Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **43** (4), 72-84 (2023).
https://doi.org/10.30546/2617-7900.43.4.2023.72

On the Dirichlet problem for a class of non-uniformly elliptic equations with measure data

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Received: 16.03.2023 / Revised: 28.06.2023 / Accepted: 06.07.2023

Abstract. In this paper, we prove the existence of the solution to the Dirichlet problem for the linear elliptic equation of the type

$$-\frac{\partial}{\partial z_i}\left(a_{ij}\left(z\right)\frac{\partial u}{\partial z_j}\right)=f,\quad z\in\Omega,\quad u\Big|_{\partial\Omega}=0$$

in an open bounded domain $\Omega\subset\mathbb{R}^N$, $N\geq 2$. The coefficients matrix $A=\left\{a_{ij}(z)\right\}_{i,j=1}^N$ satisfies the non-uniform ellipticity condition, meaning that it is positive almost everywhere in Ω and

$$c_1(\omega(x)\left|\xi\right|^2+\left|\eta\right|^2)\leq A(z)\zeta\cdot\zeta\leq c_2(\omega(x)\left|\xi\right|^2+\left|\eta\right|^2)$$

for all, $z\in\Omega,\zeta\subset\mathbb{R}^N$ with $\zeta=(\xi,\eta)$, $\xi\in\mathbb{R}^n$, $\eta\in\mathbb{R}^m$; the positive weight function $\omega\in A_2$ is of Muckenhoupt's class in $\mathbb{R}^n$ and the f is a Radon measure.

Keywords. liner elliptic equation, non-uniformly elliptic equation, degenerate elliptic equation, weak solution, Dirichlet problem, weights, Sobolev spaces.

Mathematics Subject Classification (2010): 2010 Mathematics Subject Classification: 26D10, 35B45, 42B25, 42B37

1 Introduction

This paper relates to the solvability question of the Dirichlet problem for a class of equations with principal part is a second-order divergent structure linear elliptic operator of N variables

$$-\frac{\partial}{\partial z_i}\left(a_{ij}\left(z\right)\frac{\partial u}{\partial z_j}\right)=f,\quad z\in\Omega,\quad u\Big|_{\partial\Omega}=0,\tag{1.1}$$

where the coefficients matrix $A=\left\|a_{ij}\left(z\right)\right\|\left(1\leq i,j\leq N\right)$ is of measurable functions class on an open bounded domain Ω of N -dimensional Euclidean space $\mathbb{R}^N$. Following the usual summation convention, repeated indexes indicate summation from 1 to N. The equation we consider is elliptic in Ω , since the coefficients matrix $A(z)=\{a_{ij}\left(z\right)\}$ is

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positively definite almost everywhere in Ω. Moreover, we assume that there exist positive constants c_1, c_2 such that

$$c_1(\omega(x)\,|\xi|^2+|\eta|^2)\leq A(z)\zeta\cdot\zeta\leq c_2(\omega(x)\,|\xi|^2+|\eta|^2) \tag{1.2}$$

a.e. $z\in\Omega$, with $\forall\zeta=(\xi,\eta)\in\mathbb{R}^N, N=n+m,\ \xi\in\mathbb{R}^n,\ \eta\in\mathbb{R}^m$. Throughout the paper we have taken the $m,n\geq 1$. We use the terminology non-uniformly elliptic equation for (1.1) since condition (1.2) in general does not imply the uniform ellipticity condition:

$$c_1'\,|\zeta|^2\leq A(z)\zeta\cdot\zeta\leq c_2'\,|\zeta|^2.$$

Here the μ in (1.1) is a Radon measure defined on Borelian subsets of Ω, that is a functional $f: C_0(\Omega)\to\mathbb{R}$ satisfying $|\langle f,\varphi\rangle|\leq c\,\|\varphi\|_{C(\Omega)}$ for all continuous functions with compact support in Ω. Also we may assume that $\langle f,\varphi\rangle=\int_\Omega\varphi d\mu$ with $\|f\|=\mathrm{Var}\,\mu$. The weight function $\omega(x)$ is in from $A_p(\mathbb{R}^n)$ -class. The term "weight function" is used to denote a positive measurable function receiving finite positive values a.e. $x\in\mathbb{R}^n$.

We say the positive weight function $\omega:\mathbb{R}^n\to[0,\infty)$ $(n\geq 1)$ is a function of the $A_p(\mathbb{R}^n)$-class (or simply, $A_p(\mathbb{R}^n)$-class for $p>1$ if

$$\left(\int_Q\omega\,dx\right)\left(\int_Q\omega^{-1/(p-1)}\,dx\right)^{p-1}\leq\alpha|Q|^p \tag{1.3}$$

or for $p=1$ if

$$\left(\int_Q\omega\,dx\right)\frac{1}{\inf\limits_{x\in Q}\omega}\leq\alpha|Q|$$

for all the Euclidean balls $Q\subset\mathbb{R}^n$, where $|Q|$ denotes the Lebesgue measure of the ball Q. The constant $\alpha>0$ does not depend on Q.

