Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **43** (4), 105-113 (2023).
https://doi.org/10.30546/2617-7900.43.4.2023.105

Density problem some of the fractional functional spaces via conformable fractional calculus and their application

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Received: 08.12.2022 / Revised: 24.08.2023 / Accepted: 21.10.2018

Abstract. *In this study, we provide a conformable fractional calculus generalization of the density of various functional spaces, such as spaces of continuous functions, spaces of order α derivatives, fractional spaces of Lebesgue integrals, and fractional Sobolev's spaces.*

Mathematics Subject Classification (2010): 46E35, 46E39, 42B35

1 Introduction

For nondifferential solutions, fractal calculus and fractional calculus have recently gained popularity in both mathematics and engineering (see [9]). In several disciplines of study, the usage of the fractional derivative has therefore seen a considerable improvement and increase in attention (see [16–19]).

It is well known that differentiation and integration of a random (noninteger) order are included in the definition of fractional calculus. Mathematicians including Leibniz, Liouville, Riemann, Letnikov, and Grunwald are responsible for the theory. Fractional calculus is currently one of the areas of mathematical analysis that is most actively developing. There are many definitions of fractional calculus, including Riemann-Liouville fractional calculus, Caputo fractional calculus, Grunwald-Letnikov fractional calculus, Hadamard fractional calculus, Riesz fractional calculus, Weyl fractional calculus, Kolwankar-Gangal fractional calculus, and others. Particularly, in 2014, Khalil et al, developed a new kind of fractional derivative called "conformable fractional derivative" in their publication [13]. Establishing a suitable space and variational functional for conformable fractional differential equations, such as the space of a derivative of conformable fractional, the fractional spaces of the

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Lebesgue integral, and the fractional Sobolev's spaces, is necessary for studying solutions of boundary value problems for fractional differential equations.

The structure of this essay is as follows: We examine the functional subspace by reporting Lebesgue space in sense integrals conformable to fractional scales, such as fractional Sobolev's spaces, in Section 3.

2 Preliminaries

We define and describe the characteristics of the α-differentiable and -integral in the conformal sense in the first section of the preamble, pulse to see [13].

Definition 2.1 *[13, Definition 2.1.] Let $f:[0,a]\to\mathbb{R}$ and $\alpha\in(0,1]$, with $a>0$, we define $f^{(\alpha)}(s)$ to be the number, provided it exists, such that*

$$f^{(\alpha)}(s) := \lim_{\varepsilon\to 0}\frac{f\left(s+\varepsilon s^{1-\alpha}\right)-f(s)}{\varepsilon}\qquad \text{for all } s\in[0,a]\,.$$

We frequently refer to $f^{(\alpha)}$as the conformable fractional derivative of order.
In addition, we simply state that f is α-differentiable if $f^{(\alpha)}$ exists.
If f is α-differentiable in some $s\in(0,a)$, $a>0$, and $\lim_{s\to 0^+} f^{(\alpha)}(s)$ exists, then we define

$$f^{(\alpha)}(0)=\lim_{s\to 0^+} f^{(\alpha)}(s)\,.$$

Definition 2.2 *[13, Definition 3.1.]Suppose $f:[0,a]\to\mathbb{R}$ and $\alpha\in(0,1]$. The conformable fractional integral of f of order α from to s, denoted by the abbreviation $\mathcal{I}_a^\alpha(f)(s)$, is defined as follows:*

$$\mathcal{I}_a^\alpha(f)(s) := \int_a^s \frac{f(\tau)}{\tau^{1-\alpha}}d\tau = \int_a^s f(\tau)\,d_\alpha\tau,$$

where the integral mentioned above is the typical improper Riemann integral.

