let $u \in \overset{\circ}{W}{}_p^1(\Omega)$ be a weak solution to problem* (1.1) *and w be a weak solution to problem* (3.1). *Then there exists a constant $C(n, \Lambda_1, \gamma)$ so that*

$$\fint_{\Omega_{2R}} |Du - Dw| dx \le \frac{C\|f\|_{L_1(\Omega_{2R})}}{R^{n-1}}$$

and thus

$$\fint_{\Omega_{2R}} |Dw| dx \le \int_{\Omega_{2R}} |Du| dx + \frac{C\|f\|_{L_1(\Omega_{2R})}}{R^{n-1}}.$$

We consider reverse Hölder inequality from [2] for the higher integrability property.

Lemma 3.2 *([2]) For any given $\chi_0 = \chi_0(n, \Lambda_1, \gamma) > 1$ and $q \in (0, 2]$ there exists a constant $C = C(n, \Lambda_1, \gamma f, \chi_0, q)$ such that for any $\Omega_{2s}(x_0) \subset \Omega_{2R}$ the following inequality holds*

$$\Big(\fint_{\Omega_\rho(x_0)} |Dw|^{2\chi_0} dx \Big)^{\frac{1}{2\chi_0}} \le C \Big(\fint_{Q_{2\rho}(x_0)} |Dw|^q dx \Big)^{1/q}.$$

From Hölder inequality and Lemma 3.2 we can directly obtain the following result.

Corollary 3.1 *Under the above assumptions and estimate of Lemma 3.2 the following inequality holds*

$$\fint_{\Omega_\rho(x_0)} |Dw|^p dx \le C \Big(\fint_{Q_{2\rho}(x_0)} |Dw| dx \Big)^p$$

for some constant $C = C(n, \Lambda_1, \gamma) > 0$.

Let w be a weak solution to (3.1) and consider the problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{\Omega_r}(Dv) = 0, & \text{in } \ \Omega_r, \\ v = w, & \text{on } \ \partial\Omega_r. \end{cases} \tag{3.2}$$

In the following lemmas we give some estimates for solution to problem (3.2).

Lemma 3.3 *([7]) Let v be a weak solution to* (3.2). *Then there exist C and σ_1 so that*

$$\fint_{\Omega_r} |D(w - v)|^p dx \le C[\mathbf{a}]_{p,R_0}^{\sigma_1} \Big(\fint_{\Omega_{2R}} |Dw| dx \Big)^p.$$

For the gradient of a weak solution to (3.2) the following well-known L_∞ estimate holds.

Lemma 3.4 *([24]) Let v be a weak solution to* (3.2). *Then*

$$\|Dv\|_{L_\infty(\Omega_{R/2})} \lesssim \fint_{\Omega_r} |Dv| dx.$$

In cylinder $\Omega_r(z)$ we define

$$F(f, \Omega_r(z)) = \frac{\|f\|_{L_1(\Omega_r(z))}}{r^{n-1}}.$$

Then we have the following approximation result.

Lemma 3.5 *If $u \in W^1_{p,\mathrm{loc}}(B_R)$ is a weak solution to* (1.1) *in Ω_{2R} and $2R \leq R_0$, then there is a function $v \in W^1_p(B_R)$ with $Dv \in L_\infty(\Omega_{R/2})$ such that*

$$\|Dv\|_{L_\infty(\Omega_{R/2})} \lesssim \fint_{\Omega_{2R}} |Du| dx + CF(f, \Omega_{2R}(x_0)) \tag{3.3}$$

and

$$\fint_{\Omega_{2R}} |Du - Dv| dx \lesssim F(f, \Omega_{2R}(x_0)) + [a]^{\sigma_1/2}_{p,R_0} \Big(\fint_{\Omega_{2R}} |Du| dx + F(f, \Omega_{2R}(x_0)) \Big). \tag{3.4}$$

Proof. Let w and v be as in (3.1) and (3.2). By interior regularity and Corollary 3.1 we have

$$\|Du\|_{L_\infty(\Omega_{R/2})} \lesssim \fint_{\Omega_r} |Dv| dx \lesssim \fint_{\Omega_r} |Dw| dx \lesssim \fint_{\Omega_{2R}} |Dw| dx.$$

From Lemma 3.1 we have

$$\fint_{\Omega_r} |Dw| dx \lesssim \fint_{\Omega_{2R}} |Du| dx + F(f, \Omega_{2R}(x_0)).$$