The model problem for the case is

$$div_x\left(\omega\left(x\right)\nabla_x u\right)+\Delta_y u=f\left(x;y\right),(x;y)\in\Omega, u\Big|_{\partial\Omega}=0,$$

where, $\nabla_x=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},...,\frac{\partial}{\partial x_n}\right),\Delta_y=\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+...+\frac{\partial^2}{\partial y_m^2}$.

Let $\Omega\subset\mathbb{R}^N$ be a domain and $p>1$. Define the weighted Sobelev space $\mathring{W}^{1,p}(\Omega;\omega dz)$. For that, denote the non-uniformly degenerate gradient

$$\nabla_\omega g=\left(\omega^{1/p}\nabla_x g,\nabla_y g\right),\ |\nabla_\omega g|=\left(\omega(x)|\nabla_x g|^p+|\nabla_y g|^p\right)^{1/p},$$

for a function $g(x,y)$ dependent on two variables of the function $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$. Do not mix the non-uniform gradient with partial gradients $\nabla_x g$ and $\nabla_y g$ of the total gradient vector ∇g .

Define the Banach space $W^{1,p}(\Omega;\omega dz)$ a closer of the Lipshitz continuous functions $g:\Omega\to\mathbb{R}$ under the norm

$$\|g\|_{W^{1,p}(\Omega;\omega dz)}=\|g\|_{L_p(\Omega)}+\|\nabla_\omega g\|_{L_p(\Omega)}.$$

For the case $p=2$ and $\omega\in A_2(\mathbb{R}^n)$ we deal with the following Hilbert space. Set an inner production for $\forall u,\varphi\in Lip_0(\Omega)$ as

$$\langle u;\varphi\rangle=(\nabla_\omega u,\nabla_\omega\varphi)=\int\limits_\Omega\left[\omega(x)u_{x_i}\varphi_{x_i}+u_{y_j}\varphi_{y_j}\right]dz$$

and set the corresponding norm $\|u\| := \sqrt{\langle u; u\rangle}$. Closure of $Lip_0(\Omega)$ on this norm is a Hilbert space and denote it $\overset{\circ}{W}{}^{1,2}(\Omega;\omega dz)$, the norm is equivalently $\left(\int_\Omega |\nabla_\omega g|^2\, dz\right)^{1/2}$ (see the Lemma 4.1 below).

A solution of the problem (1.1) is defined using the distributional approach. We say $u \in \overset{\circ}{W}{}^{1,1}(\Omega;\omega dz)$ is a (weak) solution of problem (1.1) if $\forall\varphi \in Lip_0(\Omega)$

$$\int\limits_\Omega a_{ij}(z)\,\frac{\partial u}{\partial z_i}\frac{\partial \varphi}{\partial z_j}dz = \int\limits_\Omega \varphi d\mu. \tag{1.4}$$

A study of non-uniform elliptic equations on the subject of boundary value problem and regularity properties of weak solutions is rising in many applications. This is explained mainly by the development of associated Sobolev and Poincare-type inequality approaches to the area. Many studies are started in this connection in the last 30 years by Franchi, Gutierrez, Wheeden, and Mamedov (see, e.g., [14], [15], [16], [23], [24], [25], [27]). On the study of regularity properties of weak solutions of the non-uniformly elliptic equations, we refer to Trudinger (see, [31]), Wang (see, [32]) and Franchi, Gutierrez, Wheeden (see, e.g., [15], [16]); the last time studies see the works by DiFazio, Fanciullo, Zamboni (see, e.g., [11], [12], [13]). The topic of this paper is a study of the measure data problems for a class of non-uniformly elliptic equations which is new and not much studied. We make a step to make attention to the case. For that, the approach by Bocardo-Gallouet is applied (see, e.g., [2], [3]). Note that the measure data regularity problems for uniformly elliptic equations (also for the nonlinear equations with small terms) were intensively studied in the 80th year by Boccardo, Benilan, Brezis, Gallouet, Kilpelainen, Pierre, Stampacchia, Vazguees, and many other authors (see, e.g., [1], [3], [4], [5], [6], [7], [8], [17], [18], [19], [21], [22], [30], [7] (see, also [26], [28], [29])).

2 A quasi-metric

In this section, we define a quasi-metric in order to propose the Sobolev-type inequality results in Lemma 4.1 below. Following the ideas of (see, [15, Proposition 2.2, 2.2a, 2.2b]) or (see, e.g., [14], [16]) set up the quasi-metric corresponding to the equation (1.1) for $p = 2$.