Lemma 2.1 *[13, Theorem 2.2.]Let $\alpha\in(0,1]$ and assume $f,g:[0,a]\to\mathbb{R}$ to be α-differentiable, for all $\lambda,\gamma\in\mathbb{R}$, we have*

$$\begin{aligned}(\lambda f+\gamma g)^{(\alpha)} &= \lambda.f^{(\alpha)}+\gamma.g^{(\alpha)},\\ (fg)^{(\alpha)} &= f.g^{(\alpha)}+g.f^{(\alpha)},\\ \left(\frac{f}{g}\right)^{(\alpha)} &= \frac{f^{(\alpha)}g-fg^{(\alpha)}}{g^2}.\end{aligned}$$

If, in addition, f is differentiable at a point $s>0$, then

$$f^{(\alpha)}(s)=s^{1-\alpha}f'(s)\,.$$

Lemma 2.2 *[13, Theorem 3.1.]Let $\alpha\in(0,1]$ and $f:[0,a]\to\mathbb{R}$, such as f is a continuous function in the domain of I_a^α, then*

$$[\mathcal{I}_a^\alpha f]^{(\alpha)}(s)=f(s)\,,\quad \text{for all } s\in[0,a]\,.$$

Definition 2.3 *[10, Definition 22]Let $p\in\mathbb{R}$ be such that $p\geq 1$ and let $f:[0,a)\to\mathbb{R}$ be a measurable function, we say that f belongs to $L_\alpha^p([0,a],\mathbb{R})$ provided that either*

$$\int_0^a |f(t)|^p\,d_\alpha t=\int_0^a |f(t)|^p\,t^{\alpha-1}dt<\infty.$$

Lemma 2.3 *[10, Theorem 24.] Let $p \in \mathbb{R}$ be such that $p \geq 1$. Then the set $L_\alpha^p([0,a],\mathbb{R})$ is a Banach space together with the norm defined for $f \in L_\alpha^p([0,a],\mathbb{R})$ as*

$$\|f\|_{\alpha,p}^p = \int_0^a |f(t)|^p \, d_\alpha t.$$

Lemma 2.4 *[10, Theorem 45]Let $p \in \mathbb{R}$ be such that $p \geq 1$ and $f : [0,a) \to \mathbb{R}$, one says that $f \in W^{\alpha,p}([0,a],\mathbb{R})$ if and only if $f \in L_\alpha^p([0,a],\mathbb{R})$ and $f^{(\alpha)} \in L_\alpha^p([0,a],\mathbb{R})$. Then the set $W^{\alpha,p}([0,a],\mathbb{R})$ is a Banach space together with the norm defined for $f \in W^{\alpha,p}([0,a],\mathbb{R})$ as*

$$\|f\|_{W^{\alpha,p}} = \|f\|_{\alpha,p} + \left\|f^{(\alpha)}\right\|_{\alpha,p}.$$

3 Main results

We need the following lemma to get to the key conclusions in this section:

Lemma 3.1 *Let $p \in \mathbb{R}$ be such that $p \geq 1$, $\alpha \in (0,1)$ and $a \in (0,\infty)$, then $L_\alpha^p([0,a],\mathbb{R})$ is dense in $L^p([0,a],\mathbb{R})$, where $L^p([0,a],\mathbb{R})$ is considered $L_1^p([0,a],\mathbb{R})$.*

Proof. First, let's show that $L_\alpha^p([0,a],\mathbb{R})$ inject $L^p([0,a],\mathbb{R})$, for all $f \in L_\alpha^p([0,a],\mathbb{R})$, since $a > 0$ and $1-\alpha > 0$, we have

$$\int_0^a |f(t)|^p \, dt = \int_0^a |f(t)|^p \, t^{\alpha-1} t^{1-\alpha} dt \leq a^{1-\alpha} \int_0^a |f(t)|^p \, t^{\alpha-1} dt < \infty,$$

which means $f \in L^p([0,a),\mathbb{R})$ and $\|f\|_p \leq a^{\frac{1-\alpha}{p}} \|f\|_{\alpha,p}$, on the other hand, for everyone $f \in L^p([0,a],\mathbb{R})$, we consider $(f_n)_n$ be a sequence defined as follows:

$$f_n(t) = \begin{cases} f(t), \text{ if } \frac{a}{n} \leq t \leq a, \\ 0, \quad \text{if } 0 \leq t < \frac{a}{n}, \end{cases}$$

then, for all $n \in \mathbb{N}$, we have

$$\|f_n\|_{\alpha,p}^p = \int_{\frac{a}{n}}^a |f(t)|^p \, t^{\alpha-1} dt \leq \left(\frac{a}{n}\right)^{\alpha-1} \int_{\frac{a}{n}}^a |f(t)|^p \, dt \leq \left(\frac{a}{n}\right)^{\alpha-1} \|f\|_p < \infty,$$

this means $(f_n)_n \in L_\alpha^p([0,a],\mathbb{R})$ and we have

$$\|f_n - f\|_p^p \leq \int_0^{\frac{a}{n}} |f(t)|^p \, dt,$$

based on the above inequality, we conclude that $(f_n)_n$ converges to f in $L^p([0,a],\mathbb{R})$.

