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On some structural properties in Banach function spaces

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Abstract. *This article deals with some structural properties and subspaces of Banach function spaces on which the additive shift operator $(T_\delta f)(x) = f(x+\delta)$ is isometric. Naturally, constructive description of these subspaces and necessary and sufficient conditions for the functions to belong to these subspaces play an exceptional role here. Note that for grand Lebesgue spaces these conditions are well known.*

Keywords. Banach function space, rearrangement-invariant spaces, additive-invariant norm, Sobolev spaces, Marcinkiewicz spaces, weak Lebesgue spaces, Morrey spaces.

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1 Introduction

The emergence of new function spaces such as Morrey space, grand Lebesgue space, etc. naturally requires the development of corresponding theory. That's why various problems in such spaces and corresponding Sobolev spaces generated by such spaces began to be intensively studied. A lot of works have been dedicated to these issues (see [1–7, 9, 10, 12–15, 17–19, 21, 22, 25]). Therefore, studying differential equations, in particular, solvability problems of elliptic equations in rearrangement-invariant Sobolev spaces generated by rearrangement-invariant Banach function spaces, takes one of the central places in such kind of research. In general, the considered Banach function spaces are not separable. Therefore, using classical methods in these spaces requires the essential modification of classical methods and a lot of preparation, concerning correctness of substitution operator, problems related to the extension operator in such spaces, etc. To this aim, based on the additive shift operator $(T_\delta f)(x) = f(x+\delta)$, corresponding separable subspaces $X_s(\Omega)$ of such spaces are introduced, in which the set of compactly supported infinitely differentiable functions is

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dense (see [4–7,9,10,21,22]). In case of rearrangement-invariant space, where every characteristic function is absolutely continuous, the considered subspace, the subspace of absolutely continuous functions and the closure of the set of simple functions coincide.

Constructive description of these subspaces, sufficient and necessary conditions for the functions to belong to these subspaces of course play an exceptional role here. Note that for grand Lebesgue space these conditions have been given in [14].

In this article, we describe these subspaces of Marcinkiewicz spaces, weak Lebesgue space L_p^w, and Morrey spaces. In Banach function spaces with additive-invariant norm, considered subspaces coincide with the set of absolutely continuous functions, which makes a description of the above conditions a little simpler. Using this fact, we also give the proof of the corresponding theorem for grand Lebesgue spaces.

2 Needful information

We will use the following standard notations: Z will be the set of integers, while Z_+ will denote the set of non-negative integers. $R_+ = [0, +\infty)$. By $m = mes\,(M) = |M|$ we will denote the Lebesgue measure of the set M, $|x| = \sqrt{x_1^2 + ... + x_n^2}$ will be the norm of $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $B_r\,(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ will denote the open ball in $\mathbb{R}^n$, $\partial\Omega$ will stand for the boundary of the domain Ω, and $\overline{\Omega} = \Omega \bigcup \partial\Omega$ will be the closure of Ω. The diameter of the set Ω will be denoted by $d(\Omega) = d_\Omega = diam\Omega$, $\rho\,(x,\ M) = dist\,(x, M)$ will be the distance between x and the set M. By $M_1 \Delta M_2$ we will denote the symmetric difference of the sets M_1 and M_2. Let

$$\Omega_r\,(x_0) = \Omega \bigcap B_r\,(x_0)\,,\ B_r = B_r\,(0)\,,$$

$$\Omega - \delta = \{x :\ x + \delta \in \Omega\}\ (\forall \delta \in \mathbb{R}^n)\,,$$

$$\Omega_\varepsilon = \{x :\ dist\,(x,\ \Omega) < \varepsilon\}\,,\ \ \Omega_{-\varepsilon} = \{x \in \Omega :\ dist\,(x,\ \partial\Omega \geq \varepsilon)\}\ ,\ (\forall \varepsilon > 0)\,.$$

$\Im(\Omega)$ will denote the set of measurable functions on $\Omega \subset \mathbb{R}^n$, and $\Im_0(\Omega)$ the set of finite-valued functions. $[X, Y]$ will be a Banach space of bounded operators acting from X to Y, while $\|T\|_{[X,Y]}$ will stand for the norm of the operator T, which acts from X to Y. Unit balls in Banach function space X and its associate space will be denoted by B_X and $B_{X'}$, respectively. By $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ we will denote a multi-index with the coordinates $\alpha_k \in Z_+,\ \forall k = \overline{1, n};\quad \partial_i = \frac{\partial}{\partial x_i}$ will denote the differentiation operator, with $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}$. For every $\xi = (\xi_1, \xi_2, ..., \xi_n)$, we assume $\xi^\alpha = (\xi_1^{\alpha_1}, \xi_2^{\alpha_2}, ..., \xi_n^{\alpha_n})$.

We will assume the following: let $K = \left\{(x_1, ..., x_n) : |x_i| < \frac{d}{2}\right\} \subset \mathbb{R}^n$ be a cube, $X(K)$ be a Banach function space defined on K with Lebesgue measure and the function norm ρ. For $\Omega \subset K :\ \overline{\Omega} \subset K$, by $X(\Omega)$ we will mean the space of restrictions of all functions from $X(K)$ to Ω with corresponding norm, i.e.

$$X(\Omega) = \Big\{ f \in \Im(K):\ \ \|f\|_{X(\Omega)} = \ \|f \chi_\Omega\|_{X(K)} < +\infty \Big\}\,.$$

Depending on circumstances, we will assume that $f \in X(\Omega)$ is extended by zero to K, or to all of $\mathbb{R}^n$.