Define the function $h_x(.) : [0,\infty) \to [0,\infty)$,

$$h_x(t) = t\left(t^{-n}\int_{Q(x,t)} \omega^{-1/(p-1)}(s)ds\right)^{1/p'},\ t > 0,\ x \in \mathbb{R}^n,$$

where $Q(x,t) = \{\xi \in \mathbb{R}^n : |\xi - x| < t\}$; the $\omega : \mathbb{R}^n \to (0,\infty)\}$ is an $A_p(\mathbb{R}^n)$- class function (satisfying the condition (1.3)). Assume that $h_x(0) = 0$ and $h_x(\infty) = \infty$. Consider also the inverse function $h_x^{-1}(.) : [0,\infty \to [0,\infty)$ defined

$$h_x^{-1}(v) = \sup\{\rho > 0 :\ h_x(\rho) \le v\},\quad v > 0.$$

Define the quasi-metric on $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ of points $z = (x,y)$ as following. Define the distance between two points $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$ as

$$\rho(z_1,z_2) = \max\Big\{|x_2 - x_1|,\ h_{x_1}^{-1}(|y_2 - y_1|),\ h_{x_2}^{-1}(|y_2 - y_1|)\Big\}. \tag{2.1}$$

Theorem 2.1 *Let $\omega \in A_p(\mathbb{R}^n)$-class function. The distance* (2.1) *makes $\mathbb{R}^{n+m}$ a homogeneous space $\Big(\mathbb{R}^{n+m}, \rho, \mu\Big)$ using on place of the doubling measure the $dz = dxdy$ or ωdz.*

See the proof e.g. in [23]. In those proofs, the main step is to show the quasi-metric $\rho : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ satisfies the triangle property

$$\rho(z_1, z_2) \le K_0\Big(\rho(z_1, z_3) + \rho(z_2, z_3)\Big) \tag{2.2}$$

with a constant $K_0 \ge 1$ independent from $z_1, z_2, z_3 \in \mathbb{R}^N$ following the ideas e.g. of [15].

Denote $B_{\mathbb{R}}^{z_0}$ the quasi-metric ball $\big\{\zeta \in \mathbb{R}^N : \rho(\zeta, z_0) < R\big\}$ with center $z_0 = (a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ of radius R, also the presentation

$$B(z_0, R) = Q(a, R) \times E\bigg(b, R\Big(R^{-n} \int_{Q(a,R)} \omega^{-1/(p-1)}(s)\, ds\Big)^{1/p'}\bigg), \tag{2.3}$$

valid for it, where

$$E(b, R) = \bigg\{y \in \mathbb{R}^m : \ |y - b| < R\Big(R^{-n} \int_{Q(a,R)} \omega^{-1/(p-1)}(\tau)\, d\tau\Big)^{1/p'}\bigg\},$$

where $Q(a, R) \subset \mathbb{R}^n$ is the n-dimensional Euclidean ball with center a of radius R.

3 Main results

Consider the Dirichlet problem

$$\begin{aligned} -\frac{\partial}{\partial z_i}\left(\sum_{i,j=1}^{N} a_{ij}(z)\, \frac{\partial u}{\partial z_j}\right) &= f(z), \quad z \in \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega, & \end{aligned} \tag{3.1}$$

whenever $f \in L_1(\Omega) \cap W^{-1,2}(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^N$. The following main result is asserted for the problem (3.1).

Theorem 3.1 *Let condition* (1.2) *be satisfied and μ be a Radon measure with support in Ω. Then there exists a weak solution $u(z)$ of the problem* (3.1) *with regularity $u \in \mathring{W}^{1,r}(\Omega; \omega dz)$ for $r \in (1,\ N/(N-1))$.*

We use the following main assertion to prove Theorem 3.1 .

Lemma 3.1 *Let $r \in (1,\ N/(N-1))$ and $f(z) \in L^1(\Omega) \cap W^{-1,2}(\Omega; \omega dz)$ with $\|f\|_{L_1(\Omega)} \le B$ for a $B > 0$. Then there exists $C > 0$ depending on the function ω, the domain Ω and c_1, c_2, α, m, n $(m, n \ge 1)$ such that for the solution $u(z)$ of problem* (3.1) *the estimate*

$$\|u\|_{\mathring{W}^{1,r}(\Omega;\omega dz)} \le C\, \|f\|_{L_1(\Omega)} \tag{3.2}$$

holds.

4 Useful assertion

Also to prove Theorem 3.1, we use several assertions for the problem equation (3.1). Also, we use the next result on non-uniform Sobolev inequality of Fabes-Kenig-Serapioni (see, [10]) and Chanillo-Wheeden (see, [9]) type.

Lemma 4.1 *Let $B(z_0, R)$ be a fixed ball of the quasi-metric* (2.1) *with $z_0 = (a, b) \in \mathbb{R}^N; a \in \mathbb{R}^n,\ b \in \mathbb{R}^{N-n}$. Let $p > 1$ and $\omega : \mathbb{R}^n \to (0, \infty)$ be positive measurable function on $\mathbb{R}^n$ satisfying the Muckenhoupt condition $A_p(\mathbb{R}^n)$ such that for a $\ q \geq p$*

$$\Big(\int_{Q(x,r)} \omega(s) ds \Big/ \int_{Q(x,R)} \omega(s) ds \Big)^{\frac{1}{p} - \frac{m}{p}\left(\frac{1}{p} - \frac{1}{q}\right)} \geq C \left(r/R\right)^{1 - \frac{m(n+p)}{p}\left(\frac{1}{p} - \frac{1}{q}\right)} \tag{4.1}$$

