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Marcinkiewicz integral with rough kernel in local Morrey-type spaces

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Abstract. *The goal of this paper is to investigate the local Guliyev estimates of Marcinkiewicz integral with rough kernel. By giving the local Guliyev estimates, the boundedness of the Marcinkiewicz integral with rough kernel on the local Morrey-type spaces (≡ local Morrey-Guliyev spaces) is obtained.*

Keywords. Local Morrey-type spaces; Marcinkiewicz integral; Hardy operator

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35

1 Introduction and notation

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $n \geq 2$ equipped with normalized Lebesgue measure $d\sigma$. Suppose $\Omega \in L^q(S^{n-1})$ with $1 < q \leq \infty$ is homogeneous of degree zero and satisfies the cancelation condition

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. The singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy.$$

Furthermore, Marcinkiewicz integral operator μ_Ω is defined by

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

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where

$$F_{\Omega,t}(x) = \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It is well known that singular integral operator and Marcinkiewicz operator play an important role in harmonic analysis. When $\Omega \in L\log L(S^{n-1})$, Calderón- Zygmund [11] proved that T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 2002, Al-Salman and Pan [2] showed that if $\Omega \in H^1(S^{n-1})$, then T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. On the other hand, Benedek et al. [3] proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 2002, Ding et al. [22] showed that if $\Omega \in L^q(S^{n-1}),\ q > 1$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

The classical Morrey spaces $L^{p,\lambda}$ were introduced by Morrey [26] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [15, 25,27] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [16,19,32]). Komori and Shirai [23] defined weighted Morrey spaces $L^{p,\kappa}_w(\mathbb{R}^n)$. Guliyev [21] gave a concept of the generalized weighted Morrey spaces $M^{p,\varphi}_w(\mathbb{R}^n)$ which could be viewed as extension of both $M^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,\kappa}_w(\mathbb{R}^n)$. In [21], the boundedness of the classical operators and their commutators in spaces $M^{p,\varphi}_w(\mathbb{R}^n)$ was also studied.

Suppose $0 < p,\ \theta \le \infty$ and w be a non-negative measurable function on $(0,\infty)$, for any function $f \in L^p_{loc}(\mathbb{R}^n)$, we denote by $LM_{p\theta,w},\ GM_{p\theta,w}$, the local Morrey-type space, the global Morrey-type space respectively with finite quasinorms

$$\|f\|_{LM_{p\theta,w}} = \|w(r)\|f\|_{L^p(B(0,r))}\|_{L^\theta(0,\infty)},\ \|f\|_{GM_{p\theta,w}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w}}.$$

For $w(r) = r^{-\frac{\lambda}{p}},\ 0 < \lambda < n$ we get the variant of Morrey-type space $GM_{p\theta,r^{-\lambda}}$ introduced by D.R. Adams [1], which were used by G. Lu [24] for studying the embedding theorems for vector fields of Hörmander type. For $\theta = \infty$, $LM_{p\infty,w} \equiv GM_{p\infty,w}$ are the generalized Morrey space $L^{p,w}(\mathbb{R}^n)$. When $\theta = \infty,\ w = r^{-\lambda/p}$, it is the classical Morrey space.

Recall that in 1994 the doctoral thesis [15] by Guliyev (see also [16]-[18]) introduced the local Morrey-type space $LM_{p\theta,w}$. In [15] intensively studied the classical operators in the local Morrey-type space $LM_{p\theta,w}$, see also [16], where these results were presented for the case when the underlying space is the Heisenberg group or a homogeneous group, respectively. The main purpose of [15] (also in [16]-[18]) is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups in local Morrey-type space $LM_{p\theta,w}$. In a series of papers by V. Burenkov, H. Guliyev and V. Guliyev (see [4]-[10]) be given some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type space $LM_{p\theta,w}$.