Example 1 Let $a \in (0,\infty)$, $\frac{1}{2} < \alpha < 1$, $a > 0$ and $f : [0,a) \to \mathbb{R}$, be defined by $f(t) = t^{-1}$, we have $f \in L_\alpha^1([0,a],\mathbb{R})$ and $f \notin L^1([0,a],\mathbb{R})$.

Remark 3.1 Deduce from 1 that the space $L_\alpha^p([0,a],\mathbb{R})$ contained strict in space $L^p([0,a],\mathbb{R})$, with $p \geq 1$, $a > 0$ and $\alpha \in (0,1)$.

Definition 3.1 *Given $\alpha \in \mathbb{R}$ and $a \in (0,\infty)$, such as $\alpha = n + \beta$, $n \in \mathbb{N}$ and $\beta \in [0,1)$, one may state that $f \in \mathcal{C}^\alpha([0,a],\mathbb{R})$ if and only if $f^{[\alpha]}$ is α-differentiable and $\left(f^{(n)}\right)^{(\beta)} \in \mathcal{C}([0,a],\mathbb{R})$.*

Remark 3.2 Let $\alpha \in \mathbb{R}$ and $a \in (0,\infty)$, such as $\alpha = [\alpha] + \beta$ and $\beta \in \beta \in [0,1)$, then $\mathcal{C}^{\alpha}([0,a],\mathbb{R})$ is a Banach space together with the norm defined for $f \in \mathcal{C}^{\alpha}([0,a],\mathbb{R})$ as

$$\|f\|_{\alpha,\infty} = \sum_{j=0}^{j=[\alpha]} \left\|f^{(j)}\right\|_{\infty} + \left\|\left(f^{(n)}\right)^{(\beta)}\right\|_{\infty},$$

with $n = [\alpha]$.

Theorem 3.1 *Let $p \in \mathbb{R}$ be such that $p \geq 1$, $\alpha \in (0,1)$ and $a \in (0,\infty)$, then $\mathcal{C}^{\alpha}([0,a],\mathbb{R})$ is dense in $L^p_{\alpha}([0,a],\mathbb{R})$.*

Proof. Let $f \in L^p_{\alpha}([0,a],\mathbb{R})$, by Definition 2.3, we get $t^{\frac{\alpha-1}{p}} f \in L^p([0,a],\mathbb{R})$, since $\mathcal{C}^1([0,a],\mathbb{R})$ is dense in $L^p([0,a],\mathbb{R})$, then there exists a sequence $(h_n)_{n\in\mathbb{N}} \in \mathcal{C}^1([0,a],\mathbb{R})$ that converges to f in $L^p([0,a],\mathbb{R})$, we define $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by:

$$f_n(t) = \begin{cases} t^{\frac{\alpha-1}{p}} h_n(t), & \text{if } \frac{a}{n} \leq t \leq a, \\ P_{n,a}(t) & \text{if } 0 \leq t < \frac{a}{n}, \end{cases}$$

we choose $P_{n,a} : [0,a] \to \mathbb{R}$ such that $P_{n,a} \in \mathcal{C}^{\alpha}\left(\left[0,\frac{a}{n}\right],\mathbb{R}\right)$,

$$\mu_{n,a} = P_{n,a}\left(\frac{a}{n}\right) = \left(\frac{a}{n}\right)^{\frac{\alpha-1}{p}} h_n\left(\frac{a}{n}\right) \quad \text{and} \quad \lim_{t\to\left(\frac{a}{n}\right)^+} P^{(\alpha)}_{n,a}(t) = \lim_{t\to\left(\frac{a}{n}\right)^+} \left(t^v h_n(t)\right)^{(\alpha)} = \lambda_{n,a}. \tag{3.1}$$