For arbitrary function $f \in X(\Omega)$ and for arbitrarily small $\delta \in \mathbb{R}^n : |\delta| < dist\,(\partial\Omega,\ \partial K)$, by T_δ we denote the additive shift operator, defined as

$$(T_\delta f)\,(x) = \begin{cases} f\,(x + \delta)\,, & x + \delta \in \Omega, \\ 0, & x + \delta \notin \Omega\,. \end{cases}$$

By $X_s\,(\Omega)$ we will denote the subspace of all functions from $X\,(\Omega)$ with the following property:

$$\|T_\delta\,(f) - f\|_{X(K)} \to 0, \delta \to 0,$$

where $\delta \in R^n$ is a shift vector.

Moreover, we assume that $X(K)$ has the following property:

Property A).

$$\forall \Omega : \overline{\Omega} \subset K\ \forall f \in X(\Omega),\ \forall\,|\delta| < dist\,(\partial\Omega,\,\partial K) \Rightarrow \|f\|_{X(K)} = \|T_\delta f\|_{X(K)}\,. \tag{2.1}$$

For example, rearrangement-invariant Banach function spaces, Morrey spaces have Property A). In the sequel, such spaces will be called *the spaces with additive-invariant norm or additive-invariant Banach function spaces*.

Let's impose the following conditions:

$$\beta)\quad \forall E_n \to \emptyset \Rightarrow \|\chi_{E_n}\|_{X(K)} \to 0, \tag{2.2}$$

$$\beta')\quad m\,(E) < \infty \Rightarrow \left\|\chi_{E\Delta(E-\delta)}\right\|_{X(K)} \underset{\delta\to 0}{\longrightarrow} 0. \tag{2.3}$$

It should be noted that Property $\beta')$ introduced above is closely connected with the relationship between X_b and X_s. Indeed, satisfaction of Property $\beta')$ guarantees that every characteristic function, consequently, every simple function belongs to $X_s(\Omega)$.

Let's introduce the following spaces of functions:

$$W_X^m(\Omega) = \left\{f \in X(\Omega) :\ \partial^p f \in X, \forall p \in Z_+^n,\ |p| \le m\right\},$$

$$W_{X_s}^m(\Omega) = \left\{f \in W_X^m(\Omega) :\ \|T_\delta f - f\|_{W_X^m(\Omega)} \to 0,\ \delta \to 0\right\},$$

$$\overset{0}{W}{}_{X_s}^m(\Omega) = \overline{C_0^\infty}(\Omega)\ \ (closure\ is\ taken\ in\ the\ space\ W_X^m(\Omega)),$$

with the corresponding norm

$$\|f\|_{W_X^m(\Omega)} = \sum_{|p|\le m} \|\partial^p f\|_{X(\Omega)}\,. \tag{2.4}$$

The shift operator is continuous on $W_X^m(\Omega)$, therefore $W_{X_s}^m(\Omega)$ is a closed subspace of $W_X^m(\Omega)$. It is evident that

$$W_{X_s}^m(\Omega) = \left\{f \in W_{X_s}^m(\Omega) :\ \partial^p f \in X_s,\ \forall p \in Z_+^n,\ |p| \le m\right\}.$$

It is also clear that every function from $\overset{0}{W}{}_{X_s}^m(\Omega) = \overline{C_0^\infty}(\Omega)$ can be extended by zero to all of K.

Corollary 2.1 *If $\beta)$ holds, then $\beta')$ holds too.*

Proof. Indeed, let E be an arbitrary measurable set. Then for arbitrary $\varepsilon > 0$ there is a some finite disjoint set of open sets $U_k,\ k = 1, ..., p$, such that

$$U = \bigcup_k U_k \Rightarrow mes\,(U\Delta E) = mes\,((E-\delta)\,\Delta\,(U-\delta)) < \varepsilon.$$

The relation

$$\begin{aligned} E\Delta\,(E-\delta) = (E\Delta U) \textstyle\bigcup ((E-\delta)\,\Delta U) = \\ = (E\Delta U) \textstyle\bigcup ((E-\delta)\,\Delta\,(U-\delta)) \textstyle\bigcup (U\Delta\,(U-\delta)) \end{aligned} \tag{2.5}$$

implies that it suffices to prove this assertion for open set U, which is obvious.

The corollary is proved.

Definition 2.1 *Let $f \in X$. If for each sequence of measurable sets $\{E_n\}_1^\infty$ with $E_n \to \emptyset$ the relation $\mu - a.e. \Rightarrow \|f\chi_{E_n}\|_X \to 0$ holds, then it is said that f has an absolutely continuous norm. The set of all functions in X with absolutely continuous norms is denoted by X_a. If $X = X_a$, then the space X is said to have an absolute norm. Let X be a Banach function space. The closure of the set of simple functions is denoted by X_b.*

The following lemma describes the relationship between the above spaces.

Lemma 2.1 *a) The following inclusions hold:*

$$L_\infty(\Omega) \subset X(\Omega) \subset L_1(\Omega),\ X_a(\Omega) \subset X_b(\Omega) \subset X(\Omega).$$

b) Subspaces $X_a(\Omega)$ and $X_b(\Omega)$ coincide if and only if for every set $E \subset \Omega$ of finite measure χ_E has an absolutely continuous norm.

Lemma 2.2 below has been proved in [9,21].

Lemma 2.2 *Let X be a rearrangement-invariant Banach function space. If $\beta)$ holds, then $X_s(\Omega) = X_a(\Omega) = X_b(\Omega) = \overline{C_0^\infty(\Omega)}$ (the closure is taken in topology of $X(\Omega)$).*

Remark 2.1 Note that in Lemma 2.2, instead of rearrangement-invariance of the space it suffices to assume that the norm is additive-invariant, i.e. to assume that the equality (2.1) holds.

This follows from the proof of Proposition 3.2 in [21].
In other words, Lemma 2.2 can be formulated in the following exact form.

