Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **43** (4), 54-64 (2023).
https://doi.org/10.30546/2617-7900.43.4.2023.54

Weak solvability of the first boundary value problem for nonuniformly and strongly degenerate second-order elliptic-parabolic equations in divergent form

Narmin R. Amanova

Received: 13.10.2022 / Revised: 28.06.2023 / Accepted: 07.07.2023

Abstract. *The paper considers the first boundary value problem for a non-uniformly and strongly degenerate second-order elliptic-parabolic equation in divergent form. A Friedrichs-type inequality is proved and conditions are found under which this problem is uniquely generalized solvable in a weighted anisotropic Sobolev space.*

Keywords. elliptic-parabolic equation, non-uniformly and strongly degenerate, Sobolev space.

Mathematics Subject Classification (2010): 35D30, 35J15, 35J70

1 Introduction

Let $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ be Euclidean spaces of points $x = (x_1, ..., x_n)$ and $(x,t) = (x_1, ..., x_n, t)$, respectively, $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^2$, $0 \in \overline{\Omega}, Q_T$ is a cylinder $\Omega \times (-T, 0)$, where , $n \geq 1$ and $T > 0$ is a constant. Denote

$$Q_0 = \left\{(x,t) : x \in \overline{\Omega}, t = -T\right\}, S_T = \partial\Omega \times [-T, 0] \text{ and } \Gamma(Q_T) = Q_0 \cup S_T.$$

Consider in Q_T the first boundary value problem

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial t}\left(\varphi(-t)\frac{\partial u}{\partial t}\right) - \frac{\partial u}{\partial t} = f(x,t), \tag{1.1}$$

$$u|_{\Gamma(Q_T)} = 0 \tag{1.2}$$

assuming that $f(x,t) \in L_2(Q_T)$, $\|a_{ij}(x,t)\| -$ is a real symmetric matrix with measurable elements in Q_T, and for all $(x,t) \in Q_T$ and $\xi \in E_n$ the condition

$$\gamma\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq \gamma^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2, \tag{1.3}$$

is fulfilled, and $\varphi(z)$ is a continuous non-negative non-decreasing function on $[-T, 0]$ and for sufficiently small $z > 0$

$$\varphi(0) = 0, \varphi(z) \geq 0,\ \varphi'(z) \geq 0, \varphi'(0) = 0, \varphi''(z) \geq 0,\ \varphi'(z) \geq \varphi(z)\varphi''(z). \tag{1.4}$$

N.R. Amanova
SABIS Sun International School
Baku Zigh Highway, 22 km, towards H. Aliyev Int. Airport, Dramland, Baku, Azerbaijan
E-mail: amanova.n93@gmail.com
namanova@ssisbaku.sabis.net

Here $\gamma \in (0,1]$ is a constant, and the functions $\lambda_i(x,t), i = 1,...,n$ are finite almost everywhere in Q_T and are positive.

Let $\delta > 0$ be a constant. We impose the following conditions on the functions $\lambda_i(x,t),\ i = 1,...,n$:

$$\lambda_i(x,t) \in L_1(Q_T),\ \ \lambda_i^{-1}(x,t) \in L_{n/2}(\Omega),\ \text{if}\ \ n \ge 2; \tag{1.5}$$

$$\lambda_1^{-1}(x_1,t) \in L_{1+\delta}(\Omega),\ \ \text{if}\ \ n = 1. \tag{1.6}$$

The aim of this paper is to find conditions on the functions $f(x,t), \varphi(z)$ and $\lambda_i(x,t),\ i = 1,...,n$ for which problem (1.1)-(1.2) is uniquely generalized solvable in the corresponding Sobolev space. We find conditions on the function $\varphi(z)$ under which the properties of solutions of problem (1.1)-(1.2) are identical to the properties of solutions of this problem for non-uniformly degenerate second-order parabolic equations (for $\varphi \equiv 0$) (see e.g. [24]).

Initially, the theory of degenerate elliptic-parabolic equations was studied in the classical work of Keldysh [1], in which, in the case of one space variable and a power type of the function $\varphi(z)$, the correct formulations of boundary value problems for second-order elliptic-parabolic equations were indicated. The results of Keldysh found their development in the work of Fichera [2], in which the weak solvability of the first boundary value problem for second-order elliptic-parabolic equations of a non-divergence structure with smooth coefficients was studied. Let us note the works of Petrushko [3–7], who studied the problems of weak solvability of boundary value problems and the behavior on the boundary of solutions of second-order elliptic-parabolic equations with a divergent structure. As for similar questions for elliptic-parabolic equations of non-divergence structure with smooth coefficients, we point out the works [8–12]. We also note the works [13–18], where the existence and uniqueness of the solution of the first boundary value problem for second-order elliptic and parabolic equations with discontinuous coefficients and Cordes-type conditions are proved. A more complete survey of results on the solvability of boundary value problems for elliptic-parabolic equations can be found in [19–23].