It is sufficient to take

$$P_{n,a}(t) = \mu_{n,a} + \lambda_{n,a}\left(t - \frac{a}{n}\right), \text{ for all } t \in \left[0,\frac{a}{n}\right].$$

By Lemma 3.2, we have

$$f^{(\alpha)}_n(t) := \begin{cases} \gamma t^{\frac{\alpha-1}{p}-\alpha} h_n(t) + t^{1+\frac{\alpha-1}{p}-\alpha} h^{(1)}_n(t), & \text{if } \frac{a}{n} \leq t \leq a, \\ \lambda_{n,a}, & \text{if } 0 \leq t < \frac{a}{n}. \end{cases}$$

By (3.1), we get to the conclusion that $(f_n)_{n\in\mathbb{N}} \in \mathcal{C}^{\alpha}([0,a],\mathbb{R})$. Hence,

$$\begin{aligned} \|f_n - f\|^p_{\alpha,p} &= \int_0^{\frac{a}{n}} |P_{n,a}(t) - f(t)|^p t^{1-\alpha} dt + \int_{\frac{a}{n}}^{a} \left|h_n(t) - t^{\frac{\alpha-1}{p}} f(t)\right|^p dt \\ &\leq \int_0^{\frac{a}{n}} |P_{n,a}(t) - f(t)| t^{1-\alpha} dt + \left\|h_n - t^{\frac{\alpha-1}{p}} f\right\|_p, \end{aligned}$$

we conclude that $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^p_{\alpha}([0,a],\mathbb{R})$.

Remark 3.3 Let E, F, G be three spaces such that $E \subset F \subset G$ and (G,τ) is a topological space, then

1) If F is dense in (G,τ) and E is dense in (F,τ), then E is dense in (G,τ).
2) If E is dense in G, then F is dense in G.

Theorem 3.1 has the following conclusions as a result.

Corollary 3.1 *Let $p \in \mathbb{R}$ be such that $p \geq 1$, $\alpha \in (0,1)$ and $a \in (0,\infty)$, then $L^p_{\alpha}([0,a],\mathbb{R})$ is dense in $L^1_{\alpha}([0,a],\mathbb{R})$.*

Proof. Let $f \in L^p_\alpha([0,a],\mathbb{R})$ and $q \in \mathbb{R}$ be such that $\frac{1}{p}+\frac{1}{q}=1$. By Hölder's inequality, we have

$$\|f\|_{1,\alpha}=\int_0^a |f(t)|\, t^{\frac{\alpha-1}{p}} t^{\frac{\alpha-1}{q}} dt \le a^{-\frac{1}{q}} \|f\|_{\alpha,p} < \infty,$$

this means $f \in L^1_\alpha([0,1])$, we get to the formula $\mathcal{C}^\alpha([0,a],\mathbb{R}) \subset L^p_\alpha([0,a],\mathbb{R}) \subset L^1_\alpha([0,a],\mathbb{R})$, by Theorem 3.1, we have $\mathcal{C}^\alpha([0,a],\mathbb{R})$ is dense in $L^1_\alpha([0,a],\mathbb{R})$. Therefore, remark 3.3, implies that $L^p_\alpha([0,a],\mathbb{R})$ is dense in $L^1_\alpha([0,a],\mathbb{R})$. As $\mathcal{C}^{[\alpha]+1}([0,a],\mathbb{R}) \subset \mathcal{C}^\alpha([0,a],\mathbb{R}) \subset \mathcal{C}^n([0,a],\mathbb{R})$, since $\mathcal{C}^{[\alpha]+1}([0,a],\mathbb{R})$ is dense in $\mathcal{C}^n([0,a],\mathbb{R})$, As a result, the remark 3.3 suggests that $\mathcal{C}^\alpha([0,a],\mathbb{R})$ is dense in $\mathcal{C}^n([0,a],\mathbb{R})$.