The spaces $LM_{p\theta,w}$ and $GM_{p\theta,w}$ are denoted, respectively, as local Morrey-type spaces and global Morrey-type spaces, though from the point of view of the role in the development of these spaces they may be also called local and global Morrey-Guliyev spaces, respectively, see for example, [29]. For Morrey-type spaces we also refer to the survey [28]. Also the spaces $LM_{p\theta,\lambda} \equiv LM_{p\theta,r^{-\lambda}}$ and $GM_{p\theta,\lambda} \equiv GM_{p\theta,r^{-\lambda}}$ may be called local and global Morrey-Adams spaces, respectively, see for example, [29–31].

In [33] the boundedness of Marcinkiewicz integral with rough kernel was proven on Morrey-Adams spaces. In this paper, we will study the boundedness of Marcinkiewicz integral with rough kernel on local and global Morrey-Guliyev spaces.

In what follows, we denote by C positive constants which are independent of the main parameters, but it may vary from line to line.

2 Integral operators with rough kernels in local Morrey-type spaces

In this section, we study the boundedness of integral operators in local Morrey-type spaces ($\equiv$ local Morrey-Guliyev spaces) and global Morrey-type spaces ($\equiv$ global Morrey-Guliyev spaces). To state the main results, we first introduce some notations.

Definition 2.1 *Let $0 < p,\ \theta \leq \infty$, we denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t > 0$,*

$$\|w(r)\|_{L^\theta(t,\infty)} < \infty.$$

Moreover, we denote by $\Omega_{p,\theta}$ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t_1,\ t_2 > 0$,

$$\|w(r)\|_{L^\theta(t_1,\infty)} < \infty,\ \|w(r)r^{n/p}\|_{L^\theta(0,t_2)} < \infty.$$

In [7], the following result was shown

Lemma 2.1 *Let $0 < p,\ \theta \leq \infty$ and w be a non-negative measurable function on $(0,\infty)$, then the following is true*

1. *If for all $t > 0,\ \|w(r)\|_{L^\theta(t,\infty)} = \infty$, then $LM_{p\theta,w} = GM_{p\theta,w} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.*
2. *If for all $t > 0,\ \|w(r)r^{n/p}\|_{L^\theta(0,t)} = \infty$, then any functions $f \in LM_{p\theta,w}$, continuous at $0,\ f(0) = 0$, and for $0 < p < \infty,\ GM_{p\theta,w} = \Theta$.*

Consequently, in the sequel, we always assume that either $w \in \Omega_\theta$ or $w \in \Omega_{p,\theta}$.
Let H denote the Hardy operator

$$Hg(r) = \int_0^r g(t)dt,\ 0 < r < \infty$$

and $L^p_v(0,\infty)$ be the weighted Lebesgue space of function g on $(0,\infty)$ for which $\|g\|_{L^p_v(0,\infty)} = \left(\int_0^\infty |g(t)|^p v(t)dt\right)^{1/p} < \infty$. Therefore, we have the following theorem

Theorem 2.1 *Let $\Omega \in L^q(S^{n-1}),\ 1 < q < \infty$, be a homogeneous of degree zero and satisfy the cancellation condition. If for any $q' < p < \infty,\ 0 < \theta_1,\ \theta_2 \leq \infty,\ w_1 \in \Omega_{\theta_1}$ an $w_2 \in \Omega_{\theta_2}$, suppose that*

$$v(r) = w_1^{\theta_1}\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}-1},\ u(r) = w_2^{\theta_2}\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}-\theta_2-1}.$$

Assume the operator H is bounded from $L^\theta_v(0,\infty)$ to $L^\theta_u(0,\infty)$ on the cone of all non-negative non-increasing functions ϕ on $(0,\infty)$ satisfying the condition $\lim\limits_{t\to\infty}\phi(t) = 0$, then the singular integral operator T_Ω is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$ (in the latter case, it is assume that $w_1 \in \Omega_{p,\theta_1}$ and $w_2 \in \Omega_{p,\theta_2}$).