Let us accept some notation and definitions. We will say that $u(x,t) \in A(Q_T)$, if there exists a compact $\overline{K}_u \subset \Omega$ such that $\operatorname{supp} u(x,t) \subset K_u \times [-T,0]$, $u(x,t) \in C^\infty(\overline{Q}_T)$, $u|_{t=-T} = 0$. Denote by $W^{1,1}_{2,\lambda}(Q_T)$, $W^{1,1}_{2,\lambda,\varphi}(Q_T)$ and $W^{2,2}_{2,\lambda,\varphi}(Q_T)$ Banach spaces of measurable functions defined on Q_T, for which the norms

$$\|u\|_{W^{1,1}_{2,\lambda}(Q_T)} = \Big(\int_{Q_T}\left(u^2 + \sum_{i=1}^{n}\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right)dxdt\Big)^{1/2},$$

$$\|u\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)} = \Big(\int_{\Omega}u^2(x,0)dx+\int_{Q_T}\Big(\sum_{i=1}^{n}\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2+\varphi(-t)\left(\frac{\partial u}{\partial t}\right)^2\Big)dxdt\Big)^{1/2},$$

and

$$\|u\|_{W^{2,2}_{2,\lambda,\varphi}(Q_T)} = \Big(\int\limits_{Q_T}\Big(u^2 + \sum_{i=1}^{n}\lambda_i(x,t)\Big(\frac{\partial u}{\partial x_i}\Big)^2 + \sum_{i,j=1}^{n}\lambda_i(x,t)\lambda_j(x,t)\Big(\frac{\partial^2 u}{\partial x_i\partial x_j}\Big)^2$$

$$+\left(\frac{\partial u}{\partial t}\right)^2 + \varphi^2(-t)\left(\frac{\partial^2 u}{\partial t^2}\right)^2 + \varphi(-t)\sum_{i=1}^{n}\lambda_i(x,t)\left(\frac{\partial^2 u}{\partial x_i\partial t}\right)^2\Big)dxdt\Big)^{\frac{1}{2}},$$

are finite, respectively, where $\lambda = (\lambda_1(x,t),...,\lambda_n(x,t))$. Let $\overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$, $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ and $\overset{\circ}{W}{}^{2,2}_{2,\lambda,\varphi}(Q_T)$ subspaces of $W^{1,1}_{2,\lambda}(Q_T)$, $W^{1,1}_{2,\lambda,\varphi}(Q_T)$ and $W^{2,2}_{2,\lambda,\varphi}(Q_T)$ are completion of the set of all functions $u(x,t) \in A(Q_T)$ with respect to the norm of the space $W^{1,1}_{2,\lambda}(Q_T)$, $W^{1,1}_{2,\lambda,\varphi}(Q_T)$ and $W^{2,2}_{2,\lambda,\varphi}(Q_T)$, respectively.

The function $u(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ is called a weak solution to problem (1.1)- (1.2) if for the function $v(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$ and $t_1 \in (-T, 0]$ the integral identity

$$\int_{Q_{t_1}} \Big(\sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \varphi(-t) \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} - u \frac{\partial v}{\partial t} \Big) dxdt$$

$$+ \int_{\Omega} u(x,t_1) v(x,t_1) dx = - \int_{Q_{t_1}} f v dxdt, \tag{1.7}$$

is valid, where $Q_{t_1} = \Omega \times (-T, t_1)$. Throughout what follows, the notation $C(\cdots)$ means that the positive constant C depends only on the contents of the brackets.

Theorem 1.1 *Let conditions* (1.5) *and* (1.6) *be satisfied. Then for any function* $u(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_{t_1})$ *and* $t_1 \in (-T, 0]$ *the following inequality*

$$\int\limits_{Q_{t_1}} u^2(x,t) dxdt \le C_{1.1}(\lambda, n, \Omega) \int\limits_{Q_{t_1}} \sum_{i=1}^{n} \lambda_i(x,t) \left(\frac{\partial u}{\partial x_i} \right)^2 dxdt. \tag{1.8}$$

holds.