Lemma 3.2 *Let $\alpha \ge 0$, $n \in \mathbb{N}$ and $a \in (0,\infty)$, such as $\alpha > n$, then $\mathcal{C}^\alpha([0,a],\mathbb{R})$ is dense in $\mathcal{C}^n([0,a],\mathbb{R})$.*

Proof. We have, $\mathcal{C}^{[\alpha]+1}([0,a],\mathbb{R}) \subset \mathcal{C}^\alpha([0,a],\mathbb{R}) \subset \mathcal{C}^n([0,a],\mathbb{R})$, since $\mathcal{C}^{[\alpha]+1}([0,a],\mathbb{R})$ is dense in $\mathcal{C}^n([0,a],\mathbb{R})$, by remark 3.3, implies that $\mathcal{C}^\alpha([0,a],\mathbb{R})$ is dense in $\mathcal{C}^n([0,a],\mathbb{R})$.

The next proposition is found in the same way.

Proposition 3.1 *Let $p \in \mathbb{R}$ be such that $p \ge 1$, $\beta \in (1,\infty)$,$\alpha \in (0,1)$ and $a \in (0,\infty)$, then $\mathcal{C}^\beta([0,a],\mathbb{R})$ is dense in $L^p([0,a],\mathbb{R})$.*

Proof. By lemma 3.2, we have $\mathcal{C}^\beta([0,a],\mathbb{R})$ is dense in $\mathcal{C}^{[\beta]-1}([0,a],\mathbb{R})$, as $[0,a]$ is bounded, then $\mathcal{C}^\beta([0,a],\mathbb{R})$ is dense in $\mathcal{C}^{[\beta]-1}([0,a],\mathbb{R})$ provided with the induced topology of $L^p([0,a],\mathbb{R})$, since $\mathcal{C}^{[\beta]-1}([0,a],\mathbb{R})$ is dense in $L^p([0,a],\mathbb{R})$. Therefore, by remark 3.3, implies that $\mathcal{C}^\beta_0([0,a],\mathbb{R})$ is dense in $L^p([0,a],\mathbb{R})$.

Corollary 3.2 *Let $p \in \mathbb{R}$ be such that $p \ge 1$, $\alpha \in (0,1)$ and $a \in (0,\infty)$, then $W^{\alpha,p}([0,a],\mathbb{R})$ is dense in $L^p_\alpha([0,a],\mathbb{R})$.*

Proof. We have $\mathcal{C}^\alpha([0,1]) \subset W^{1,p}_\alpha([0,1]) \subset L^P_\alpha([0,1])$, by Theorem 3.1, we have $\mathcal{C}^\alpha([0,a],\mathbb{R})$ is dense in $L^p_\alpha([0,a],\mathbb{R})$, by remark 3.3, we have $W^{1,p}_\alpha([0,a],\mathbb{R})$ is dense in $L^p_\alpha([0,a],\mathbb{R})$.

Corollary 3.3 *Let $p \in \mathbb{R}$ be such that $p \ge 1$, $\alpha \in (0,1)$ and $a \in (0,\infty)$, then $W^{\alpha,p}([0,a],\mathbb{R})$ is dense in $L^1_\alpha([0,a],\mathbb{R})$.*

Proof. The proof is the same as Corollary 3.2.

Theorem 3.2 *Let $p \in \mathbb{R}$ be such that $p \ge 1$, $\alpha \in (0,1)$ and $\gamma \in \left(1,\frac{1}{1-\alpha}\right)$, then $L^{\frac{p\gamma}{\gamma-1}}([0,a],\mathbb{R})$ is dense in $L^p_\alpha([0,a],\mathbb{R})$.*