Lemma 2.3 *Let $X(K)$ be an additive-invariant Banach function space with Property $\beta)$ and $\Omega :\ \overline{\Omega} \subset K$ be any domain. Then*

$$X_s(\Omega) = X_a(\Omega) = X_b(\Omega) = \overline{C_0^\infty(\Omega)}.$$

It is clear that under conditions of Lemma 2.3 we have

$$f \in X_s(\Omega) \Rightarrow T_\delta f \in X\left(\Omega - \delta\right).$$

3 Some general properties

In this section, we are going to formulate some general structural properties about the subspaces $X_s(\Omega)$, $X_a(\Omega)$, $X_b(\Omega)$.

It is obvious that if $X(K)$ has Property $\beta)$, then every compactly supported continuous function belongs to $X_s(\Omega)$, i.e. $C_0^\infty(\Omega) \subset X_s(\Omega)$ and $C\left(\overline{\Omega}\right) \subset X_a(\Omega)$.

Proposition 3.1 *Let $X(K)$ be an additive-invariant Banach function space with Property $\beta)$ and $\Omega :\ \overline{\Omega} \subset K$ be any domain. Then*

a) $X_s(\Omega)$ is separable;

b) if $X_1(\Omega)$ and $X_2(\Omega)$ are Banach function spaces and the inclusion $X_1 \subset X_2$ is true, then the inclusions $(X_1(\Omega))_s \subset (X_2(\Omega))_s$, $(X_1(\Omega))_a \subset (X_2(\Omega))_a$, $(X_1(\Omega))_b \subset (X_2(\Omega))_b$ are also continuous.

c) $X_a(\Omega) \subset X_s(\Omega)$.

Proof. *a)* This is a consequence of Lemma 2.3.

b) This is obvious. Indeed,

$$f \in (X_1)_s \Rightarrow \|T_\delta f - f\|_{X_2} \le c\,\|T_\delta f - f\|_{X_1} \to 0,\ \delta \to 0.$$

Other inclusions are proved similarly.

c) Let $f \in X_a$ be any function. Using Lusin's C-property, we have

$$\forall \delta > 0\, \exists \Omega\,(\delta) \subset \Omega :\ \overline{\Omega\,(\delta)} = \Omega\,(\delta)\,,\ |\Omega \backslash \Omega\,(\delta)| < \delta \Rightarrow f|_{\Omega(\delta)} \in C\,(\Omega\,(\delta))\,.$$

Hence,

$$\|f\,(x+z) - f\,(x)\|_{X(\Omega)} \le \|\varphi_1\,(x,z)\|_{X(K)} + \|\varphi_2\,(x)\|_{X(K)}\,, \tag{3.1}$$

where

$$\varphi_1\,(x,z) = \begin{cases} f\,(x+\ z) - f\,(x)\,, & \text{if } x+z \wedge x \in \Omega\,(\delta)\,, \\ 0, & \text{if } x+z\, \vee x \notin \Omega\,(\delta)\,. \end{cases}$$
$$\varphi_2\,(x,z) = f\,(x+z) - f\,(x) - \varphi_1\,(x,z)\,.$$

It is clear that $f\,(x+z) - f\,(x) = \varphi_1\,(x,z) + \varphi_2\,(x,z)$. From the compactness of $\Omega(\delta)$ and the continuity of $f\,(x)$ on $\Omega(\delta)$ it follows that the first term on the right-hand side of (3.1) is sufficiently small due to uniform continuity of $f|_{\Omega(\delta)}$. Consider the second term. It is obvious that

$$\begin{aligned}
&\mathrm{supp}\varphi_2\,(x,z) = \{x \in K : x+z \in \Omega \wedge x+z \notin \Omega\,(\delta)\} \bigcup \{x \in \Omega :\ x \in \Omega \backslash \Omega\,(\delta)\} \\
&= \{x \in (\Omega - z) \setminus (\Omega\,(\delta) - z)\} \bigcup \{x \in \Omega :\ x \in \Omega \backslash \Omega\,(\delta)\} \\
&\Rightarrow |\mathrm{supp}\varphi_2\,(x,z)| \le |\{x \in (\Omega - z) \setminus (\Omega\,(\delta) - z)\}| \\
&+ |\{x \in \Omega :\ x \in \Omega \backslash \Omega\,(\delta)\}| \le 2\delta.
\end{aligned}$$

From the absolute continuity of the function f it follows that the second term is also small for small δ. Therefore, $f \in X_s \Rightarrow X_a \subset X_s$. The proposition is proved.

Now let Property $\beta')$ holds instead of Property $\beta)$. In this case, for arbitrary E : $m\,(E) < \infty$ we have

$$\left\|\chi_{E\Delta(E-\delta)}\right\|_X \underset{\delta \to 0}{\longrightarrow} 0 \Rightarrow \|T_\delta \chi_E - \chi_E\|_X \underset{\delta \to 0}{\longrightarrow} 0 \Rightarrow \chi_E \in X_s.$$

Consequently, $X_b \subset X_s$.

By Proposition 3.2 of [21], the relation $X_s \subset \overline{C^\infty\left(\overline{\Omega}\right)}$ also holds. On the other hand, every function $\forall f \in C_0^\infty\left(\overline{\Omega}\right)$ is uniformly continuous. Therefore,

$$\begin{aligned}
&\forall \varepsilon > 0\, \exists \delta_\varepsilon > 0\ \forall x_1, x_2 \in \Omega :\ |x_1 - x_2| < \delta_\varepsilon \Rightarrow |f\,(x_1) - f\,(x_2)| < \varepsilon \\
&\Rightarrow \forall z :\ |z| < \delta_\varepsilon\ \Rightarrow \|f\,(x+z) - f\,(x)\|_{X(K)} \le c\,\|f\,(x+z) - f\,(x)\|_{C(K)} < c\,\varepsilon,
\end{aligned}$$

where $c = const$ is independent of x and defined by the embedding $L_\infty(K) \subset X(K)$. Therefore, the inclusion $C_0^\infty(\Omega) \subset X_b$ also holds. Consequently, the inclusion $X_s \subset X_b$ is valid.