Proof. Let $n \ge 2$. Obviously, it suffices to prove (1.8) for the function $u \in A(Q_T)$. We will use the following classical embedding theorem (see e.g. [21]): for any function $u(x,t) \in C_0^\infty(\Omega)$ for $1 \le p < n$ the inequality

$$\|u\|_{L_{\frac{pn}{n-p}}(\Omega)} \le C_{1.2}(n,p,\Omega) \, \|\nabla u\|_{L_p(\Omega)}, \tag{1.9}$$

holds. Setting $p = \frac{2n}{n+2}$ in (1.9), we obtain

$$\|u\|_{L_2(\Omega)} \le C_{1.2}(n,\Omega) \, \|\nabla u\|_{L_{\frac{2n}{n+2}}(\Omega)}. \tag{1.10}$$

But on the other hand

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(\Omega)} = \Big(\int_{\Omega} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{\frac{2n}{n+2}} \Big)^{\frac{2+n}{2n}}$$

$$= \Big(\int_{\Omega} \sum_{i=1}^{n} \lambda_i^{-q}(x,t) \lambda_i^{q}(x,t) \left| \frac{\partial u}{\partial x_i} \right|^{\frac{2n}{n+2}} dx \Big)^{\frac{n+2}{2n}}$$

$$\le \Big(\sum_{i=1}^{n} \left(\int_{\Omega} \lambda_i^{qS}(x,t) \left| \frac{\partial u}{\partial x_i} \right|^{\frac{2nS}{n+2}} dx \right)^{\frac{1}{S}} \left(\int_{\Omega} \lambda_i^{-qS'}(x,t) dx \right)^{\frac{1}{S'}} \Big)^{\frac{n+2}{2n}},$$

where $q > 0$ and $S > 1$ are arbitrary numbers and, $S' = \frac{S}{S-1}$. Let us now set $S = \frac{n+2}{n}$, $q = \frac{n}{n+2}$. Then $S' = \frac{n+2}{2}$ and therefore

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(\Omega)} \le \Big(\sum_{i=1}^{n} \Big(\int_{\Omega} \lambda_i(x,t) \left| \frac{\partial u}{\partial x_i} \right|^2 dx \Big)^{\frac{n}{n+2}} \Big(\int_{\Omega} \lambda_i^{-n/2}(x,t) dx \Big)^{2/(n+2)} \Big)^{\frac{n+2}{2n}}. \tag{1.11}$$

By virtue of condition (1.5), we have

$$\Big(\int_\Omega \lambda_i^{-n/2}(x,t)dx\Big)^{1/n} \leq C_{1.3}(\lambda, n, \Omega), i=1,...,n.$$

Thus, from (1.11) we conclude that

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(\Omega)} \leq C_{1.4}(\lambda, n, \Omega)\Big(\sum_{i=1}^n \int_\Omega \lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2 dx\Big)^{1/2}. \qquad (1.12)$$

Then from (1.10) and (1.12) it follows

$$\Big(\int_\Omega u^2(x,t)dx\Big)^{1/2} \leq C_{1.2}\cdot C_{1.4}\Big(\sum_{i=1}^n \int_\Omega \lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2 dx\Big)^{1/2}.$$

We integrate the last inequality with respect to t from $-T$ to t_1. Thus, the required estimate (1.8) follows from this expression if $n \geq 2$.

Let, now $n=1$. We will use the following embedding theorem (see e.g. [21]):for any function $u(x,t) \in C_0^\infty(\Omega)$ for $1<p<2$ the inequality

$$\sup_\Omega |u(x_1,t)| \leq C_{1.5}(p,\Omega)\left\|\frac{\partial u}{\partial x_1}\right\|_{L_p(\Omega)},$$

holds.

Then

$$\Big(\int_\Omega u^2(x_1,t)dx_1\Big)^{1/2} \leq \sup_\Omega |u(x_1,t)| \leq C_{1.5}\Big(\int_\Omega \lambda_1^{-p/2}(x_1,t)\lambda_1^{p/2}\left|\frac{\partial u(x_1,t)}{\partial x_1}\right|^p dx_1\Big)^{1/p}$$

$$\leq C_{1.5}\Big(\int_\Omega \lambda_1(x_1,t)\left(\frac{\partial u}{\partial x_1}\right)^2 dx_1\Big)^{1/2}\Big(\int_\Omega \lambda_1^{-p/(2-p)}(x_1,t)dx_1\Big)^{\frac{2-p}{2p}}.$$

Let $\frac{p}{2-p}=1+\delta$, then $p=\frac{1+\delta}{1+\delta/2}$ and if $n=1$ then the required estimate (1.8) is proved. Theorem 1.1 is proved.