for all $r \in (0, R),\ x \in Q(a, R)$. Then

$$\begin{aligned} \left(\frac{1}{|B(z_0, R)|} \int\limits_{B(z_0,R)} |f(z)|^q dz \right)^{1/q} &\leq CR \left(\fint_{Q(a,R)} \omega^{-1/(p-1)}(s) ds \right)^{1/p'} \\ &\times \left(\frac{1}{\left|B(z_0, R)\right|} \int_{B(z_0,R)} \Big(\omega(x) |\nabla_x f|^p + |\nabla_y f|^p \Big) dz \right)^{1/p} \end{aligned} \tag{4.2}$$

holds for all Lipschitz continuous functions f in the ball $B(z_0, R) \subset \mathbb{R}^N$ vanishing on $\partial B(z_0, R)$ (of Sobolev type) or with zero average $f_{B(z_0,R)} = \int_{B(z_0,R)} f(z) dz = 0$ (of Poincare type); the constant C_0 depends on n, m, q and C, δ from condition $A_p(\mathbb{R}^n)$.

The proof of Lemma 4.1 is obtained from the general results of [24] (or see, [25]) by the way e.g. of [23, Remark 2.1 as $v \equiv 1$].

Throughout the paper, we denote by C, C_1, C_2, C_3 different positive constants which may change their values at each appearance and which may depend on the $c_1, c_2, q, n, m, \alpha, \Omega$ and the weight function $\omega \in A_2(\mathbb{R}^n)$.

5 Lax-Milgram solution

Lemma 5.1 *Let* (1.2) *be satisfied, the positive weight function ω is of $A_2(\mathbb{R}^n)$ -class function and for that the condition* (4.1) *is satisfied by $p = 2$ and $q \geq 2$. Suppose $f(z) \in L^1(\Omega) \cap W^{-1,2}(\Omega; \omega dz)$. Then, there exists a unique solution $u(z)$ of the problem* (3.1) *in space $W^{1,2}(\Omega; \omega dz)$.*

Proof. Apply Lax-Milgram principle (see, [20]) to prove Lemma 5.1. Solution of the problem (3.1) due to the understanding

$$\int\limits_{\Omega} a_{ij}(z) \frac{\partial u}{\partial z_i} \frac{\partial \varphi}{\partial z_j} dz = \int\limits_{\Omega} f(z) \varphi dz, \quad \forall \varphi \in \mathring{W}^{1,2}(\Omega; \omega dz).$$

Set the bilinear form

$$B(u, \varphi) = \int\limits_{\Omega} a_{ij}(z) \frac{\partial u}{\partial z_i} \frac{\partial \varphi}{\partial z_j} dz$$

and establish that the bilinear form is coercive and bounded on space $\overset{\circ}{W}{}^{1,2}(\Omega;\omega dz)$. Using (1.2) and that,

$$\begin{aligned} |B(u,\varphi)| &= \left| \int_{\Omega} a_{ij}(z) \frac{\partial u}{\partial z_i} \frac{\partial \varphi}{\partial z_j} dz \right| \\ &\leq \left(\int_{\Omega} a_{ij}(z) \frac{\partial u}{\partial z_i} \frac{\partial u}{\partial z_j} dz \right)^{1/2} \left(\int_{\Omega} a_{ij}(z) \frac{\partial \varphi}{\partial z_i} \frac{\partial \varphi}{\partial z_j} dz \right)^{1/2} \\ &\leq c_2 \left\| \nabla_\omega u \right\|_{L_2(\Omega)} \left\| \nabla_\omega \varphi \right\|_{L_2(\Omega)} = c_2 \left\| u \right\| \left\| \varphi \right\|, \end{aligned} \tag{5.1}$$

the boundedness is ready. Using the assumption (4.1) for the function $\omega \in A_2(\mathbb{R}^n)$ we have the inequality (4.2) with $p = 2$ and $q \geq 2$ for a function $u \in \overset{\circ}{W}{}^{1,2}(\Omega;\omega dz)$:

$$\|u\|_{L_2(\Omega)} \leq |\Omega|^{1-1/q} \|u\|_{L_q(\Omega)} \leq C_1 |\Omega|^{1-1/q} \|\nabla_\omega u\|_{L_2(\Omega)}, \tag{5.2}$$

where $C_1 > 0$ depends on $\omega, \Omega, n, m, \alpha$ and the constant C from (4.1). On basis of (5.2) and (6.4) we get

$$B(u,u) = \int_{\Omega} a_{ij}(z) \frac{\partial u}{\partial z_i} \frac{\partial u}{\partial z_j} dz \geq c_1 \left\| \nabla_\omega u \right\|^2_{\overset{\circ}{W}{}^{1,2}(\Omega;\omega dz)} \geq C_2 \|u\|^2_{\overset{\circ}{W}{}^{1,2}(\Omega;\omega dz)}.$$

Show, the functional

$$\langle f, \varphi \rangle = \int_{\Omega} f(z) \varphi(z) dz$$

is bounded on $\overset{\circ}{W}{}^{1,2}(\Omega;\omega dz)$. On basis of the assumptions,

$$|\langle f, \varphi \rangle| \leq \|f\|_{\overset{\circ}{W}{}^{-1,2}(\Omega;\omega dz)} \|\varphi(z)\|_{\overset{\circ}{W}{}^{1,2}(\Omega;\omega dz)} = \|f\|_{\overset{\circ}{W}{}^{-1,2}(\Omega;\omega dz)} \|\varphi\|.$$

In order to have the bounded norm $\|f\|_{W^{-1,2}(\Omega;\omega dz)}$ propose a summability condition, where $W^{-1,2}(\Omega;\omega dz)$ is the conjugate space of $W^{1,2}(\Omega;\omega dz)$.