In other words, the following is true.

Proposition 3.2 *Let X be an additive-invariant Banach function space with Property $\beta')$ and $\Omega :\ \overline{\Omega} \subset K$ be any domain. Then*

$$X_s(\Omega) = X_b(\Omega) = \overline{C_0^\infty(\Omega)}.$$

The following proposition shows that Properties $\beta)$ and $\beta')$ are equivalent in the rearrangement-invariant space.

Proposition 3.3 *Let X be a rearrangement-invariant Banach function space. Then Properties $\beta)$ and $\beta')$ are equivalent.*

Proof. As we proved above, $\beta)$ implies $\beta')$.

Now let $\beta')$ hold. We are going to prove that $\beta)$ holds. Taking into account that the space $X(\Omega)$ is rearrangement-invariant, we have

$$\forall E,\, F \subset \Omega :\ |E| = |F| \Rightarrow \|\chi_E\|_{X(\Omega)} = \|\chi_F\|_{X(\Omega)} .$$

From this assertion it follows that it suffices to prove

$$\exists E_n :\ |E_n| \to 0 \Rightarrow \|\chi_{E_n}\|_{X(\Omega)} \to 0.$$

But it is obvious. Consider any cube $E \subset \Omega$. It is evident that

$$\forall \delta > 0 :\ E - \delta \subset \Omega \Rightarrow |E \Delta (E - \delta)| > 0 \text{ and } \ |E\Delta(E - \delta)| \to 0,\, \delta \to 0.$$

The proposition is proved.

Proposition 3.4 *Let $X(\Omega)$ be a rearrangement-invariant Banach function space and $X_a(\Omega) = \{0\}$. Then $X(\Omega)$, furthermore, $X_b(\Omega)$ is non-separable.*

Proof. Let's prove that

$$X_a = \{0\} \Rightarrow \exists m > 0\, \forall E \subset \Omega :\ |E| > 0 \Rightarrow \|\chi_E\|_{X(\Omega)} \geq m.$$

Indeed, otherwise Property $\beta)$ would hold. Therefore, $X_a(\Omega) = X_b(\Omega) \neq \emptyset$, which contradicts the condition of the proposition.

Consider any cube $E \subset \Omega$. Then, for arbitrary pair of vectors $z_1 = t_1 z \neq z_2 = t_2 z,\ \forall t_1, t_2 \in \mathbb{R},\ E + z_1 \subset \Omega,\ E + z_2 \subset \Omega$, we have

$$\|\chi_{E+z_1} - \chi_{E+z_2}\|_{X(K)} = \left\|\chi_{(E+z_1)\Delta(E+z_2)}\right\|_{X(K)} \geq m > 0.$$

It is clear that the set of such pairs of vectors is uncountable.

The following lemma was proved in [9,21].

Lemma 3.1 *Let X be a rearrangement-invariant Banach function space with Property $\beta)$ on the domain $\Omega \subset \mathbb{R}^n$. Then, $\forall \varphi \in L_\infty(\Omega)$, $\varphi \cdot f \in X_s(\Omega)$ implies $\varphi f \in X_s$.*

From Remark 2.1 it follows that this lemma can be reformulated in the following exact form.

Lemma 3.2 *Let $X(K)$ be an additive-invariant Banach function space with Property $\beta)$ and $\Omega :\ \overline{\Omega} \subset K$ be any domain. Then $\varphi f \in X_s$, $\forall \varphi \in L_\infty(\Omega)$, $\forall f \in X_s(\Omega)$.*

Proof. Indeed, in this case, by Lemma 2.3 we have $X_s(\Omega) = X_a(\Omega) = X_b(\Omega) = \overline{C_0^\infty(\Omega)}$. It is clear that $\forall \varphi \in L_\infty(\Omega) \Rightarrow \varphi X_s(\Omega) = \varphi X_a(\Omega) \subset X_a(\Omega)$.

The lemma is proved.

The following proposition shows that the converse of the above assertion is also true.

Proposition 3.5 *Let $X(K)$ be an additive-invariant Banach function space with Property $\beta)$ and $\Omega :\ \overline{\Omega} \subset K$ be any domain. Then $f \in X_s$ if and only if*

$$\exists \varphi \in L_\infty(\Omega) :\ \operatorname*{ess\,inf}_{\Omega} |\varphi| = m > 0,\, \varphi f \in X_s(\Omega).$$

Proof. Let $\operatorname{ess\,sup}_{\Omega} |\varphi| = M$. Then, $m|f| \le |\varphi f| \le M|f|$, or $\frac{1}{M}|\varphi f| \le |f| \le \frac{1}{m}|\varphi f|$. Consequently, f and φf are both absolutely continuous.

Consider Lebesgue measure case again. Let $X(K)$ be an additive-invariant Banach function space. Let's prove the following proposition.

Proposition 3.6 *Let $X(K)$ be an additive-invariant Banach function space with Property β). Let the mapping $\varphi : K \to K$ be one-to-one, the composition operator ϕ defined as*

$$(\phi f)(.) = f(\varphi(.)), \forall f \in \Im_0(\Omega) \tag{3.2}$$

and its inverse be bounded operators from $X(K)$ to $X(K)$, and $X_s(\Omega) = X_a(\Omega) = X_b(\Omega)$ for every domain $\Omega : \overline{\Omega} \subset K$. Then

$$\phi(X_a(\Omega)) = \phi(X_b(\Omega)) = \phi(X_s(\Omega)) = X_a(D) = X_b(D) = X_s(D),$$

where $D = \varphi(\Omega) \subset\subset K$.