Theorem 1.2 *Let the coefficients of the operator L satisfying conditions* (1.3)-(1.6) *be defined in the cylindrical region $Q_T \subset \mathbb{R}^{n+1}$. Then the first boundary value problem* (1.1)-(1.2) *is uniquely generalized solvable in the space $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ for any $f(x,t) \in L_2(Q_T)$. Moreover, for the solution $u(x,t)$ the following estimate is true:*

$$\|u\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)} \leq C_{1.6}(\gamma,\lambda,n,\Omega)\,\|f\|_{L_2(Q_T)}. \qquad (1.13)$$

Proof. Suppose $\partial\Omega \in C^2$. Let us introduce the following notation for natural numbers $m, (x,t) \in Q_T$ and $i=1,...,n$:

$$\lambda_i^m(x,t) = \begin{cases} \frac{1}{m}, \text{ if } \lambda_i(x,t) < \frac{1}{m}; \\ \lambda_i(x,t), \text{ if } \frac{1}{m} \leq \lambda_i(x,t) \leq m, \\ m, \text{if } \lambda_i(x,t) > m. \end{cases}$$

Let $\left\|a_{ij}^m(x,t)\right\|$ be a real symmetric matrix with measurable elements in Q_T and for $i,j=1,...,n$ as $m\to\infty$ in Q_T $a_{ij}^m(x,t)\to a_{ij}(x,t)$, and for $(x,t)\in Q_T$ and $\xi \in E_n$

$$\gamma\sum_{i=1}^n \lambda_i^m(x,t)\xi_i^2 \leq \sum_{i,j=1}^n a_{ij}^m(x,t)\xi_i\xi_j \leq \gamma^{-1}\sum_{i=1}^n \lambda_i^m(x,t)\xi_i^2.$$

Denote by $(a_{ij})_h$ the Friedrichs averaging of the function $a_{ij}^m(x,t)$ with the parameter $h>0$. Further, by $\lambda_i^h(x,t)$ and $u^h(x,t)$ we denote the Friedrichs averaging of the function $\lambda_i^m(x,t)$ and $u^m(x,t)$ with parameter $h>0$, respectively.

Consider for $h>0$ the family of the following first boundary value problems

$$L^h u^h = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left((a_{ij})_h \frac{\partial u^h}{\partial x_j}\right) + \frac{\partial}{\partial t}\left(\varphi(-t)\frac{\partial u^h}{\partial t}\right) - \frac{\partial u^h}{\partial t} = f(x,t), \tag{1.14}$$

$$u^h\Big|_{\Gamma(Q_T)} = 0, \tag{1.15}$$

where φ satisfies conditions (1.4).

It is clear that $(a_{ij})_h \in C^\infty(\overline{Q}_T)$, and for all $h>0$ with respect to $(a_{ij})_h$ a condition of type (1.3) with constant γ is satisfied. Then, according to [23], there exists a uniquely strong solution $u^h(x,t) \in \overset{\circ}{W}{}^{2,2}_{2,\lambda,\varphi}(Q_T)$ of problem (1.14)-(1.15). It is obvious that $u^h(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$.

We multiply both sides of equation (1.14) by the functions $\upsilon(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$, and then integrate it over the domain Q_T:

$$\int_{Q_T} L^h u^h \upsilon dxdt = \int_{Q_T} f\upsilon dxdt. \tag{1.16}$$

Since $u^h \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$, we can substitute $\upsilon = u^h$ in (1.16). Then we have

$$\int\limits_{Q_T} \sum_{i,j=1}^{n} (a_{ij})_h \frac{\partial u^h}{\partial x_i}\frac{\partial u^h}{\partial x_j} dxdt - \int\limits_{Q_T} u^h \frac{\partial u^h}{\partial t} dxdt$$

$$+\int\limits_{\Omega} (u^h(x,0))^2 dx + \int\limits_{Q_T} \varphi(-t)\left(\frac{\partial u^h}{\partial t}\right)^2 dxdt = -\int\limits_{Q_T} f u^h dxdt. \tag{1.17}$$

On the other hand, it follows from (1.3) that

$$\gamma \int\limits_{Q_T} \sum_{i=1}^{n} \lambda_i^h(x,t)\left(\frac{\partial u^h}{\partial x_i}\right)^2 dxdt \le \int\limits_{Q_T} \sum_{i,j=1}^{n} (a_{ij})_h \frac{\partial u^h}{\partial x_i}\frac{\partial u^h}{\partial x_j} dxdt.$$

Let us represent the second term on the left-hand side of equality (1.17) as follows

$$\int_{Q_T} u^h \cdot \frac{\partial u^h}{\partial t} dxdt = \frac{1}{2}\int_{\Omega}\Big(u^h(x,t)\Big)^2 dx\Big|_{t=-T}^{t=0} = \frac{1}{2}\int_{\Omega}\Big(u^h(x,0)\Big)^2 dx.$$