Setting in Lemma 4.1 the $p = 2$ we set the condition (4.1) in order to have the inclusion $\overset{\circ}{W}{}^{1,2}(\Omega;\omega dz) \subset L_q(\Omega)$, i.e. to be valid the inequality (4.2).

Propose a summability condition on the function $f(z)$ in order to have $f(z) \in W^{-1,2}(\Omega;\omega dz)$. On basis of Holder's inequality,

$$|\langle f, \varphi \rangle| \leq \|\varphi\|_{L_{q,\omega}(\Omega)} \|f\|_{L_{q',\omega^{-1/(q-1)}}(\Omega)}$$

It remains to request to be finite $\|f\|_{L_{q',\omega^{-1/(q-1)}}(\Omega)}$ for some $q > 2$. Therefore and using (4.2) as $p = 2$ we get

$$|\langle f, \varphi \rangle| \leq c_1 \|\varphi\|_{\overset{\circ}{W}{}^{1,2}(\Omega;\omega dz)} \|f\|_{L_{q',\omega^{-1/(q-1)}}(\Omega)} = c_1 \|\varphi\| \|f\|_{L_{q,\omega^{-1/(q-1)}dz}(\Omega)},$$

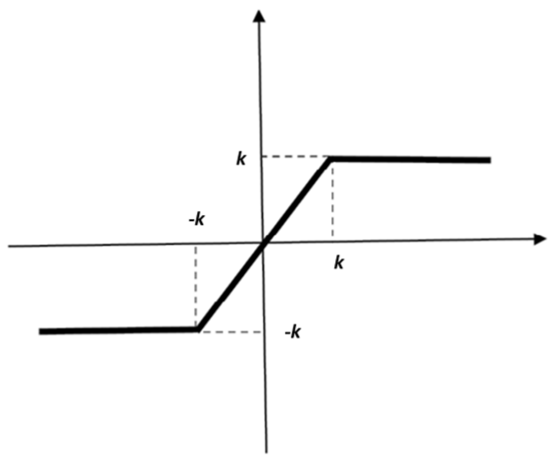
i.e. $L_{q',\omega^{-1/(q-1)}}(\Omega) \subset \overset{\circ}{W}{}^{-1,2}(\Omega;\omega dz)$ for any bounded domain $\Omega \subset \mathbb{R}^N$ if $\omega \in A_2$ over the n-dimensional balls of $\mathbb{R}^n$ and (4.1) is satisfied as $p = 2$. Also, assume that $L_{q',\omega^{-1/(q-1)}(\Omega)} \subset L_1(\Omega)$. This is fulfilled e.g. by using Holder's inequality, $\|f\|_{L_1(\Omega)} \leq \omega(\Omega)^{1/q} \|f\|_{L_{q',\omega^{-1/(q-1)}}(\Omega)}$. Applying now Lax-Milgram's principle we obtain a unique solution to the problem (3.1).

6 Proof of Lemma 3.1

Proof. We may approximate the functions of $L_1(\Omega)$ with smooth functions $f_k \to f$ a.e. in Ω. Hence the request lies on summability of the function $\omega^{-1/(q-1)} \subset L_1(\Omega)$ for some $q > 2$. Let $f_k(z)$ be an element of $L^1(\Omega) \cap W^{1,-2}(\Omega; \omega dz)$ and $u(z)$ be the corresponding solution of (3.1), and suppose that $\|f\|_{L^1(\Omega)} \leq B$.

Let k be a fixed integer and define ψ as

$$\psi(s) = \begin{cases} k & if \quad s > k, \\ s & if \quad -k \leq s \leq k, \\ -k & if \quad s < -k. \end{cases}$$



where $s \in R$. The using of $\psi(u)$ as test function in (1.4) yields

$$\int_\Omega \psi(u)' a_{ij}(z) \frac{\partial u}{\partial z_i} \frac{\partial u}{\partial z_j} dz = \int_\Omega f(z) \psi(u) dz.$$

We have

$$c_1 \int_\Omega \psi(u)' |\nabla_\omega u|^2 dz \leq \int_\Omega f(z) \psi(u) dz. \tag{6.1}$$