Proof. For any ball $B = B(x_0,r)$, function $f(x)$ can be divided into two parts: $f = f\chi_{4B} + f\chi_{\mathbb{R}^n\setminus 4B} := f_1 + f_2$, thus we have

$$\|T_\Omega f\|_{L^p(B)} \leq \|T_\Omega f_1\|_{L^p(B)} + \|T_\Omega f_2\|_{L^p(B)} \equiv I_1 + I_2. \tag{2.1}$$

For I_1, by $L^p(\mathbb{R}^n)$ boundedness of T_Ω in [11,14], we have

$$I_1 \leq C\|f\|_{L^p(4B)} \leq Cr^{\frac{n}{p}}\int_r^\infty \|f\|_{L^p(B(x,t))}\frac{dt}{t^{\frac{n}{p}+1}}. \tag{2.2}$$

For I_2, we first estimate $Tf_2(x)$ for any $x \in B$, since $y \in \mathbb{R}^n \setminus 4B$, it has the following inequality: $|x-y| > |y-x_0| - |x-x_0| > \frac{1}{2}|y-x_0| > 3r$, therefore we obtain

$$\begin{aligned} |T_\Omega f_2(x)| &= \Big| \int_{\mathbb{R}^n \setminus 4B} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \Big| \\ &\le \int_{\mathbb{R}^n \setminus B(0,3r)} \frac{|\Omega(z)|}{|z|^n} |f(x-z)|dz \\ &= C \int_{\mathbb{R}^n \setminus B(0,3r)} |\Omega(z) f(x-z)| \int_{|z|}^\infty \frac{dt}{t^{n+1}} dz \\ &\le C \int_{3r}^\infty \int_{B(0,t)} |\Omega(z) f(x-z)| dz \frac{dt}{t^{n+1}} \\ &\le C \|\Omega\|_{L^q(S^{n-1})} \int_{3r}^\infty \left(\int_{B(0,t)} |f(x-z)|^{q'} dz \right)^{\frac{1}{q'}} \frac{dt}{t^{\frac{n}{p}+1}}, \end{aligned}$$

since $q' < p < \infty$, for any $|x-x_0| < r,\ |z| < t$, it has the following inequality: $|x-z-x_0| \le |z| + |x-x_0| < 2t$, hence we have

$$\begin{aligned} |T_\Omega f_2(x)| &\le C \|\Omega\|_{L^q(S^{n-1})} \int_{3r}^\infty \Big(\int_{B(x_0,2t)} |f(y)|^p dy \Big)^{\frac{1}{p}} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\le C \|\Omega\|_{L^q(S^{n-1})} \int_r^\infty \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned}$$

Thus for I_2, we have

$$I_2 \le C \|\Omega\|_{L^q(S^{n-1})} r^{\frac{n}{p}} \int_r^\infty \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{2.3}$$

Finally, by the definition of local Morrey-type space ($\equiv$ local Morrey-Guliyev spaces) and inequalities of (2.1) $-$ (2.3), we show

$$\begin{aligned} \|T_\Omega f\|_{LM_{p\theta_2 w_2}} &= \|w_2(r) \|T_\Omega f\|_{L^p(B(0,r))} \|_{L^{\theta_2}(0,\infty)} \\ &\le C \|w_2(r) r^{\frac{n}{p}} \int_r^\infty t^{-n/p-1} \|f\|_{L^p(B(0,t))} dt \|_{L^{\theta_2}(0,\infty)} \\ &= C \|w_2(r^{-\frac{p}{n}}) \frac{1}{r} \int_0^r \|f\|_{L^p(B(0,t^{-\frac{p}{n}}))} dt r^{-\frac{p}{n\theta_2} - \frac{1}{\theta_2}} \|_{L^{\theta_2}(0,\infty)}. \end{aligned}$$

Let $g(t) = \|f\|_{L^p(B(0,t^{-\frac{p}{n}}))}$, $u(r) = w_2^{\theta_2}\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n} - \theta_2 - 1}$, then

$$\|T_\Omega f\|_{LM_{p\theta_2 w_2}} \le C \|Hg(r)\|_{L_u^{\theta_2}(0,\infty)}. \tag{2.4}$$