As a result we have the following inequality

$$\gamma \int_{Q_T} \sum_{i=1}^{n} \lambda_i^h(x,t)\left(\frac{\partial u^h}{\partial x_i}\right)^2 dxdt + \frac{1}{2}\int_{\Omega}\Big(u^h(x,0)\Big)^2 dx$$

$$+\int_{Q_T} \varphi(-t)\left(\frac{\partial u^h}{\partial t}\right)^2 dxdt \le \frac{\sigma}{2}\int_{Q_T}\Big(u^h\Big)^2 dxdt + \frac{1}{2\sigma}\int_{Q_T} f^2 dxdt,$$

where $\sigma > 0$ will be chosen later.

By inequality (1.8), we have

$$\int_{Q_T} (u^h)^2 dxdt \le C_{1.7}(\lambda, n, \Omega) \int_{Q_T} \sum_{i=1}^{n} \lambda_i^h(x,t) \left(\frac{\partial u^h}{\partial x_i}\right)^2 dxdt.$$

Thus, the number σ can be chosen so small that the inequality

$$\left\| u^h \right\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)} \le C_{1.8}(\lambda, n, \Omega) \left\| f \right\|_{L_2(Q_T)}, \tag{1.18}$$

is fulfilled. It follows from (1.18) that the sequence $\left\{u^h(x,t)\right\}$ is strongly bounded in $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$. Thus, this sequence is weakly compact in $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$. In other words, there is a subsequence $\left\{u^{h_k}(x,t)\right\}$, $h_k \to 0$ for $k \to \infty$ and the function $u(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ such that for any $\psi(x,t) \in C_0^\infty(\overline{Q_T})$

$$\lim_{k\to\infty} \left(Lu^{h_k}, \psi\right) = (Lu, \psi). \tag{1.19}$$

Moreover, the function $u(x,t)$ satisfies the estimate

$$\|u\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)} \le C_{1.8} \left\| f \right\|_{L_2(Q_T)}.$$

Let us now show that the function $u(x,t)$ satisfies equality (1.7) for any $\upsilon(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$. Since the function $u^{h_k} \in \overset{\circ}{W}{}^{2,2}_{2,\lambda,\varphi}(Q_T)$ is a weak solution of equation (1.14) (see [22]), then for any $\upsilon(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$ and $t_1 \in (-T, 0]$ the following equality holds

$$\int_{Q_{t_1}} \Big(\sum_{i,j=1}^{n} (a_{i,j})_{h_k} \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} + \varphi(-t) \frac{\partial u^{h_k}}{\partial t} \frac{\partial \upsilon}{\partial t} - u^{h_k} \frac{\partial \upsilon}{\partial t}\Big) dxdt$$

$$+ \int_{\Omega} u^{h_k}(x,t_1) \upsilon(x,t_1) dx = - \int\limits_{Q_{t_1}} f \upsilon dxdt. \tag{1.20}$$

Hence if we pass to the limit as $k \to \infty$, then by virtue of (1.19) it remains to prove that

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} (a_{ij})_{h_k} \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt \to \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt,$$

for $k \to \infty$. We have

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} (a_{ij})_{h_k} \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt = \int_{Q_{t_1}} \sum_{i,j=1}^{n} ((a_{ij})_{h_k} - a_{ij}) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt$$

$$+ \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt. \tag{1.21}$$

The first term on the right-hand side of equality (1.21) tends to zero as $k \to \infty$. Indeed

$$\Big| \int_{Q_{t_1}} \sum_{i,j=1}^{n} \left((a_{ij})_{h_k} - a_{ij}\right) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt \Big|$$

$$\leq \int_{Q_{t_1}} \sum_{i,j=1}^{n} \left| ((a_{ij})_{h_k} - a_{ij}) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} \right| \sqrt{\lambda_j(x,t)} \sqrt{\lambda_j^{-1}(x,t)} dxdt$$

$$\leq \sum_{i,j=1}^{n} \sup_{Q_{t_1}} |(a_{ij})_{h_k} - a_{ij}| \cdot \left(\int_{Q_{t_1}} \lambda_i^{-1}(x,t) \left(\frac{\partial \upsilon}{\partial x_i} \right)^2 dxdt \right)^{\frac{1}{2}}$$

$$\times \left(\int_{Q_{t_1}} \lambda_j(x,t) \left(\frac{\partial u^{h_k}}{\partial x_j} \right)^2 dxdt \right)^{\frac{1}{2}} \to 0, k \to \infty$$

due to estimate (1.18).