Proof. Let $f \in L^{\frac{p\gamma}{\gamma-1}}([0,a],\mathbb{R})$, by Hölder's inequality, we obtain

$$\begin{aligned}
\|f\|^p_{\alpha,p} &= \int_0^a |f(t)|^p t^{\alpha-1} dt \le \left(\int_0^a |f(t)|^{\frac{p\gamma}{\gamma-1}} dt\right)^{1-\frac{1}{\gamma}} \left(\int_0^a t^{\gamma(\alpha-1)} dt\right)^{\frac{1}{\gamma}} \\
&\le C_{a,\alpha,\gamma} \left(\int_0^\alpha |f(t)|^{\frac{p\gamma}{\gamma-1}} dt\right)^{1-\frac{1}{\gamma}} \\
&= C_{a,\alpha,\gamma} \|f\|^p_{\frac{p\gamma}{\gamma-1}} < \infty,
\end{aligned}$$

with $C_{a,\alpha,\gamma} = a^{\frac{1}{\gamma}-(1-\alpha)} (1-\gamma(1-\alpha))^{\frac{-1}{\gamma}} > 0$. Then $L^{\frac{p\gamma}{\gamma-1}}([0,a],\mathbb{R})$ is injected dense in $L^p_\alpha([0,a],\mathbb{R})$. Therefore, let $f \in L^p_\alpha([0,a],\mathbb{R})$, by Definition 2.3, we have $t^{\frac{\alpha-1}{p}} f \in$

$L^p([0,a],\mathbb{R})$, since $\mathcal{C}([0,1])$ is dense in $L^p([0,a],\mathbb{R})$, then there exists a sequence $(h_n)_{n\in\mathbb{N}} \in \mathcal{C}([0,a],\mathbb{R})$ that converges to $t^{\frac{\alpha-1}{p}} f$ in $L^p([0,a],\mathbb{R})$, we pose

$$f_n(t) := t^{\frac{1-\alpha}{p}} h_n(t), \qquad \text{for all } t \in [0,a].$$

Then

$$\|f_n\|_{L^{\frac{p\gamma}{\gamma-1}}}^{\frac{p\gamma}{\gamma-1}} = \int_0^1 t^{\frac{(1-\alpha)\gamma}{(\gamma-1)}} |h_n(t)|^{\frac{p\gamma}{\gamma-1}} dt \leq (2-\alpha)^{-\frac{1}{\gamma}} \left(\int_0^1 |h_n(t)|^{\frac{p}{\gamma-1}} dt\right)^{1-\frac{1}{\gamma}} < \infty,$$

Thus, $(f_n)_n \in L^{\frac{p\gamma}{\gamma-1}}([0,a],\mathbb{R})$, on the other hand, we have

$$\|f_n - f\|_{L^p} = \left\|f_n - t^{\frac{\alpha-1}{p}} f\right\|_{L^p_\alpha},$$

this means $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^p_\alpha([0,a],\mathbb{R})$.

Example 2 Let $p \in \mathbb{R}$ be such that $p \geq 1$, if $\alpha = \frac{1}{2}$, for all $\gamma \in (1,2)$, we have $L^{3p}([0,a],\mathbb{R})$ is dense in $L^p_{1/2}([0,a],\mathbb{R})$.

Lemma 2.2 yields the subsequent Corollary.

Corollary 3.4 *Let $\alpha,\beta \in (0,1]$, such that $\alpha \leq \beta$, then $\mathcal{C}^\beta([0,a],\mathbb{R})$ is injected in $\mathcal{C}^\alpha([0,a],\mathbb{R})$.*

Proposition 3.2 *Let $\alpha,\beta \in (0,1]$, such that $\alpha \leq \beta$, then $\mathcal{C}^\beta([0,a],\mathbb{R})$ is dense in $\mathcal{C}^\alpha([0,a],\mathbb{R})$.*

Proof. Let $f \in \mathcal{C}^\alpha([0,a],\mathbb{R})$, then $f^{(\alpha)} \in \mathcal{C}([0,a],\mathbb{R})$, since $\mathcal{C}^1([0,a],\mathbb{R})$ is dense in $\mathcal{C}([0,a],\mathbb{R})$, then there exists a sequence $(h_n)_{n\in\mathbb{N}} \in \mathcal{C}^1_0([0,a],\mathbb{R})$ that converges to $f^{(\alpha)}$ in $\mathcal{C}([0,a],\mathbb{R})$, by Lebesgue dominated convergence theorem and Lemma 2.2, we get $(I^\alpha(h_n))_n$ converges to $I^\alpha\left(f^{(\alpha)}\right) = f$, let $(f_n)_b$ be a sequence defined by:

$$f_n(t) := \mathcal{I}^\alpha(h_n)(t), \qquad \text{for all } t \in [0,a].$$

From Lemma 2.2, we have

$$f_n^{(\beta)}(t) = t^{1-\beta}(I^\alpha(h_n))^{(1)}(t) = t^{\alpha-\beta} h_n(t), \quad \text{for all } t \in (0,a].$$