Let $v(r) = w_1^{\theta_1}(r^{-n}) r^{-n}$, by the weighted L^p boundedness of Hardy operator H and inequality (2.4), we have

$$\begin{aligned} \|T_\Omega f\|_{LM_{p\theta_2 w_2}} &\lesssim \|g(r)\|_{L_v^{\theta_1}(0,\infty)} \\ &= \left(\int_0^\infty \|f\|^{\theta_1}_{L^p(B(0,r^{-\frac{p}{n}}))} w_1^{\theta_1}\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}-1} dr \right)^{\frac{1}{\theta_1}} \\ &= \left(\int_0^\infty \|f\|^{\theta_1}_{L^p(B(0,r))} w_1^{\theta_1}(r) dr \right)^{\frac{1}{\theta_1}} \\ &= \| \|f\|_{L^p(B(0,r))} w_1(r) \|_{L^{\theta_1}(0,\infty)} = \|f\|_{LM_{p\theta_1 w_1}}. \end{aligned}$$

On the other hand, by the definition of global Morrey-type spaces, it only need to $g(t) = \|f\|_{L^p(B(x_0,t^{-\frac{p}{n}}))}$, just like local Morrey-type spaces, we also obtain the boundedness in global Morrey-type spaces ($\equiv$ global Morrey-Guliyev spaces).

Theorem 2.2 *Let $\Omega \in L^q(S^{n-1}),\ 1 < q < \infty$, be a homogeneous of degree zero and satisfy the cancellation condition. If for any $q' < p < \infty,\ 0 < \theta_1,\ \theta_2 \leq \infty,\ w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, suppose that*

$$v(r) = w_1^{\theta_1}\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}-1},\ u(r) = w_2^{\theta_2}(r^{-\frac{p}{n}})r^{-\frac{p}{n}-\theta_2-1}.$$

Assume the operator H is bounded from $L_v^\theta(0,\infty)$ to $L_u^\theta(0,\infty)$ on the cone of all n *non-negative nonincreasing functions ϕ on $(0,\infty)$ satisfying the condition $\lim_{t\to\infty}\phi(t) = 0$, then the Marcinkiewicz integral operator μ_Ω is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$ (in the latter case, it is assume that $w_1 \in \Omega_{p,\theta_1}$ and $w_2 \in \Omega_{p,\theta_2}$).*

Proof. As before, we can write

$$\|\mu_\Omega f\|_{L^p(B)} \leq \|\mu_\Omega f_1\|_{L^p(B)} + \|\mu_\Omega f_2\|_{L^p(B)} := I_3 + I_4.$$

The first part, by the L^p boundedness of μ_Ω in [3], we yield

$$I_3 \lesssim \|f\|_{L^p(4B)} \lesssim r^{\frac{n}{p}} \int_r^\infty \|f\|_{L^p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}}.$$

On the other hand, let us deal with the term I_4, for any fixed $x \in B$, by the Minkowski inequality, we have

$$\begin{aligned}|\mu_\Omega f_2(x)| &\leq \int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_2(y)|\left(\int_{|x-y|}^\infty\frac{dt}{t^3}\right)^{\frac{1}{2}}dy\\ &= C\int_{\mathbb{R}^n\setminus 4B}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|dy.\end{aligned}$$

Now, the same as theorem 2.1, we also obtain

$$I_4 \lesssim \|\Omega\|_{L^q(S^{n-1})}r^{\frac{n}{p}}\int_r^\infty \|f\|_{L^p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}}.$$

By the definition of local Morrey-type spaces ($\equiv$ local Morrey-Guliyev spaces) and global Morrey-type spaces ($\equiv$ global Morrey-Guliyev spaces) , and the Hardy operator H, the same as theorem 2.1, we complete the proof.