The second term on the right-hand side of equality (1.21) can be represented as

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt = \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \left(\frac{\partial u^{h_k}}{\partial x_j} - \frac{\partial u}{\partial x_j} \right) \frac{\partial \upsilon}{\partial x_i} dxdt$$

$$+ \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt.$$

We have

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \left(\frac{\partial u^{h_k}}{\partial x_j} - \frac{\partial u}{\partial x_j} \right) \frac{\partial \upsilon}{\partial x_i} dxdt$$

$$= \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial}{\partial x_j} (u^{h_k} - u) \frac{\partial \upsilon}{\partial x_i} dxdt \to 0,\ k \to \infty$$

due to the weak convergence of the sequence $\{u^{h_k}(x,t)\}$ to the function $u(x,t)$ in space $W_{2,\lambda,\varphi}^{1,1}(Q_T)$.

Consequently

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} (a_{ij}(x,t))_{h_k} \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt \to \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt,\ k \to \infty.$$

Thus, the existence of a weak solution to problem (1.1)-(1.2) for $\partial \Omega \in C^2$ is proved.

Now let $\partial \Omega \overline{\in} C^2$. Consider a sequence of domains $\Omega_m, m = 1, 2, ...,$ for which $\partial \Omega_m \in C^2$; $\overline{\Omega}_m \subset \Omega_{m+1} \subset \overline{\Omega}_{m+1} \subset \Omega,\ \lim\limits_{m \to \infty} \Omega_m = \Omega$. Assume $Q_T^m = \Omega_m \times (-T, 0)$.

Let u^m- be the solution of the boundary value problem

$$Lu^m = f(x,t), (x,t) \in Q_T^m; \quad u^m|_{\Gamma(Q_T^m)} = 0.$$

By what was proved above, for every natural number m such a solution exists, and

$$\|u^m\|_{W_{2,\lambda,\varphi}^{1,1}(Q_T^m)} \leq C_{1.10} \|f\|_{L_2(Q_T^m)},$$

holds, where the constant $C_{1.10}$ is independent of m.

Let us extend the function u^m by zero in $Q_T \backslash Q_T^m$ and denote the extended function again by u^m. It is clear that $u^m \in \overset{\circ}{W}{}_{2,\lambda,\varphi}^{1,1}(Q_T)$ and

$$\|u^m\|_{W_{2,\lambda,\varphi}^{1,1}(Q_T)} \leq C_{1.10} \|f\|_{L_2(Q_T)}.$$

Thus the sequence $\{u^m\}$ is strongly bounded in $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ and therefore, it is weakly compact in the same space, i.e., there is a function $u(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ and a sequence $\{m_k\}, m_k \to 0$ as $k \to \infty$ such that the corresponding sequence $\{u^{m_k}(x,t)\}$ weakly converges to the function $u(x,t)$ in $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ as $k \to \infty$. It remains to show that $u(x,t)$ is a solution of the equation $Lu = f$. This is done quite similarly to the previous one.

Let us now prove the uniqueness of the solution of the problem (1.1)-(1.2). To do this, it suffices to prove that the homogeneous boundary value problem $Lu = 0$, $u|_{\Gamma(Q_T)} = 0$ has only the zero solution.

In equality (1.7) we set $f = 0$, and then as $\upsilon(x,t)$ we take the function

$$\upsilon_{(\overline{h})}(x,t) = \frac{1}{h}\int_{t-h}^{t} \upsilon(x,\tau)d\tau, \tag{1.22}$$

where $\upsilon(x,t)$ is an arbitrary element of $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T^{-h})$, equal to zero for $t \geq -h$ and for $t \leq -T$ (see [21]), and fix $h > 0$. Here $Q_T^{-h} = \Omega \times (-h, 0)$. Therefore, we have

$$\int_{Q_{-h}} \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial(\upsilon_{(\overline{h})})}{\partial x_i}dxdt - \int_{Q_{-h}} u\frac{\partial(\upsilon_{(\overline{h})})}{\partial t}dxdt$$

$$+\int_{Q_{-h}} \varphi(-t)\frac{\partial u}{\partial t}\frac{\partial(\upsilon_{(\overline{h})})}{\partial t}dxdt = 0. \tag{1.23}$$