By virtue of the non-uniformly ellipticity condition this yields

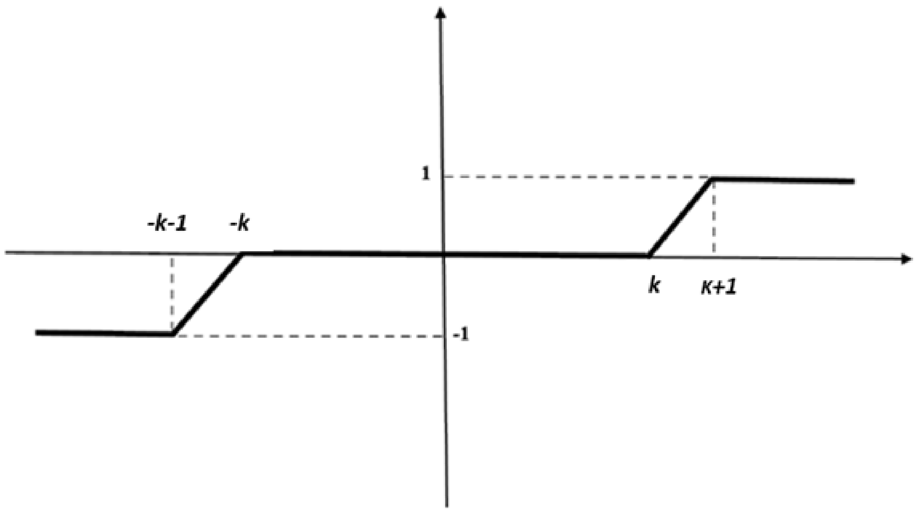
$$\begin{aligned} \int_{D_n} |\nabla_\omega u|^2 dz \leq \frac{1}{c_1} \left| \int_\Omega f(z)\psi(u) dz \right| \\ \leq \int_\Omega |f(z)| \, |\psi(u)| \, dz \leq \frac{k}{c_1} \int_\Omega |f(z)| \, dz = \frac{k}{c_1} \|f\|_{L_1(\Omega)} = k\tilde{c_1} \end{aligned} \tag{6.2}$$

with

$$D_k = \{z \in \Omega, \, |u(z)| \leq k, \, |\nabla_\omega u| \geq M\}. \tag{6.3}$$

Now we define ψ as

$$\psi(s)=\begin{cases}1 & if \quad s>k+1,\\ s-k & if \quad k\le s\le k+1,\\ 0 & if \quad -k<s<k,\\ s+k & if \quad -k-1\le s\le -k,\\ -1 & if \quad s<-k-1\end{cases}$$



Then, if we denote $(1/c_1)\,\|f\|_{L_1(\Omega)}$ by $\tilde{c_1}$,

$$\int\limits_{B_k}|\nabla_\omega u|^2dz\le\tilde{c_1} \tag{6.4}$$

with

$$B_k=\{z\in\Omega,k\le|u(z)|\le k+1,|\nabla_\omega u|\ge M\}. \tag{6.5}$$

The using (6.2) we have

$$D_k=B_0\cup B_1\cup\cdots\cup B_{k-1}$$

For any $r<2$, Applying Holder's inequality

$$\sum_{i=1}^{\infty}|a_ib_i|=\left(\sum_{i=1}^{\infty}|a_i|^{\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^{\infty}|b_i|^{\beta}\right)^{\frac{1}{\beta}},\ 1<\alpha,\,\beta<\infty,\ 1/\alpha+1/\beta=1,$$

it follows that

$$\int\limits_{B_k}|\nabla_\omega u|^r dz\le\left(\int\limits_{B_k}|\nabla_\omega u|^2dz\right)^{r/2}|B_k|^{(2-r)/2},\quad |B_k|=\mathrm{meas}_N B_k.$$

If we take $1/q = 1/r - 1/N$ for $1 < r < N/(N-1)$ and using the inequality $|B_k| \le (1/k^q) \int\limits_{B_k} |u|^q dz$, we get

$$\int\limits_{B_k} |\nabla_\omega u|^r dz \le \tilde{c}_2 \left(\int\limits_{B_k} |u|^q dz \right)^{(2-r)/2} \frac{1}{k^{(2-r)q/2}}, \quad \tilde{c}_2 = \tilde{c}_1^{q/2}. \tag{6.6}$$

Now applying Holder's inequality for the number series with the exponents $2/(2-r)$ and $2/r$ we obtain for all positive integers n_0

$$\sum_{k=k_0}^{\infty} \int\limits_{B_k} |\nabla_\omega u|^r dz \le \tilde{c}_2 \left(\sum_{l=l_0}^{\infty} \int\limits_{B_n} |u|^q dz \right)^{(2-r)/2} \left(\sum_{k=k_0}^{\infty} \frac{1}{n^{(2-r)q/r}} \right)^{r/2}$$

where $\frac{q(2-r)}{r} > 1$.

The last estimate, together with (6.5), yields

$$\int\limits_{\Omega} |\nabla_\omega u|^r dz \le c_3 + c_4 k_0^{r/2} + \tilde{c}_2 \|u\|_{L^q}^{q(2-r)/2} \left(\sum_{k=k_0}^{\infty} \frac{1}{k^{(2-r)q/r}} \right)^{r/2}, \tag{6.7}$$

where $c_3 = M^r |\Omega|, c_4 = \tilde{c}_1^{r/2} |\Omega|^{(2-r)/2}$.