Since $(h_n)_{n\in\mathbb{N}} \in \mathcal{C}^1_0([0,a],\mathbb{R})$, then there exists a neighborhood $\mathcal{V} \subset [0,a]$ of 0 and $\varepsilon : \mathcal{V} \to \mathbb{R}$, such as $\lim\limits_{t\to 0}\varepsilon(t) = 0$ and

$$h_n(t) = h_n^{(1)}(0)\, t + \varepsilon(t), \quad \text{for all } t \in \mathcal{V},$$

by the last equality, there exists a neighborhood $\mathcal{V}_1 \subset [0,a]$ of 0 and $\varepsilon_1 : \mathcal{V}_1 \to \mathbb{R}$, such as $\lim\limits_{t\to 0}\varepsilon_1(t) = 0$ and

$$f_n^{(\beta)}(t) = h_n^{(1)}(0)\, t^{\alpha+1-\beta} + \varepsilon_1(t), \quad \text{for all } t \in \mathcal{V}_1,$$

Thus,

$$f_n^{(\beta)}(0) = \lim_{t\to 0} f_n^{(\beta)}(t) = 0.$$

Finall, we obtain $(f_n)_n \in \mathcal{C}^\beta([0,a],\mathbb{R})$.

4 Conclusion and Application

Utilization of density qualities: In order to demonstrate some findings pertaining to a given function f, it is occasionally helpful to approach the issue from a distance by positioning oneself in an appropriate functional space and utilizing the density properties of specific function subclasses.

We are prompted to illustrate the required attribute for less complex functions as a result. We provide an example of an application that may be attacked using this strategy (Lemma 4.1).

Lemma 4.1 *If* $f \in L^1_\alpha\left([0,a],\mathbb{R}\right)$*, such as*

$$\int_0^a f(t)\,\varphi(t)\,d_\alpha t = 0, \qquad \textit{for } \varphi \in \mathcal{C}\left([0,a],\mathbb{R}\right), \tag{4.1}$$

with $\alpha \in (0,1)$ *and* $a > 0$*, then* $f(t) = 0$ *a.e in* $[0,a]$.

Proof. Let $f \in \mathcal{C}\left([0,a],\mathbb{R}\right)$, such as $\int_0^a f(t)\,\varphi(t)\,d_\alpha t = 0$, for $\varphi \in \mathcal{C}\left([0,a],\mathbb{R}\right)$. I can take $\varphi = f$, we obtain $\|f\|^2_{L^2_\alpha} = \int_0^a |f(t)|^2\,d_\alpha t = 0$, which implies $f(t) = 0$, a.e in $[0,a]$, such that the attribute (4.1) is validated, such as $f \in \mathcal{C}\left([0,a],\mathbb{R}\right)$. As $\mathcal{C}\left([0,a],\mathbb{R}\right)$ is dense in $L^1_\alpha\left([0,a],\mathbb{R}\right)$, then, if $f \in L^1_\alpha\left([0,a],\mathbb{R}\right)$, for $\varepsilon > 0$, then there is $f_\varepsilon \in \mathcal{C}\left([0,a],\mathbb{R}\right)$, such as $\lim\limits_{\varepsilon\to 0} f_\varepsilon = f$ in $L^1_\alpha\left([0,a],\mathbb{R}\right)$ and $\int_0^a f_\varepsilon(t)\,\varphi(t)\,d_\alpha t = 0$, for $\varphi \in \mathcal{C}\left([0,a],\mathbb{R}\right)$, as ε approaches zero, we get the outcome of this lemma.
The proof is finished.