Then we get the estimate (3.3). On the other hand, from Lemma 3.3 and Hölder inequality we find

$$\fint_{\Omega_r} |D(w - v)| dx \lesssim [\mathbf{a}]^{\sigma_1/2}_{p,R_0} \fint_{\Omega_{2R}} |Dw| dx.$$

Thus we have

$$\fint_{\Omega_r} |D(u - v)| dx \lesssim \fint_{\Omega_r} |D(u - w)| dx + [\mathbf{a}]^{\sigma_1/2}_{p,R_0} \fint_{\Omega_{2R}} |Dw| dx.$$

Then inequality (3.4) follows from Lemma 3.1.

4 Boundary estimates

In this section we consider boundary estimates. Fix $x_0 \in \partial\Omega$ and $0 < R < R_0/10$. Let u be a weak solution to problem (1.1) and consider the problem

$$\begin{cases} \operatorname{div} \mathbf{a}(x, w) = 0, & \text{in} \quad K_{10,R}(x_0), \\ w = u, & \text{on} \quad \partial K_{10R}(x_0). \end{cases} \tag{4.1}$$

The problem (4.1) has a unique solution and up to the boundary holds comparison estimate.

Lemma 4.1 *Let w be a weak solution to problem* (4.1)*. Then there exists a constant $C(n, \Lambda_1, 8)$ such that the inequalities*

$$\fint_{K_{10R}(x_0)} |Du - Dw| dx \le CF(f, K_{10R}(x_0))$$

and

$$\fint_{K_{10R}(x_0)} |Dw| dx \le \fint_{K_{10R}(x_0)} |Du| dx + CF(f, K_{10R}(x_0))$$

hold.

We now take $\rho = R(1 - \delta)$ so that $0 < \frac{\rho}{1-\delta} < \frac{R_0}{10}$. From the definition of Reifenberg flat domains there exists a coordinate system $\{z_1, z_2, \ldots, z_n\}$ with the origin $0 \in \Omega$ such that in this coordinate system $x_0 = (0, \ldots, 0, -\frac{\rho\delta}{1-\delta})$ and

$$B_\rho^+ \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \Big\{ z = (z_1, z_2, \ldots, z_n) : z_n > \frac{-2\rho\delta}{1-\delta} \Big\}.$$

Thus if $\delta < \frac{1}{2}$, then we have

$$B_\rho^+ \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{ z = (z_1, z_2, \ldots, z_n) : z_n > -4\delta\rho \},$$

where $B_\rho^+(0) = B_\rho(0) \cap \{ z = (z_1, z_2, \ldots, z_n) : z_n > 0 \}$.

In the new coordinate system we define a function v as the unique weak solution to the problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{\Omega_\rho(0)}(Dv) = 0, & \text{in} \quad K_\rho(0), \\ v = w, & \text{on} \quad \partial K_\rho(0). \end{cases} \tag{4.2}$$

For the solution of problem (4.2) we have the following.

Lemma 4.2 *([7]) Let v be a weak solution to problem* (4.2)*. Then there exist $C > 0$ and σ_2 so that*

$$\fint_{K_\rho(0)} |D(w - v)|^p dx \le C\, [\mathbf{a}]_{p,R_0}^{\sigma_2} \Big(\fint_{K_{2\rho}} |Dw|^p dx \Big)^{\frac{1}{p}}.$$

Because of the lack of smoothness condition on the boundary of Ω, we cannot expect that L_∞ norm of Dv is finite near the boundary. To overcome this difficulty consider its associated problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{\Omega_\rho(0)}(DV) = 0, & \text{in} \quad \Omega_\rho^+(0), \\ V = 0, & \text{on} \quad \Omega_\rho(0) \cap \{ x = (x', x_n) : x_n = 0 \}. \end{cases} \tag{4.3}$$

Lemma 4.3 *([38]) Let V be a weak solution to* (4.3)*. Then we have*

$$\|DV\|_{L_\infty(\Omega^+_{\rho/2}(0))} \lesssim \Big(\fint_{\Omega^+_\rho(0)} |DV|^p dx \Big)^{1/p}.$$

We now consider a scaled version of (4.2)

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{\Omega_1(0)}(Dv) = 0, & \text{in} \quad K_1(0), \\ v = 0, & \text{on} \quad \partial K_1(0) \end{cases} \tag{4.4}$$

under the geometric setting

$$B_1^+(0) \subset \Omega_1 \subset B_1(0) \cap \{x_n > -4\delta\}. \tag{4.5}$$

We give two results from [11] in the following.