By virtue of Sobolev imbedding theorem we have,

$$\|u\|_{L^q}^r \le c_5 \left(k_0^{r/2} + \|u\|_{L^q}^{(2-r)q/2} \left(\sum_{k=k_0}^{\infty} \frac{1}{k^{(2-r)q/r}} \right)^{r/2} \right). \tag{6.8}$$

Also $\frac{(2-r)q}{r} > 1$ and $r \ge (2-r)q/2$ as $r < N/(N-1)$. Therefore the relevant choice of k_0 in estimate (6.8) implies

$$\|u\|_{L^q} \le c_6. \tag{6.9}$$

Then, due to (6.7),

$$\|\nabla_\omega u\|_{L^r} \le c_7,$$

which proves the main Lemma 3.1.

7 Proof of Theorem 3.1

Now we are ready to prove Theorem 3.1 basing on the obtained estimates (3.2), (6.9). Let f be a Radon measure and $\nabla_\omega u \in L_1(\Omega)$ satisfies (1.4) and the non-uniform ellipticity condition (1.2) is satisfied. A sequence $(f_k) \subset W^{-1,2}(\Omega; \omega dz) \cap L^1(\Omega)$ and converges to f in the distribution sense, meaning that for $\forall \varphi \subset \mathring{W}^{1,2}(\Omega; \omega dz) \cap L^\infty(\Omega)$ and $\langle f_k, \varphi \rangle \rightharpoonup \int_\Omega \varphi d\mu$, therefore

$$\int\limits_{\Omega} a_{ij}(z) \frac{\partial u_k}{\partial z_i} \frac{\partial \varphi}{\partial z_j} dz = \int\limits_{\Omega} f_k(z) \varphi dz \tag{7.1}$$

Let u_k be the solution of (3.1) with $f = f_k$. Then for every integer k,

$$-\frac{\partial}{\partial z_i} \left(a_{ij}(z) \frac{\partial u_k}{\partial z_j} \right) = f_k(z) \tag{7.2}$$

has a solution $u_k \in \mathring{W}^{1,2}(\Omega;\omega dz)$ in the distribution sense, by virtue of the Lemma 5.1. On basis of Lemma 5.1 there exists $M_1 > 0$ such that $\|u_k\|_{\mathring{W}^{1,r}(\Omega;\omega dz)} \leq M_1$. Using the Banach-Aloglu theorem, there exist an $u \in \mathring{W}^{1,r}(\Omega;\omega dz)$ and some subsequence $\{u_k\}$ satisfying $u_k \rightharpoonup u$ in the weak topology of $\mathring{W}^{1,r}(\Omega;\omega dz)$. Therefore, $u_k \to u$ in $L^1(\Omega)$. This follows from the compact imbedding $\mathring{W}^{1,r}(\Omega;\omega dz) \subset\subset L_s(\Omega)$, where $1 \leq s < rN/(N-r)$. Thus $\|u_k\|_s \leq c\,\|u_k\|_{\mathring{W}^{1,r}(\Omega;\omega dz)}$, and $u_k \to u$ in the sense of almost everywhere convergence.

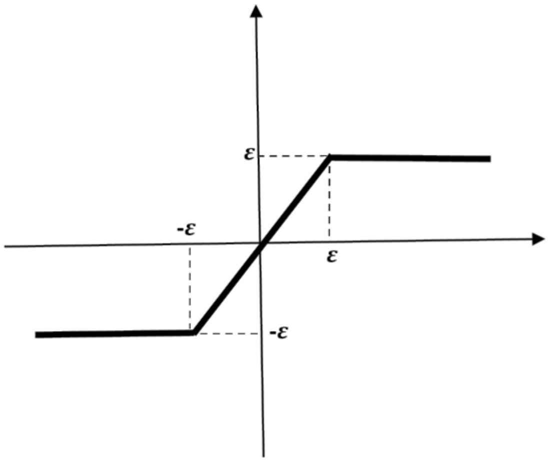
The assumption,

$$\sum_{i,j=1}^{N} a_{ij}\zeta_i\zeta_j \geq c_1\left(\omega(x)\,|\xi|^2 + |\eta|^2\right)$$

plays a central role in proving such a convergence, moreover, the following result holds true.

Let the conditions (1.2) is fulfilled, $\omega \in A_2$, (f_k) be a sequence of $W^{1,-2}(\Omega;\omega dz) \cap L^1(\Omega)$ for which u_k is a solution of (3.1) with $\mu = f_k dz$. We get the boundedness of the sequence in $L^1(\Omega)$. We get u_k is relatively compact in $\mathring{W}^{1,r}(\Omega;\omega dz)$ as r is in $[1, N/(N-1))$.