Proof. Indeed, the transformation (3.2) preserves the set of characteristic functions. Consequently, it also preserves the set of simple functions. Let $f \in X_b(\Omega)$ be any function, and $f = \lim_{n\to\infty} f_n$, where $\{f_n\} \subset X(\Omega)$ is a sequence of simple functions and

$$g = \phi(f) \in X(D), \, g_n = \phi(f_n), \forall n.$$

From the boundedness of the composition operator (3.2) it follows that $\lim_{n\to\infty} g_n = g \in X_b(\Omega)$. Consequently, we have $\phi(X_b(\Omega)) \subset X_b(D)$. And, similarly, from the boundedness of the inverse operator ϕ^{-1} we have $\phi^{-1}(X_b(D)) \subset X_b(\Omega)$. Thus, $\phi(X_b(\Omega)) = X_b(D)$.

The proposition is proved.

4 Description of subspaces $X_s(\Omega)$ of some Banach function spaces

We are going to use Lemmas 2.1-2.3 for description of the subspaces $X_s(\Omega)$ of grand Lebesgue spaces, Marcinkiewicz spaces, weak Lebesgue spaces L_p^w and Morrey spaces. In the sequel, we assume that the function from $X(\Omega)$ is extended by zero on all $\mathbb{R}^n$.

4.1. The grand Lebesgue space $X = L_{p)}(\Omega)$, $1 < p < +\infty$

This is a Banach function space of measurable (in Lebesgue sense) functions $f : \Omega \to C$ with the norm

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_{\Omega} |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}, \; f \in L_{p)}(\Omega).$$

It is well known that the space $L_{p)}(\Omega)$ is a non-separable rearrangement-invariant Banach function space, and Property β) holds. Indeed,

$$E \downarrow 0 \Rightarrow \|\chi_E\|_{p)} = \sup_{0<\varepsilon<p-1} (\varepsilon |E|)^{\frac{1}{p-\varepsilon}} \le ((p-1)|E|)^{\frac{1}{p}} \to 0.$$

Therefore, in this case the relation $X_s = X_a = X_b = \overline{C_0^\infty(\Omega)}$ holds (the closure is taken in topology of $L_{p)}(\Omega)$).

The following theorem is well known (see, for example, [14]). Here we give the proof of this theorem based on Lemmas 2.1-2.3.

Theorem 4.1 *The closure $\overline{C_0^\infty(\Omega)}$ in $L_{p)}(\Omega)$ consists of the functions f which satisfy*

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega |f|^{p-\varepsilon}\,dx=0. \tag{4.1}$$

Proof. $\Rightarrow$. It is clear that the relation (4.1) is satisfied for any function $f\in C_0^\infty(\Omega)$. Indeed,

$$f\in C_0^\infty(\Omega)\Rightarrow \varepsilon\int_\Omega |f|^{p-\varepsilon}\,dx=\varepsilon\,(\max|f|)^{p-\varepsilon}\le \varepsilon\max\left\{1,\,\max_\Omega|f|^p\right\}\underset{\varepsilon\to 0}{\longrightarrow}0.$$

Let $g\in X_s$ be arbitrary function, $\delta>0$ be some positive number and $f\in C_0^\infty(\Omega)$, $\varepsilon>0$: $\|g-f\|_{L_{p)}(\Omega)}<\delta,\ \delta>0,\ \left(\varepsilon\int_\Omega |f|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}}<\delta.$
Then, by Minkowski inequality, we obtain

$$\varepsilon\int_\Omega |g|^{p-\varepsilon}\,dx\le\left((\varepsilon\int_\Omega |f|^{p-\varepsilon}\,dx)^{\frac{1}{p-\varepsilon}}+(\varepsilon\int_\Omega |f-g|^{p-\varepsilon}\,dx)^{\frac{1}{p-\varepsilon}}\right)^{p-\varepsilon}<$$

$$<(2\delta)^{p-\varepsilon}<2\delta\to 0,\ \delta\to 0.$$

"$\Leftarrow$". Let $f\notin X_a=X_b=X_s$, but $\lim\limits_{\varepsilon\to 0}\varepsilon\int_\Omega |f|^{p-\varepsilon}\,dx=0.$
So we have

$$\exists m>0\,\exists E_n\downarrow 0\Rightarrow \|f\chi_{E_n}\|_{p)}\ge m\Leftrightarrow$$
$$\Leftrightarrow \exists\varepsilon_n:\ \left(\varepsilon_n\int_{E_n}|f|^{p-\varepsilon_n}\,dx\right)^{\frac{1}{p-\varepsilon_n}}\ge m.$$

Without loss of generality, it can be assumed that $\lim\varepsilon_n=\varepsilon_0\ne 0$. Taking into account that for every fixed set E the function $\left(z\int_E\left|f(x)^{p-z}\right|dx\right)^{\frac{1}{p-z}}$ is continuous with respect to z, we can take sufficiently large positive integer n_0 such that

$$\forall n>n_0\Rightarrow\left(\varepsilon_{n_0}\int_{E_{n_0}}|f|^{p-\varepsilon_{n_0}}\,dx\right)^{p-\varepsilon_0}\ge const\left(\varepsilon_n\int_{E_{n_0}}|f|^{p-\varepsilon_n}\,dx\right)^{p-\varepsilon_n}\ge$$
$$\ge\left(\varepsilon_n\int_{E_n}|f|^{p-\varepsilon_n}\right)^{p-\varepsilon_n}\ge const\,m\,.$$

Therefore, for example, we can assume

$$\varepsilon_0\int_{E_n}|f|^{p-\varepsilon_0}\,dx>\frac{m}{2}\Rightarrow\int_{E_n}|f|^{p-\varepsilon_0}\,dx\ge const\,m>0,\ \forall n>n_0.$$

On the other hand, the function $f^{p-\varepsilon_0}$ is an absolutely continuous function in the classical sense. Therefore, $\int_{E_n}|f|^{p-\varepsilon_0}\,dx\to 0$. But this contradicts our assumption.

The theorem is proved.