Lemma 4.4 *([11]) For any $\varepsilon > 0$ there exists a small $\delta > 0$ such that if v is a weak solution to* (4.4) *with*

$$\fint_{K_1(0)} |Dv|^2 dx \le 1,$$

then there exists a weak solution V with $\rho = 1$,

$$\fint_{\Omega_1^+(0)} |v - V|^p dx \le \varepsilon^p.$$

Lemma 4.5 *([11]) For any $\varepsilon > 0$ there exists a small $\delta(n, \Lambda, \gamma, \varepsilon) > 0$ such that if v is a weak solution to* (4.4) *along with the geometric setting* (4.5) *and the bound*

$$\fint_{K_1(0)} |Dv|^p dx \le 1,$$

then there exists a weak solution V to (4.3) *with $\rho = 1$, whose zero extension to $\Omega_1(0)$ satisfies*

$$\|DV\|_{L_\infty(\Omega_{1/4}(0))} \le C(n, \Lambda_1, \gamma)$$

and

$$\fint_{K_{1/8}(0)} |D(v - V)|^p dx \le \varepsilon^p.$$

Now we give following scaled version.

Lemma 4.6 *([38]) For any $\varepsilon > 0$ there exists a small $\delta(n, \Lambda_1, \gamma, \varepsilon)$ such that if v is a weak solution to*

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{\Omega_\rho(0)}(Dv) = 0, & \text{in} \quad K_\rho(0), \\ v = 0, & \text{on} \quad \partial K_\rho(0) \end{cases}$$

along with the geometric setting

$$B_\rho^+(0) \subset \Omega_\rho \subset B_\rho(0) \cap \{x_n > -4\rho\delta\},$$

then there exists a weak solution V of (4.3), *whose zero extension to $\Omega_\rho(0)$ satisfies*

$$\|DV\|_{L_\infty(K_{\rho/4}(0))} \lesssim \fint_{K_\rho(0)} |Dv|^p dx$$

and

$$\fint_{K_{\rho/8}(0)} |D(v-V)|^p dx \le \varepsilon^p \fint_{K_\rho(0)} |Dv|^p dx.$$

Lemma 4.7 *For any ε there exits a small $\delta(n, \Lambda_1, \gamma, \varepsilon) > 0$ such that the following holds. If Ω is (δ, R_0)-Reifenberg flat and $u \in \mathring{W}^1_p(\Omega)$ is a weak solution to* (1.1) *with $x_0 \in \partial\Omega$ and $0 < R < \frac{R_0}{10}$, then there is a function V such that*

$$\|DV\|_{L_\infty(K_{\rho/10}(x_0))} \lesssim \fint_{K_{10R}(x_0)} |Du| dx + F(f, K_{10R}(x_0))$$

and

$$\fint_{K_{R/10}(x_0)} |D(u-V)| dx \lesssim (\varepsilon + [a]^{\sigma_2}_{p,R_0}) \fint_{K_{10R}(x_0)} |Du| dx + (\varepsilon + [a]^{\sigma_2}_{p,R_0}) F(f, K_{10R}(x_0)).$$

Proof. Set $\rho = R(1-\delta)$ and let $x_0 \in \partial\Omega$, $0 < R < \frac{R_0}{10}$. We may assume that $0 \in \Omega$, $x_0 = (0, \dots, -\frac{\delta\rho}{1-\delta})$ and

$$B^+_\rho(0) \subset \Omega_\rho \subset B_\rho(0) \cap \{x_n > -4\delta\}.$$

Moreover, we observe that

$$B_\rho(0) \subset \Omega_{2\rho} \subset B_{10R}(x_0) \quad \text{and} \quad B_{R/10} \subset B_{\rho/8}(0)$$

provided $\delta < \frac{1}{4\delta}$. Next we choose w and v as in (4.1) and (4.2) corresponding to these R and ρ. Then we have

$$\fint_{K_\rho(0)} |Dv|^p dx \lesssim \fint_{K_\rho(0)} |Dw|^p dx. \tag{4.6}$$

We observe that for V

$$\fint_{K_\rho(0)} |D(u-V)| dx = \fint_{K_\rho(0)} |D(u-w)| dx + \fint_{K_\rho(0)} |D(w-v)| dx + \fint_{K_\rho(0)} |D(v-V)| dx.$$