In order to obtain sufficient conditions of the singular integral operator and Marcinkiewicz integral operator, we shall apply the known necessary and sufficient conditions ensuring boundedness of the Hardy operator H from one weighted Lebesgue space to another one for any non-negative nonincreasing function g (see, for example [12, 13]).

Lemma 2.2 *Let g be a non-negative nonincreasing function and u, v weight functions on $(0,\infty)$.*

(a) If $1<\theta_1\le\theta_2<\infty$, *then the inequality*

$$\left(\int_0^\infty (Hg)^{\theta_2}(t)u(t)dt\right)^{1/\theta_2}\le C\left(\int_0^\infty (g)^{\theta_1}(t)v(t)dt\right)^{1/\theta_1} \tag{2.5}$$

holds if any only if

$$B_{11}:=\sup_{t>0}\left(\int_0^t u(r)r^{\theta_2}dr\right)^{-\frac{1}{\theta_2}}\left(\int_0^t v(r)dr\right)^{\frac{1}{\theta_1}}<\infty,$$

and

$$B_{12}:=\sup_{t>0}\left(\int_t^\infty u(r)dr\right)^{\frac{1}{\theta_2}}\left(\int_0^t \frac{v(r)r^{\theta_1'}}{\left(\int_0^r v(\rho)d\rho\right)^{\theta_1'}}dr\right)^{\frac{1}{\theta_1'}}<\infty.$$

(b) If $0<\theta_1\le 1, 0<\theta_1\le\theta_2<\infty$, *then the inequality* (2.5) *holds if any only if* $B_{11}<\infty$ *and*

$$B_{22}:=\sup_{t>0}\left(\int_t^\infty u(r)dr\right)^{\frac{1}{\theta_2}}\left(\int_0^t v(r)dr\right)^{-\frac{1}{\theta_1'}}<\infty.$$

(c) If $1<\theta_1\le\infty,\ 0<\theta_2<\theta_1<\infty,\ \theta_2\ne 1$, *then the inequality* (2.5) *holds if any only if*

$$B_{31}:=\left(\int_0^\infty\left(\frac{\int_0^t u(r)r^{\theta_2}dr}{\int_0^t v(r)dr}\right)^{\frac{\theta_2}{\theta_1-\theta_2}}u(t)t^{\theta_2}dt\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}}<\infty,$$

and

$$B_{32}:=\left(\int_0^\infty\left[\left(\int_t^\infty u(r)dr\right)^{\frac{1}{\theta_2}}\left(\int_0^t \frac{v(r)r^{\theta_1'}}{\left(\int_0^r v(\rho)d\rho\right)^{\theta_1'}}dr\right)^{\frac{\theta_2-1}{\theta_2}}\right]^{\frac{\theta_1\theta_2}{\theta_1-\theta_2}}\right.$$

$$\left.\times\frac{v(t)t^{\theta_1'}}{\left(\int_0^t v(\rho)d\rho\right)^{\theta_1'}}dt\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}}<\infty.$$

(d) If $1=\theta_2<\theta_1<\infty$, *then the inequality* (2.5) *holds if any only if*

$$B_{41}:=\left(\int_0^\infty\left(\frac{\int_0^t u(r)rdr}{\int_0^t v(r)dr}\right)^{\frac{1}{\theta_1-1}}u(t)tdt\right)^{\frac{\theta_1-1}{\theta_1}}<\infty,$$

and $B_{42}:=\sup_{t>0}\left[\left(\frac{\int_0^t u(r)rdr+t\int_t^\infty u(r)dr}{\int_0^t v(r)dr}\right)^{\theta_1'-1}\times\left(\int_t^\infty u(r)dr\right)dt\right]^{\theta_1'}<\infty.$