4.2. Marcinkiewicz space $X=M^{p,\lambda}(\Omega)$, $1\le p<+\infty$, $0<\lambda<n$

This is a Banach function space of measurable (in Lebesgue sense) functions on Ω with the norm

$$\|f\|_{p,\lambda}=\sup_I\left(\frac{1}{|I|^{\frac{\lambda}{n}}}\int_I|f|^p\,dt\right)^{\frac{1}{p}},$$

where $I \subset \mathbb{R}^n$ is an arbitrary measurable subset. In particular, if $I = \Omega$, then we have

$$\left(\frac{1}{|\Omega|^{\lambda}} \int_{\Omega} |f|^p \, dt \right)^{\frac{1}{p}} \le \|f\|_{p,\lambda} \Leftrightarrow \|f\|_p \le const \, \|f\|_{p,\lambda},$$

i.e. the continuous inclusion $M^{p,\lambda}(\Omega) \subset L^p(\Omega)$ holds.

Recall that in the classical Morrey space $L^{p,\lambda}(\Omega)$ sup is got on $I = B \bigcap \Omega$, where $B \subset \mathbb{R}^n$ is an arbitrary ball. Unlike $L^{p,\lambda}(\Omega)$, Marcinkiewicz space is rearrangement-invariant. It is clear that the inclusion $M^{p,\lambda}(\Omega) \subset L^{p,\lambda}(\Omega)$ holds. Let's prove that Property $\beta)$ holds in $M^{p,\lambda}(\Omega)$. Let $E: \, |E| \to 0$ and fix any measurable subset $I \subset \Omega$. We have

$$\left(\tfrac{1}{|I|^{\frac{\lambda}{n}}} \int_I \chi_E^p dt \right)^{\frac{1}{p}} = \left(\tfrac{|I \bigcap E|}{|I|^{\lambda/n}} \right)^{\frac{1}{p}} \le \left(\tfrac{|I \bigcap E|}{|I \bigcap E|^{\lambda/n}} \right)^{\frac{1}{p}} = \left(|I \bigcap E|^{1-\frac{\lambda}{n}} \right)^{\frac{1}{p}} \le |E|^{\frac{n-\lambda}{np}} \Rightarrow$$
$$\Rightarrow \|\chi_E\|_{M^{p,\lambda}(\Omega)} \le |E|^{\frac{n-\lambda}{np}} \to 0, \; E \to 0 \, .$$

This space is a rearrangement-invariant Banach function space. Consequently, the relation $X_s(\Omega) = X_a(\Omega) = X_b(\Omega) = \overline{C_0^{\infty}(\Omega)}$ holds.

Theorem 4.2 *The set $C_0^{\infty}(\Omega)$ of finite and infinitely differentiable functions in Ω is not dense in $M^{p,\lambda}(\Omega)$. The closure $\overline{C_0^{\infty}(\Omega)}$ in $M^{p,\lambda}(\Omega)$ consists only of the functions f which satisfy the relation*

$$\frac{1}{|E|^{\lambda/n}} \int_E |f|^p \, dx \to 0, \; E \to 0. \tag{4.2}$$

Proof. It should be noted that if the function has the property (4.2), then it belongs to $M^{p,\lambda}(\Omega)$. Indeed, from (4.2) it follows that

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall E: \, |E| < \varepsilon \Rightarrow \frac{1}{|E|^{\lambda}} \int_E |f|^p \, dx < \varepsilon.$$

Now let's consider the case where $|E| > \delta$. The following is evident:

$$\frac{1}{|E|^{\lambda}} \int_E |f|^p \, dx \le \frac{1}{\delta^{\lambda}} \int_E |f|^p \, dx \le \frac{1}{\delta^{\lambda}} \int_{\Omega} |f|^p \, dx = \frac{1}{\delta^{\lambda}} \|f\|_p{}^p.$$

It follows that $\|f\|_{p,\lambda} \le max\{\varepsilon^{\frac{1}{p}}, \frac{1}{\delta^{\frac{\lambda}{p}}} \|f\|_p\}$.

Now let's prove the assertion.

" $\Rightarrow$ " Let $f \in C_0^{\infty}(\Omega)$. In this case we have

$$\frac{1}{|E|^{\lambda/n}} \int_E |f|^p \, dx \to 0 \le \max |f|^p \, |E|^{1-\frac{\lambda}{n}} \to 0, \; E \to 0.$$

Let $g \in X_s(\Omega)$ be an arbitrary function, $\delta > 0$ be some positive number, $E \subset \Omega$ be an arbitrary measurable subset and

$$f \in C_0^{\infty}(\Omega), \, \varepsilon > 0: \; \|g - f\|_{M^{p,\lambda}(\Omega)} < \delta, \; \left(\frac{1}{|E|^{\lambda/n}} \int_E |f|^p \, dx \right)^{\frac{1}{p}} < \delta.$$

By Minkowski inequality, we obtain

$$\frac{1}{|E|^{\lambda/n}}\int_E |g|^p\,dx \le \left((\frac{1}{|E|^{\lambda/n}}\int_E |f|^p\,dx)^{\frac{1}{p}} + (\frac{1}{|E|^{\lambda/n}}\int_E |f-g|^p\,dx)^{\frac{1}{p}} \right)^p < 2\delta \to 0,\ \delta \to 0.$$

$'' \Leftarrow ''$Assume the contrary. Let the relation (4.2) hold for the function $f \notin X_a(\Omega) = X_s(\Omega) = X_b(\Omega)$, i.e.

$$\frac{1}{|E|^{1-\lambda}}\int_E |f|^p\,dx \to 0,\ \ E \to 0.$$

But

$$\exists m > 0\, \exists I_n :\ |I_n| \underset{n\to\infty}{\longrightarrow} 0\,\&\ \|f\chi_{I_n}\|_{X_s} > m.$$

This means that

$$\exists E_n \Rightarrow \frac{1}{|E_n|^{\lambda/n}}\int_{E_n} |f|^p\,\chi_{I_n}dx = \frac{1}{|E_n|^{\lambda/n}}\int_{E_n\bigcap I_n} |f|^p\,dx \ge m > 0.$$

It should be noted that the last inequality allows us to say that $|E_n \bigcap I_n| = mes\,(E_n \bigcap I_n) > 0$. On the other hand, by (4.2) we have

$$m \le \frac{1}{|E_n|^{\lambda/n}}\int_{E_n\bigcap I_n} |f|^p\,\chi_{I_n}dx \le \frac{1}{|I_n \bigcap E_n|^{\lambda/n}}\int_{I_n\bigcap E_n} |f|^p\,dx \to 0,$$

which shows that our assumption is impossible.