By Lemma 4.1

$$\fint_{K_\rho(0)} |D(u-w)| dx \le \fint_{K_{10R}(x_0)} |D(u-w)| dx \lesssim F(f, K_{10R}(x_0)).$$

By Lemma 4.2 we find

$$\fint_{K_\rho(0)} |D(w-v)| dx \lesssim [\mathbf{a}]^{\sigma_2/2}_{p,R_0} \fint_{K_{2\rho}(0)} |Dw| dx.$$

Also by Lemma 4.6 for any $\varepsilon > 0$ there exists a small positive $\delta(n, \Lambda, \gamma, \varepsilon) < 1$ such that there is a function V such that

$$\|DV\|_{L_\infty(K_{\rho/4}(0))} \lesssim \fint_{K_\rho(0)} |Dv|^p dx \lesssim \fint_{K_\rho(0)} |Dw|^p dx$$

and

$$\fint_{K_{\rho/8}(0)} |D(v-V)|^p dx \le \varepsilon^p \fint_{K_\rho(0)} |Dv|^p dx \lesssim \fint_{K_\rho(0)} |Dw|^p dx.$$

Note that by Lemma 4.1 we have

$$\Big(\fint_{K_\rho(0)} |Dw|^p dx\Big)^{1/p} \lesssim \fint_{K_{10R}(x_0)} |Du| dx + F(f, K_{10R}(x_0)).$$

Thus the proof of the lemma is completed.

5 Calderón-Zygmund type estimates

In this section we give Calderón-Zygmund type estimates for weak solutions to a class of nonlinear elliptic equations and prove our main result Theorem 1.1.

Proposition 5.1 *Suppose that* **a** *satisfies the conditions* (1.2) *and* (1.3). *Then there is a constant $\Lambda(n, \Lambda_1, \gamma) > 0$ so that for any $\varepsilon > 0$ if there exists a small $\delta(\varepsilon) > 0$ such that for the weak solution u to problem* (1.1)*, where Ω is $(\delta, R_0)-$Reifenberg flat,* **a** *satisfies small $(\delta, R_0) - BMO$ condition and*

$$\Omega_\rho(y) \cap \{x \in K_r : M(|Du|)(x) \le 1\} \cap \{x \in K_r : (M_1 f)^{\frac{1}{p-1}}(x) \le \delta\} \ne \varnothing \qquad (5.1)$$

for some $\Omega_\rho(y)$ with $\rho < R_0$ and ρ sufficiently small, then we have

$$|\{x \in K_r : M(|Du|)(x) > \Lambda\} \cap \Omega_\rho(y)| < \varepsilon |\Omega_\rho(y)|. \qquad (5.2)$$

Proof. By (5.1) there exists $x_0 = (x_0, t_0) \in \Omega_\rho(y)$ such that for any $\tilde r > 0$,

$$\fint_{\Omega_{\tilde r}(x_0)} |Du| dx \le 1 \quad \text{and} \quad \tilde r \fint_{\Omega_{\tilde r}(x_0)} d|f| \le \delta. \qquad (5.3)$$

Here we recall that both u and f are extended by zero outside Q. Then for $z \in \Omega_\rho(y)$ we have

$$M(|Du|)(z) \le \max\{M(\chi \Omega_{2\rho}(y)|Du|)(z), 3^n\}. \qquad (5.4)$$

Indeed for $\tilde r \le \rho$, $\Omega_{\tilde r}(z) \subset \Omega_{2\rho}(y)$ and

$$\fint_{\Omega_{\tilde r}(z)} |Du| dx = \fint_{\Omega_{\tilde r}(z)} \chi_{\Omega_{2\rho}(y)} |Du| dx. \qquad (5.5)$$

For $\tilde r > \rho$, $\Omega_{\tilde r}(z) \subset \Omega_{3\tilde r}(x_0)$ and

$$\fint_{\Omega_{\tilde r}(z)} |Du| dx \le 3^n \fint_{\Omega_{3\tilde r}(x_0)} |Du| dx \le 3^n.$$