Now let $\psi \in C(R,R)$ be such that, for fixed $\varepsilon > 0$,



$$\psi(s) = \begin{cases} \varepsilon & if \quad s > \varepsilon, \\ s & if \quad -\varepsilon \leq s \leq \varepsilon, \\ -\varepsilon & if \quad s < -\varepsilon. \end{cases}$$

Then, using (1.4) with $d\mu = f_k dz$ and $f_m dz$, $u = u_k$ and u_m, as well as $v = \psi(u_k - u_m)$ we get

$$\begin{aligned} &\int_\Omega \left(\sum_{i,j=1}^{N} a_{ij}(z)\left(\frac{\partial u_n}{\partial z_i} - \frac{\partial u_m}{\partial z_i}\right)\left(\frac{\partial u_n}{\partial z_j} - \frac{\partial u_m}{\partial z_j}\right)\right)\psi'(u_n - u_m) \\ &= \int_\Omega (f_n - f_m)\psi(u_n - u_m). \end{aligned} \tag{7.3}$$

Since $\|f_n\|_{L^1(\Omega)} \leq B, \forall B > 0$, by virtue of above assumption,

$$\sum_{i,j=1}^{N} a_{ij}(z)\left(\zeta_i - \zeta_i'\right)\left(\zeta_j - \zeta_j'\right) \geq c_1 \left(\omega(x)\left|\xi - \xi'\right|^2 + \left|\eta - \eta'\right|^2\right)$$

and (7.3) we have

$$\int\limits_{D_{k,m,\varepsilon}} \left|\nabla_\omega u_k - \nabla_\omega u_m\right|^2 \leq 2\varepsilon B, \quad D_{k,m,\varepsilon} = \left\{z \in \Omega, \left|u_k(z) - u_m(z)\right| \leq \varepsilon\right\}. \tag{7.4}$$

Thus,

$$\int\limits_{D_{k,m,\varepsilon}} \left(\omega(x)\left|\nabla_x(u_k - u_m)\right|^2 + \left|\nabla_y(u_k - u_m)\right|^2\right) dz \leq 2\varepsilon B. \tag{7.5}$$

Using (7.5) and Holder's inequality we get

$$\int\limits_{D_{k,m,\varepsilon}} \left|\nabla_\omega u_k - \nabla_\omega u_m\right| \leq \tilde{c}_1 \varepsilon^{1/2} |D_{k,m,\varepsilon}|^{1/2}, \tag{7.6}$$

where $\tilde{c}_1 = (2B)^{1/2}$

Estimate (7.6) is used to prove that $(\nabla_\omega u_k)$ is Cauchy sequence in $L^1(\Omega)$. We have

$$\int\limits_{\Omega} \left|\nabla_\omega\left(u_k - u_m\right)\right| = \int\limits_{D_{k,m,\varepsilon}} \left|\nabla_\omega\left(u_k - u_m\right)\right| + \int\limits_{\Omega \backslash D_{k,m,\varepsilon}} \left|\nabla_\omega\left(u_k - u_m\right)\right|.$$

Then using (7.6),

$$\int\limits_{\Omega} \left|\nabla_\omega\left(u_n - u_m\right)\right| \leq \tilde{c}_1 \varepsilon^{1/2} + \tilde{c}_2 \ \left|\left\{z \in \Omega; \left|u_n(z) - u_m(z)\right| > \varepsilon\right\}\right|^{1-1/q}, \tag{7.7}$$

where q is in $(1,\ N/(N-1))$.

Let u_k be a Cauchy sequence in measure, (7.7) implies that for some $k_0(\varepsilon)$ depending on ε

$$\int\limits_{\Omega} \left|\nabla_\omega\left(u_k - u_m\right)\right| \leq \tilde{c}_1 \varepsilon^{1/2} + \varepsilon \quad \text{for all} \quad k, m \geq k_0(\varepsilon),$$

which proves that $(\nabla_\omega u_k)$ is a Cauchy sequence in $L^1(\Omega)$, means that

$$\nabla_\omega u_k \to \nabla_\omega u \quad in \quad L^1(\Omega).$$

By (6.9), we get the convergence

$$\nabla_\omega u_n \to \nabla_\omega u \quad \text{in} \quad L^r(\Omega), \ \text{ for all } \ r \in [1, N/(N-1)).$$

Thus, u_k is relatively compact in $\mathring{W}^{1,r}(\Omega; \omega dz)$. By (3.1) together with Vitali's theorem, we have

$$\left(\sum_{j=1}^{N} a_{ij}(z)\frac{\partial u_k}{\partial z_j}\right) \to \left(\sum_{j=1}^{N} a_{ij}(z)\frac{\partial u}{\partial z_j}\right) \text{ in } L^r(\Omega)$$

for all r in $[1, N/(N-1))$. Now we can pass to limit in (7.1) and conclude that

$$-\frac{\partial}{\partial z_i}\left(a_{ij}\left(z\right)\frac{\partial u}{\partial z_j}\right) = \mu.$$

Thus, u is a weak solution of (1.4), i.e. it is a solution of problem (1.1), this completes the proof of Theorem 3.1.

Acknowledgements. I would like to thank my supervisor, Professor Farman Mamedov, for setting the problem and for the valuable discussions throughout writing this article.

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