In all terms of equality (1.23), we transfer the averages $(\cdot)_{\overline{h}}$ from υ by the factors in front of it, in addition, in the second term we will integrate by parts over t. Then we obtain

$$\int_{Q_{-h}} \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)_{(h)}\frac{\partial \upsilon}{\partial x_i}dxdt + \int_{Q_{-h}} \frac{\partial(u_{(h)})\upsilon}{\partial t}dxdt$$

$$+\int_{Q_{-h}} \left(\varphi(-t)\frac{\partial u}{\partial t}\right)_{(h)}\frac{\partial \upsilon}{\partial t}dxdt = 0, \tag{1.24}$$

where

$$u_{(h)}(x,t) = \frac{1}{h}\int_{t}^{t+h} u(x,\tau)d\tau.$$

We have

$$\frac{\partial(u_{(h)})}{\partial t} = \frac{\partial}{\partial t}\Big(\frac{1}{h}\int_{t}^{t+h} u(x,\tau)d\tau\Big) = \frac{1}{h}(u(x,t+h) - u(x,t)).$$

Consequently, $u_{(h)} \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$. Therefore, in equality (1.24), instead of υ we can take the function $u_{(h)}$. Then

$$\int\limits_{Q_{-h}} \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)_{(h)}\frac{\partial(u_{(h)})}{\partial x_i}dxdt + \int\limits_{Q_{-h}} \frac{\partial(u_{(h)})u_{(h)}}{\partial t}dxdt$$

$$+\int\limits_{Q_{-h}} \left(\varphi(-t)\frac{\partial u}{\partial t}\right)_{(h)}\left(\frac{\partial u}{\partial t}\right)_{(h)}dxdt = 0.$$

Since
$$\int_{Q_{-h}} \frac{\partial(u_{(h)})u_{(h)}}{\partial t}dxdt = \frac{1}{2}\int_{\Omega}\left(u_{(h)}(x,0)\right)^2 dx \geq 0,$$
then
$$\int\limits_{Q_{-h}} \sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)_{(h)}\frac{\partial(u_{(h)})}{\partial x_i}dxdt + \int\limits_{Q_{-h}}\left(\varphi(-t)\frac{\partial u}{\partial t}\right)_{(h)}\left(\frac{\partial u}{\partial t}\right)_{(h)}dxdt \leq 0.$$

Fix an arbitrary $h_0 \in (-T,0)$. Then in the previous inequality the domain Q_{-h} can be replaced by the domain Q_{-h_0}, where $h \leq h_0$. Thus
$$\int_{Q_{-h_0}} \sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)_{(h)}\frac{\partial(u_h)}{\partial x_i}dxdt + \int_{Q_{-h_0}}\left(\varphi(-t)\frac{\partial u}{\partial t}\right)_{(h)}\left(\frac{\partial u}{\partial t}\right)_{(h)}dxdt \leq 0.$$
Hence as $h \to 0$, we have
$$\int_{Q_{-h_0}} \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}dxdt + \int_{Q_{-h_0}}\varphi(-t)\left(\frac{\partial u}{\partial t}\right)^2 dxdt \leq 0.$$
Taking into account condition (1.3), we have
$$\int_{Q_{-h_0}}\left(\gamma\sum_{i=1}^{n}\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2 + \varphi(-t)\left(\frac{\partial u}{\partial t}\right)^2\right)dxdt \leq 0. \tag{1.25}$$
From (1.25) it follows that $\int\limits_{Q_{-h_0}} \sum\limits_{i=1}^{n}\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2 dxdt = 0.$

On the other hand
$$\int\limits_{Q_{-h_0}} u^2 dxdt \leq C_{1.11}\int\limits_{Q_{-h_0}} \sum_{i=1}^{n}\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2 dxdt = 0.$$
Thus, the function $u(x,t) = 0$ almost everywhere in Q_{-h_0}. Since h_0 is arbitrary, it follows that $u(x,t) = 0$ almost everywhere in Q_T. Theorem 1.2 is proved.

Acknowledgements. We thank the referee(s) for careful reading the paper and useful comments. The author thanks prof. F.I. Mammadov for attention to the work and useful discussions.

References

1. Keldysh, M.V.: *On certain cases of degeneration of equations of elliptic type on the boundry of a domain*, (Russian) Doklady Akad. Nauk SSSR (N.S.) **77** (2), 181–183 (1951).
2. Fichera, G.: *On a unified theory of boundary value problems for elliptic-parabolic equations of second order*, Mathematica **7** (6), 99–122 (1963).
3. Petrushko, I.M.: *Boundary values in L_p, $p > 1$, of solutions of elliptic equations in domains with Lyapunov boundary*, (Russian) Mat. Sb. (N.S.) **120(162)** (4), 569-588 (1983).