We consider separately the cases $B_{4\rho}(y) \subset \Omega$ and $\overline{B_{4\rho}(y)} \cap \partial\Omega \neq \emptyset$. Let $\overline{B_{4\rho}(y)} \cap \partial\Omega \neq \emptyset$ and $y_0 \in \overline{B_{4\rho}(y)} \cap \partial\Omega \neq \emptyset$. Then

$$B_{2\rho}(y) \subset B_{4\rho}(y) \subset B_R(y_0) \subset B_{R+\rho}(x_0), \tag{5.6}$$

where R is big enough than 4ρ. Since $\rho < R$ for any $\eta \in (0,1)$ there is a small $\delta(\eta) > 0$ such that if Ω is (δ, R_0)-Reinfenberg flat, then one can find a function V with

$$\|DV\|_{L_\infty(\Omega_{4\rho}(y_0))} \lesssim \fint_{\Omega_r(y_0)} |Du| dx + F(f, \Omega_r(y_0)),$$

and

$$\fint_{\Omega_{4\rho}(y_0)} |D(u-V)| dx \lesssim (\eta + [\mathbf{a}]^{\sigma_2}_{p,R_0}) \fint_{\Omega_r(y_0)} |Du| dx \\ + (\eta + 1 + [\mathbf{a}]^{\sigma_2}_{p,R_0}) F(f, \Omega_r(y_0)).$$

Thus we have

$$\|DV\|_{L_\infty(\Omega_{2\rho}(y))} \lesssim \fint_{\Omega_{R+\rho}(x_0)} |Du| dx + F(f, \Omega_{R+\rho}(x_0)) \leq C_0 \tag{5.7}$$

and

$$\fint_{\Omega_{2\rho}(y)} |Du - DV| dx \lesssim (\eta + [\mathbf{a}]^{\sigma_2}_{p,R_0}) + \delta(\eta + 1 + [a]^{\sigma_2}_{p,R_0}). \tag{5.8}$$

From (5.4) and (5.7) the following inequality holds

$$\begin{aligned} &\left|\{x \in K_r : M(|Du|)(x) > \lambda\} \cap \Omega_\rho(y)\right| \\ &\leq \left|\left\{x \in K_r : M(\chi_{\Omega_{2\rho}(y)}|DV|)(x) > \frac{\Lambda}{2}\right\} \cap \Omega_\rho(y)\right| \\ &+ \left|\left\{x \in K_r : M(\chi_{\Omega_{2\rho}(y)}|Du - DV|)(x) > \frac{\Lambda}{2}\right\} \cap \Omega_\rho(y)\right|. \end{aligned}$$

Thus by weak-type (1.1) bound for the Hardy-Littlewood maximal function and inequality (5.8) we have (5.2) in the case $\overline{B_{4\rho}(y)} \cap \partial\Omega \neq \emptyset$ for any given $\varepsilon > 0$, provided η, δ are appropriately chosen. The case $B_{4\rho}(y) \Subset \Omega$ can be done in a similar way.

We can also give the contrapositive of Proposition 5.1.

Proposition 5.2 *Suppose that* $\mathbf{a}$ *satisfies the condition* (1.2)-(1.3). *Then there is a constant* $\Lambda(n, \Lambda, \gamma) > 0$, *so that for any* $\varepsilon > 0$ *if there exists a small* $\delta(\varepsilon) > 0$ *such that for the weak solution* u *to problem* (1.1), *where* Ω *is* (δ, R_0)*- Reifenberg flat,* $\mathbf{a}$ *satisfies small* $(\delta, R_0) - BMO$ *condition and*

$$|\{x \in K_r : M(|Du|)(x) > \Lambda\} \cap \Omega_\rho(y)| \geq \varepsilon |\Omega_\rho(y)|$$

for some $\Omega_\rho(y)$ *with* $\rho < R$, *then we have*

$$K_r \cap \Omega_\rho(y) \subset \left\{x \in K_r : M(|Du|)(x) > 1\right\} \cap \left\{x \in K_r : (M_1 f)^{\frac{1}{p-1}}(x) > \delta\right\}.$$

For any weak solution u to problem (1.1) and for fixed $K_r = \Omega_r(y_0, \tau_0) \cap Q$ we set

$$E = \big\{x \in K_r : M(|Du|)(x) > \Lambda\big\} \tag{5.9}$$

and

$$F = \big\{x \in K_r : M(|Du|)(x) > 1\big\} \cap \big\{x \in K_r : (M_1 f)^{\frac{1}{p-1}}(x) > \delta\big\} \tag{5.10}$$

with Λ and δ are as in Proposition 5.2. Now set

$$\varepsilon_1 = \Big(\frac{10}{1-\delta}\Big)^n \varepsilon.$$

By an iteration argument we can derive the following decay of the size the level sets of maximal operator for the spatial gradient of weak solution.