(e) If $0<\theta_2<\theta_1=1$, *then the inequality* (2.5) *holds if any only if*

$$B_{51}:=\left(\int_0^\infty\left(\frac{\int_0^t u(r)r^{\theta_2}dr}{\int_0^t v(r)dr}\right)^{\frac{\theta_2}{1-\theta_2}}u(t)t^{\theta_2}dt\right)^{\frac{1-\theta_2}{\theta_2}}<\infty,$$

and

$$B_{52}:=\left(\int_0^\infty\left(\int_t^\infty u(r)dr\right)^{\frac{\theta_2}{1-\theta_1}}\left(\inf_{0<s<t}\frac{1}{s}\int_0^s v(\rho)d\rho\right)^{\frac{\theta_2}{\theta_2-1}}\times u(t)dt\right)^{\frac{1-\theta_2}{\theta_2}}<\infty.$$

(f) If $0<\theta_2<\theta_1<1$*, then the in equality* (2.5) *holds if any only if* $B_{31}<\infty$ *and*

$$B_{62}:=\left(\int_0^\infty\sup_{0<s\le t}\frac{s^{\frac{\theta_1\theta_2}{\theta_1-\theta_2}}}{\left(\int_0^s v(\rho)d\rho\right)^{\frac{\theta_2}{\theta_1-\theta_2}}}\left(\int_t^\infty u(r)dr\right)^{\frac{\theta_1\theta_2}{\theta_1-\theta_2}}\times u(t)dt\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}}<\infty.$$

(g) If $0<\theta_1\le 1,\ \theta_2=\infty$*, then the inequality* (2.5) *holds if any only if*

$$B_7:=\operatorname*{ess\,sup}_{0<s\le t}\frac{su(t)}{\left(\int_0^s v(r)dr\right)^{\frac{1}{\theta_1}}}<\infty.$$

(h) If $1<\theta_1<\infty,\ \theta_2=\infty$*, then the inequality* (2.5) *holds if any only if*

$$B_8\ :=\operatorname*{ess\,sup}_{t>0}u(t)\left(\int_0^t\frac{r^{\theta_1'-1}}{\int_0^r v(s)}dr\right)^{\frac{1}{\theta_1{}'}}<\infty.$$

(i) If $\theta_1=\infty,\ 0<\theta_2<\infty$*, then the inequality* (2.5) *holds if any only if*

$$B_9:=\left(\int_0^\infty\left(\int_0^t\frac{dr}{\operatorname*{ess\,sup}_{0<y<r}v(y)}\right)^{\theta_2}u(t)dt\right)^{\frac{1}{\theta_2}}<\infty.$$

(j) If $\theta_1=\theta_2=\infty$*, then the inequality* (2.5) *holds if any only if*

$$B_{10}:=\operatorname*{ess\,sup}_{t>0}u(t)\int_0^t\frac{dr}{\operatorname*{ess\,sup}_{0<y<r}v(y)}<\infty.$$

From Theorems 2.1, 2.2 and Lemma 2.2, we obtain the following results

Corollary 2.1 *Let* $\Omega\in L^q(S^{n-1})$*, for any* $q'<p<\infty,\ 0<\theta_1,\ \theta_2\le\infty,\ w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$*, suppose that any of condition* $(a)-(j)$ *is satisfied, then the singular integral operator* T_Ω *is bounded from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GM_{p\theta_2,w_2}$ *(in the latter case, it assumes that* $w_1\in\Omega_{p,\theta_1}$ *and* $w_2\in\Omega_{p,\theta_2}$*).*

Corollary 2.2 *Let* $\Omega\in L^q(S^{n-1})$*, for any* $q'<p<\infty,\ 0<\theta_1,\ \theta_2\le\infty,\ w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$*, suppose that any of condition* $(a)-(j)$ *is satisfied, then the singular integral operator* μ_Ω *is bounded from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GM_{p\theta_2,w_2}$ *(in the latter case, it assumes that* $w_1\in\Omega_{p,\theta_1}$ *and* $w_2\in\Omega_{p,\theta_2}$*).*

Note that if $\theta_1=\theta_2=\infty$, that is, condition (j) is satisfied, then T_Ω and μ_Ω are bounded from generalized Morrey space M^{p,ω_1} to generalized Morrey space M^{p,ω_2}, which extend to the result of Guliyev et al. in [20].

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