The theorem is proved.

Corollary 4.1 $f \in \left(M^{p,\lambda}(\Omega)\right)_s \Leftrightarrow \lim\limits_{\varepsilon\to 0}\varepsilon^{-\lambda/n}\int_{\{E:|E|=\varepsilon\}} |f|^p\,dx = 0.$

4.3. Weak Lebesgue spaces $L_p^w(\Omega)$

Weak Lebesgue space $L_p^w(\Omega)$ $(1 \le p < \infty,\ 0 < \lambda < n)$ is a space of all functions

$$L_p^w(\Omega) = \left\{ f \in \Im(\Omega) :\ \sup_{0<\lambda<+\infty} \lambda^p m_f(\lambda) < +\infty \right\},$$

where $\Im(\Omega)$ is a set of measurable functions on Ω, and $m_f(\lambda)$ is a distribution function, i.e.

$$m_f(\lambda) = m\,\{x \in M :\ |f(x)| > \lambda\}\,.$$

In [23], the space $M_r(\Omega)$ $(r > 1)$ of measurable functions with the following norm has been considered:

$$\|f\|_{M_r} = \sup_{E\subset\Omega}\frac{1}{|E|^{1-\frac{1}{r}}}\int_E |f|\,dx, \tag{4.3}$$

where sup is taken over all measurable subsets $E \subset \Omega$. The lemma below was proved in [11,20].

Lemma 4.1 *For arbitrary $r > 1$, the spaces $L_r^w(\Omega)$ and $M_r(\Omega)$ coincide with each other: $L_r^w(\Omega) = M_r(\Omega)$.*

In line with our notations, we obtain $M^{1,\lambda}(\Omega) = M_{\frac{n}{n-\lambda}}(\Omega)$, $0 < \lambda < n$. Consequently, we have $L_r^w(\Omega) = M^{1,\lambda}(\Omega)$, where $r = \frac{n}{n-\lambda}$.

Taking into account all these facts, we can formulate the following corollaries.

Corollary 4.2

$$\left(L^{w}_{\frac{n}{n-\lambda}}(\Omega)\right)_s = \left\{ f : \frac{1}{|I|^{\lambda/n}} \int_I |f| \, dx, \; I \to 0 \right\}. \tag{4.4}$$

Corollary 4.3

$$f \in \left(L^{w}_{\frac{n}{n-\lambda}}(\Omega)\right)_s \Leftrightarrow \frac{1}{\varepsilon^{\lambda/n}} \int_{E:|E|=\varepsilon} |f| \, dx \to 0, \; \varepsilon \to 0. \tag{4.5}$$

4.4. Morrey space $L^{p,\lambda}(\Omega), (1 \le p < \infty, \; 0 < \lambda < n)$

The norm in this space is defined as

$$\|f\|_{p,\lambda} = \sup_{B_r \subset \mathbb{R}^n} \left(\frac{1}{r^\lambda} \int_{B_r} |f|^p \, dx \right)^{\frac{1}{p}}, \tag{4.6}$$

where sup is taken from all balls from $\mathbb{R}^n$. Recall that we consider the function that is continued by zero to all of $\mathbb{R}^n$. It is easy to see that this space has Property β). Indeed,

$$\|\chi_E\|_{p,\lambda} = \sup_B \left(\frac{1}{r^\lambda} \int_{B \bigcap E} dx \right)^{\frac{1}{p}}$$

$$\le const \sup_B \left| B \bigcap E \right|^{\frac{n-\lambda}{p}} \le const \, |E|^{\frac{n-\lambda}{p}} \to 0, \; E \to 0.$$

Consequently, $X_a(\Omega) = X_b(\Omega)$. But it is well known that this space is non-separable and non- rearrangement-invariant. By Remark 2.1, every function from $X_s(\Omega)$ can be approximated by the functions from $C_0^\infty(\Omega)$. Consequently, the relation $X_a(\Omega) = X_b(\Omega) = X_s(\Omega) = \overline{C_0^\infty(\Omega)}$ holds.

Theorem 4.3

$$\left(L^{p,\lambda}(\Omega)\right)_s = \left\{ f : \frac{1}{r^\lambda} \int_{B_r} |f|^p \, dx \to 0, \; r \to 0 \right\}, \; (0 < \lambda < n), \tag{4.7}$$

where $B_r \subset \mathbb{R}^n$ is an arbitrary ball.

Proof. It is clear that the relation (4.7) holds for every function from $C_0^\infty(\Omega)$. Indeed,

$$f \in C_0^\infty(\Omega) \Rightarrow \frac{1}{r^\lambda} \int_B |f|^p \, dx \le const \max_B |f|^p \, r^{n-\lambda} \underset{r \to 0}{\longrightarrow} 0.$$

Similar to the proof of Theorem 4.1, we can show that this relation holds for every function from $\left(L^{p,\lambda}(\Omega)\right)_s$.