Theorem 5.1 *Assume that* $u \in \mathring{W}^1_p(\Omega)$ *is a weak solution to problem* (1.1), $p \in (1, \infty)$, **a** *satisfies the small* $(\delta, R_0) - BMO$ *condition and* Ω *is* (δ, R_0)*-Reifenberg flat, where* k *be a positive integer. For each* $y \in K_r$*, if the inequality*

$$|E \cap \Omega_1(y)| < \varepsilon |\Omega_1(y)| \tag{5.11}$$

holds with E *as in* (5.9)*, then for each* $k = 1, 2, \dots$ *the following inequality holds*

$$\big|\big\{x \in K_r : M(|Du|)(x) > \Lambda^k\big\}\big| \le \sum_{i=1}^{k} \varepsilon_1^k \big|\big\{x \in K_r : (M_1 f)^{\frac{1}{p-1}}(x) > \delta \Lambda^{k-i}\big\}\big| + \varepsilon_1^k \big|\big\{x \in K_r : M(|Du|)(x) > 1\big\}\big|. \tag{5.12}$$

If the point $y_0 \in \Omega$ is arbitrary choice, then estimate (5.12) holds locally for any $K_r = \Omega_r(y) \cap \Omega$ with $y \in \Omega$.

We can now give the proof of our main result, Theorem 1.1.

Proof of Theorem 1.1. Under a normalization in Lemma 2.4 giving the scaling invariance property of problem (1.1) we assume the norm of $M_1 f$ is small enough. In fact, normalizing u to

$$u_\lambda = \frac{u}{\lambda} \quad \text{and} \quad f_\lambda = \frac{f}{\lambda} \quad \text{for} \quad \lambda = \frac{1}{\delta} \|M_1 f\|_{M_{q,\varphi_2}(\Omega)},$$

we have $\|M_1 f_\lambda\|_{M_{q,\varphi_2}(\Omega)} = \delta$.

Because of the properties of the maximal function (see Corollary 2.1), it is enough to get

$$\|M(|Du|)\|_{M_{q,\varphi_2}(\Omega)} \le C.$$

Let E be the set defined in (5.9) and corresponding to the solution u_λ. For each $y \in E$

$$\frac{|E \cap \Omega_1(y)|}{|\Omega_1(y)|} \le C|E| = C|\{x \in K_r : M(|Du_\lambda|)(x) > \Lambda\}| \le C \int_\Omega |Du_\lambda(x)| dx.$$

From L_q estimate $\int_\Omega |Du|^q dx \le C\|f\|_{L_1(\Omega)}$, for any $q \in \big[1, 1 + \frac{1}{n}\big]$, where constant C depends only on n, γ, q, Ω (see [3]) we get

$$\int_\Omega |Du_\lambda(x)| dx \le C \int_\Omega M_1 f_\lambda(x) dx \le C \int_\Omega |f_\lambda(x)| dx.$$

Hence it follows that

$$\int_{\Omega} |Du_\lambda(x)|dx \leq C\int_{\Omega} M_1 f_\lambda(x)dx \leq C\|M_1 f_\lambda\|_{M_{q,\varphi_2}(\Omega)} \leq C\delta.$$

Taking δ small enough, we have

$$\frac{|E \cap \Omega_1(y)|}{|\Omega_1(y)|} \leq C\delta < \varepsilon \tag{5.13}$$

which ensures (5.11). On the other hand from Lemma 2.3 we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\Lambda^{qk}|\{x \in K_r : (M_1 f_\lambda)^{\frac{1}{p-1}}(x) > \delta\Lambda^k\}|}{\varphi_2(\Omega_r(y))^q\, r^n} \\
&\leq C\delta^{-q}\|(M_1 f_\lambda)^{\frac{1}{p-1}}\|^q_{M_{q,\varphi_2}(\Omega)} \leq C,
\end{aligned} \tag{5.14}$$

where $C > 0$ depends on q, n, Λ_1, γ.