4. Petrushko, I.M.: *Boundary values of solutions that degenerate on the boundary of the domain of elliptic equations*, (Russian) Mat. Sb. (N.S.) **136(178)** (2), 241259 (1988).
5. Petrushko, I.M.: *Boundary values of solutions of degenerate elliptic equations of Keldysh type with weak degeneracy*, (Russian) Non-classical equations and equations of mixed type, (Russian) 166-182, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk (1990).
6. Petrushko, I.M.: *On the existence of boundary values of solutions of degenerate elliptic equations*, (Russian) Mat. Sb. **190** (7), 4172 (1999); translation in Sb. Math. **190** (7-8), 973-1004 (1999).
7. Petrushko, I.M.: *Boundary and initial conditions in L_p, $p > 1$, for solutions of parabolic equations*, (Russian) Mat. Sb. (N.S.), **125(167)** (4), 489521 (1984).
8. Gasanov, I.G.: *The behavior near the boundary of solutions of second-order elliptic-parabolic equations*, Izv. Akad. Nauk Azerbadzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **7** (3), 31-40 (1986).
9. Gasanov, I.G.: *Generalized solvability of the first boundary value problem for second order nondivergent elliptic-parabolic equations*, Den. at VINITI, **314-B**, 28 pp. (1988).
10. Alvino, A., Trombetti, G.: *Second order elliptic equation whose coefficients have their first derivatives weakly- L^n*, Ann. Mat. Pura Appl **138** (4), 331-340 (1984).
11. Franciosi, M.: *On an elliptic-parabolic equation with discontinuous coefficients*, (Italian) Boll. Un. Mat. Ital. **2(6)** (1), 63-75 (1983).
12. Franciosi, M.: *An existence and uniqueness theorem for the solution of an elliptic-parabolic equation, with discontinuous coefficients, in nondivergence form*, (Italian) Boll. Un. Mat. Ital. **4(6)** (1), 253-263 (1985).
13. Alkhutov, Yu.A., Mamedov, I.T.: *Some properties of the solutions of the first boundary value problem for parabolic equations with discontinuous coefficients*, (Russian) Dokl. Akad. Nauk SSSR **284** (1) 11-16 (1985).
14. Alkhutov, Yu.A., Mamedov, I.T.: *The first boundary value problem for second-order nondivergence parabolic equations with discontinuous coefficients*, (Russian) Sb. Math. (N.S.) **131(173)** (4), 477500 (1986); translation in Math. USSR-Sb. **59** (2), 471495 (1988).
15. Fiorito, G.: *A contribution to the solution of the Cauchy-Dirichlet problem*, (Italian) Matematiche (Catania) **35** (1-2), 53-60 (1980).
16. Wen, G.C.: *Oblique derivative problems for second order equations of mixed type in multiply connected domains*, Proceedings of the Second ISAAC Congress, Vol. 2 (Fukuoka, 1999), 15391546, Int. Soc. Anal. Appl. Comput., 8, Kluwer Acad. Publ., Dordrecht, (2000).
17. Gadjiev, T.S., Zulfaliyeva, G.M., Mamedova, K.N.: *The solvability degenerate elliptic-parabolic problem with nonlinear boundary conditions*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **42** (1), Mathematics, 120126 (2022).
18. Mamedov, F.I., Amanova, N.R.: *On Harnack's inequality for positive solutions of linear elliptic equations with discontinuous coefficients*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **42** (4), Mathematics, 112124 (2022).
19. Glushko, V.P., Savchenko, Yu.B.: *High order degenerate elliptic equations: spaces, operators, boundary value problems*, (Russian) Mathematical analysis, Vol. 23, 125218, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, (1985).
20. Oleinik, O.A., Radkevich, E.V.: *Second order equations with nonnegative characteristic form*, (Russian) Mathematical analysis, 1969 (Russian), pp. 7252. (errata insert) Akad. Nauk SSSR Vsesojuzn. Inst. Naun. i Tehn. Informacii, Moscow, (1971).
21. Ladyzhenskaya, O.A., Solonnikov, V.A., Uraltseva, N.N.: Linear and quasilinear equations of parabolic type, (Russian) Translated from the Russian by S. Smith Transla-

tions of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. (1968).
22. Salmanova, Sh.Yu.: *Existence and uniqueness of a weak solution of the first boundary value problem for a class of degenerate elliptic-parabolic equations of the second order*, Bulletin of BSU, ser. Phys.-Math. Sciences, **1**, 106-112 (2002).
23. Mamedov, I.T.: *The first boundary value problem for second-order elliptic-parabolic equations with discontinuous coefficients*, J. Math. Sci. (N.Y.) **190** (1), 104134 (2013).
24. Mushtagov, F.M.: *Boundary properties of solutions of the second order parabolic equations with non-uniform degree degeneration*, Proc. Inst. Math. Mech. Acad. Sci. Azerb. **7**, 87-95 (1997).