Let's prove that if the condition (4.7) holds for the function f, then $f \in X_a$. Assume the contrary. Let the relation (4.7) hold for some function $f \notin X_a(\Omega) = X_s(\Omega) = X_b(\Omega)$. From $f \notin X_a(\Omega)$ it follows that

$$\exists m > 0 \, \exists E_n : |E_n| \to 0 \Rightarrow \|f \chi_{E_n}\|_{p,\lambda} > m.$$

In view of (4.6), we have

$$\exists B_n = B_{r_n} \subset \mathbb{R}^n \Rightarrow \frac{1}{{r_n}^\lambda} \int_{B_{r_n}} |f|^p \, \chi_{E_n} dx \ge m > 0.$$

Without loss of generality, it suffices to consider two cases:

Case 1. $|B_n| \to 0$. In this case, we have

$$m \le \frac{1}{{r_n}^\lambda} \int_{B_n} |f|^p \chi_{E_n} dx = \frac{1}{{r_n}^\lambda} \int_{B_n \bigcap E_n} |f|^p \, dx \le \frac{1}{{r_n}^\lambda} \int_{B_n} |f|^p \, dx^{(\text{by } (4.7))} \!\!\to 0,\ n \to 0.$$

Case 2. $\exists \delta > 0\, \forall n \Rightarrow |B_n| > \delta$. In this case, we have

$$m \le \frac{1}{{r_n}^\lambda} \int_{B_n} |f|^p \chi_{E_n} dx = \frac{1}{{r_n}^\lambda} \int_{B_n \bigcap E_n} |f|^p \, dx \le \frac{1}{\delta^{1-\lambda}} \int_{B_n \bigcap E_n} |f|^p \, dx,\ \forall n. \quad (4.8)$$

On the other hand, taking into account that $\forall f \in L^p(\Omega)$ has an absolutely continuous norm and $|B_n \bigcap E_n| \to 0,\ n \to \infty$, it follows that the right-hand side of (4.8) converges to zero. Thus, we arrive at a contradiction again.

The theorem is proved.

Remark 4.1 i) It should be noted that the space $L^{p,\lambda}(\Omega)$ can be defined for $\forall \lambda \ge 0$. But it is well known that $L^{p,\lambda}(\Omega)$ is trivial when $\lambda > n$, i.e. $L^{p,\lambda}(\Omega) = \{0\}$, $L^{p,0}(\Omega) = L^p(\Omega)$, and $L^{p,n}(\Omega) \tilde{=} L^\infty(\Omega)$ (see, for example, [24]).

ii) Also note that $M_s^{p,\lambda}(\Omega) \ne L_s^{p,\lambda}(\Omega)$. Consider the case $\Omega = (0;1)$ and the subsets

$$E_n = \bigcup_{k=\overline{1,n}} (a_{nk}; b_{nk})\,,\ a_{n1} = 0,\ b_{nn} =$$

$$= 1,\ b_{nk} = a_{nk} + x_n,\ a_{n(k+1)} = b_{nk} + y_n,\ n\,(x_n + y_n) = 1.$$

Let's calculate the norms of characteristic functions $\chi_{E_n}$of these subsets. Taking into account that $M^{p,\lambda}$ is rearrangement-invariant, we have

$$\|\chi_{E_n}\|_{M^{p,\lambda}(0;1)} = \left\|\chi_{(0;nx_n)}\right\|_{M^{p,\lambda}(0;1)} = \left(\frac{1}{(nx_n)^\lambda} \int_0^{nx_n} dx\right)^{\frac{1}{p}} = (nx_n)^{\frac{1-\lambda}{p}}\,,$$

where we used the relation

$$\frac{1}{a^\lambda} \int_0^a dx = a^{1-\lambda} < b^{1-\lambda} = \frac{1}{b^\lambda} \int_0^b dx,\ a < b\,.$$

In Morrey space case, we have the following estimates:

$0 < z \le x_n \Rightarrow \frac{1}{z^\lambda} \int_0^z dx = z^{1-\lambda} \le {x_n}^{1-\lambda} = \frac{1}{{x_n}^\lambda} \int_0^{x_n} dx,$

$\frac{1}{(a_{k+1}+z)^\lambda} \int_0^{a_{k+1}+z} \chi_{(0;a_k+z) \bigcap E_n} dx = \frac{kx_n+z}{(k(x_n+y_n)+z)^\lambda} = (kx_n + z)^{1-\lambda} \left(\frac{kx_n+z}{kx_n+ky_n+z}\right)^\lambda =$

$= (kx_n + z)^{1-\lambda} \left(1 - \frac{ky_n}{kx_n+ky_n+z}\right)^\lambda \le ((k+1)x_n)^{1-\lambda} \left(\frac{(k+1)x_n}{(k+1)x_n+ky_n}\right)^\lambda = \frac{(k+1)x_n}{((k+1)x_n+ky_n)^\lambda}.$

Let $y_n \ge t_n x_n :\ n^{1-\lambda} < {t_n}^\lambda,$ or $(nt_n)^\lambda > n,\ t_n > 2$. Then we have

$$\frac{(k+1)\,x_n}{((k+1)\,x_n + ky_n)^\lambda} \le \frac{(k+1)\,x_n}{(t_n\,(k+1)\,x_n)^\lambda} \le \frac{n^{1-\lambda}}{{t_n}^\lambda} {x_n}^{1-\lambda} < {x_n}^{1-\lambda},$$

from which it follows that $\|\chi_{E_n}\|_{L^{p,\lambda}(0;1)} = {x_n}^{1-\lambda}$. Finally, as a result, we have

$$\frac{\|\chi_{E_n}\|_{M^{p,\lambda}(0;1)}}{\|\chi_{E_n}\|_{L^{p,\lambda}(0;1)}} = n^{\frac{1-\lambda}{p}} \to \infty,\ n \to \infty,$$

i.e. the embedding $L^{p,\lambda} \subset M^{p,\lambda}$ is impossible.

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