In view of (5.13), we employ Theorem 5.1 to Du_λ and f_λ, to have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \Lambda^{qk}\frac{|\{x \in K_r : M(|Du_\lambda|)(x) > \Lambda^k\}|}{\varphi_2(\Omega_r(y))^q\, r^n} \\
&\leq C\sum_{k=1}^{\infty} \Lambda^{qk}\frac{|\{x \in K_r : (M_1 f_\lambda)^{\frac{1}{p-1}}(x) > \delta\Lambda^{k-i}\}|}{\varphi_2(\Omega_r(y))^q\, r^n} \\
&+ \varepsilon_1^k\,\frac{|\{x \in K_r : M(|Du_\lambda|)(x) > 1\}|}{\varphi_2(\Omega_r(y))^q\, r^n} \\
&= \sum_{i=1}^{\infty}(\Lambda^q\varepsilon_1)^i\Big[\sum_{k=1}^{\infty}\Lambda^{q(k-i)}\frac{|\{x \in K_r : (M_1 f_\lambda)^{\frac{1}{p-1}}(x) > \delta\Lambda^{k-i}\}|}{\varphi_2(\Omega_r(y))^q\, r^n} \\
&+ \sum_{k=1}^{\infty}(\Lambda^q\varepsilon_1)^k\frac{|\{x \in K_r : M(|Du_\lambda|)(x) > 1\}|}{\varphi_2(\Omega_r(y))^q\, r^n}\Big] \\
&\leq C\Big(1 + \frac{|K_r|}{\varphi_2(\Omega_r(y))^q r^n}\Big)\sum_{k=1}^{\infty}(\Lambda^q\varepsilon_1)^k \leq C\sum_{k=1}^{\infty}(\Lambda^q\varepsilon_1)^k,
\end{aligned}$$

where $C = C(q, n, \varphi, \Lambda, \gamma) > 0$. By using (5.3) and due to the monotonicity condition (2.4) we get

$$\sup_{y\in\Omega, r>0} \frac{|K_r|}{\varphi_2(\Omega_r(y))^q\, r^n} = C < \infty.$$

We select $\varepsilon > 0$ in ε_1, and take $\delta(q, n, \Lambda_1, \Lambda_2, \gamma) > 0$ sufficiently small enough to satisfy

$$\Lambda^q\varepsilon_1 = (10)^n\Lambda^q\frac{\varepsilon}{(1-\delta)^n} < 1.$$

Consequently for some constant $C(q, n, \Lambda_1, \gamma) > 0$, we get

$$\sum_{k=1}^{\infty} \frac{\Lambda^{qk}|\{x \in K_r : M(|Du_\lambda|)(x) > \Lambda^k\}|}{\varphi_2(\Omega_r(y))^q\, r^n} \leq C.$$

Taking the supremum over $y \in \Omega$ and $r > 0$ in the above estimate and use Lemmas 2.1 and 2.3 we get that

$$Du_\lambda \in M_{q,\varphi_2}(\Omega)$$

with the estimate

$$\|Du_\lambda\|_{M_{q,\varphi_2}(\Omega)} \leq C.$$

Returning from u_λ back to u, we finally obtain

$$Du \in M_{q,\varphi_2}(\Omega)$$

with the estimate

$$\|Du\|_{M_{q,\varphi_2}(\Omega)} \leq C\|(M_1 f)^{\frac{1}{p-1}}\|_{M_{q,\varphi_2}(\Omega)}. \tag{5.15}$$

Now if we use estimate (2.3) in Corollary 2.1 about the boundedness of the fractional maximal operator in generalized Morrey spaces, then we get

$$\begin{aligned}\|(M_1 f)^{\frac{1}{p-1}}\|_{M_{q,\varphi_2}(\Omega)} &= \|M_1 f\|_{M_{\frac{q}{p-1},\varphi_2}(\Omega)} \\ &\lesssim \|f\|_{M_{\frac{q_1}{p-1},\varphi}(\Omega)} = \||f|^{\frac{1}{p-1}}\|_{M_{q_1,\varphi}(\Omega)}.\end{aligned} \tag{5.16}$$

Then from 5.15 and 5.16 we have

$$\|Du\|_{M_{q,\varphi_2}(\Omega)} \lesssim \||f|^{\frac{1}{p-1}}\|_{M_{q_1,\varphi}(\Omega)}.$$

Thus the proof of Theorem 1.1 is completed.

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