In the graphic below, we show the density between some of the functional spaces based on the findings from the article.

$$\begin{array}{ccccc}
\mathcal{C}^\beta\left([0,a],\mathbb{R}\right) & \longrightarrow & L_\alpha^{\frac{p\gamma}{\gamma-1}}\left([0,a],\mathbb{R}\right) & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
\mathcal{C}^\alpha\left([0,a],\mathbb{R}\right) & \longrightarrow & L^p_\alpha\left([0,a],\mathbb{R}\right) & \longrightarrow & L^1_\alpha\left([0,a],\mathbb{R}\right) \\
\uparrow & \nearrow & \uparrow & \nearrow & \\
\mathcal{C}\left([0,a],\mathbb{R}\right) & \longrightarrow & W^{\alpha,p}\left([0,a],\mathbb{R}\right) & &
\end{array}$$

where $p \in [1,\infty)$, $\beta,\alpha \in (0,1]$, $\gamma \in (1, 1/(1-\alpha))$ and $\alpha \le \beta$.
Since $\mathcal{C}\left([0,a],\mathbb{R}\right)$ is dense in $L^p_\alpha\left([0,a],\mathbb{R}\right)$, we may generalize aspects of the classical ideas (density and injection) and contribute something new. It could be interesting to expand on this topic of fractional Morrey spaces in the follow-up.

Final thought: Using the operators defined in [14, 15], which include as a specific instance the derivative and conformable integral of [13], the findings provided may be generalized.

Definition 4.1 *[14] Let* $f : [0,+\infty) \to \mathbb{R}$*,* $\alpha \in (0,1)$ *and* $F(.,\alpha)$ *be some function. Then, the* $\mathcal{N}$*-derivative of* f *of order* α *is defined by*

$$\mathcal{N}^\alpha_F f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon F(t,\alpha)) - f(t)}{\varepsilon}, \quad \textit{for all} \quad t > 0.$$

Here we will use some cases of F *defined in function of* $E_{a,b}(.)$*, the classic definition of Mittag–Leffler function with* $Re(a)$*,* $Re(b) > 0$*. Also we consider* $E_{a,b}(t^{-\alpha})_k$ *is the k-th term of* $E_{a,b}(.)$*.*
If f *is* α*-differentiable in some* $0 < \alpha \le 1$*, and* $\lim\limits_{t\to 0^+} \mathcal{N}^\alpha_F f(t)$ *exists, then define*

$$\mathcal{N}^\alpha_F f(0) = \lim_{t\to 0^+} \left[\mathcal{N}^\alpha_F f(t)\right].$$

Definition 4.2 *[15]Let $I \subseteq \mathbb{R}$ be an interval, $a, t \in I$ and $\alpha \in \mathbb{R}$. The integral operator $\mathcal{J}^{\alpha}_{F,a+}$, right and left, is defined for every locally integrable function f on I as*

$$\mathcal{J}^{\alpha}_{F,a+} f(t) = \int_a^t \frac{f(s)}{F(t-s,\alpha)} ds, \quad \textit{for all} \quad t > a,$$

and

$$\mathcal{J}^{\alpha}_{F,b-} f(t) = \int_t^b \frac{f(s)}{F(s-t,\alpha)} ds, \quad \textit{for all} \quad b > t.$$

We will also use the "central" integral operator defined by

$$\mathcal{J}^{\alpha}_{F,a} f(b) = \int_a^b \frac{f(t)}{F(t,\alpha)} dt, \quad b > a. \tag{4.2}$$

For instance, Lemma 3.1 may be expressed as follows using these operators (its proof is identical to the one previously provided and is left as an exercise for the reader).

Lemma 4.2 *If F in (4.2) is an absolutely continuous and strictly increasing function with respect to t, let $p \in \mathbb{R}$ be such that $p \geq 1$ and $a \in (0,\infty)$, then $L^p_{F,\alpha}([0,a], \mathbb{R})$ is dense in $L^p([0,a], \mathbb{R})$, where $L^p([0,a], \mathbb{R})$ is considered $L^p_1([0,a], \mathbb{R})$. With $L^p_{F,\alpha}([0,a], \mathbb{R})$ the Banach space endowed with the norm*

$$\|f\|^p_{F,\alpha} = \int_0^a |f(t)|^p \, d^F_\alpha t = \int_0^a |f(t)|^p \frac{dt}{F(t,\alpha)}.$$

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