Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **44** (1), 3-19 (2024).
https://doi.org/10.30546/2617-7900.44.1.2024.3

Fractional maximal operator associated with Schrödinger operator and its commutators on vanishing generalized Morrey spaces

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Received: 21.06.2023 / Revised: 19.01.2024 / Accepted: 02.03.2024

Abstract. Let $\mathcal{L} = -\triangle + V$ be a Schrödinger operator, where the non-negative potential V belongs to the reverse Hölder class $RH_{n/2}$, let b belong to a new $BMO_\theta(\rho)$ space which is larger than the classical BMO space, and let $M^\theta_{\beta,V}$ be the fractional maximal operator associated with $\mathcal{L}$. In this paper, we study the boundedness of the operator $M^\theta_{\beta,V}$ and its commutators $[b, M^\theta_{\beta,V}]$ with $b \in BMO_\theta(\rho)$ on generalized Morrey spaces $M^{\alpha,V}_{p,\varphi}$ associated with Schrödinger operator and vanishing generalized Morrey spaces $VM^{\alpha,V}_{p,\varphi}$ associated with Schrödinger operator. We find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators $M^\theta_{\beta,V}$ from one vanishing generalized Morrey space $VM^{\alpha,V}_{p,\varphi_1}$ to another $VM^{\alpha,V}_{q,\varphi_2}$, $1/p - 1/q = \beta/n$.

Keywords Schrödinger operator; fractional maximal operator; commutator; BMO; generalized Morrey space.

MR(2010) Subject Classification 42B35, 35J10

1 Introduction and results

In this paper, we consider the Schrödinger differential operator

$$\mathcal{L} = -\Delta + V(x) \text{ on } \mathbb{R}^n,\ n \geq 3,$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class RH_q for $q \geq n/2$.

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A nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to RH_q, $1 < q \le \infty$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \le \left(\frac{C}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \tag{1.1}$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a nonnegative polynomial, then $V \in RH_\infty$. Obviously, $RH_{q_2} \subset RH_{q_1}$, if $q_1 < q_2$. It is worth pointing out that the RH_q class is such that, if $V \in RH_q$ for some $q > 1$, then there exists an $\epsilon > 0$, which depends only n and the constant C in (1.1), such that $V \in RH_{q+\epsilon}$. Throughout this paper, we always assume that $0 \ne V \in RH_{n/2}$.

For $x \in \mathbb{R}^n$, the function $\rho(x)$ is defined by

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \ne 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim 1 + |x|$ with $V(x) = |x|^2$.

According to [3], the new BMO space $BMO_\theta(\rho)$ with $\theta \ge 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_B|dy \le C\Big(1 + \frac{r}{\rho(x)}\Big)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(y)dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

The classical Morrey spaces were originally introduced by Morrey in [16] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7,8,11,16]. The classical version of Morrey spaces is equipped with the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,r))},$$

where $0 \le \lambda < n$ and $1 \le p < \infty$. The generalized Morrey spaces are defined with r^λ replaced by a general non-negative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [2,9–11,15,17,18] and etc).

The vanishing Morrey space $VM_{p,\lambda}$ in its classical version was introduced in [24], where applications to PDE were considered. We also refer to [5] and [19] for some properties of such spaces. This is a subspace of functions in $M_{p,\lambda}(\mathbb{R}^n)$, which satisfy the condition

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n, 0<t<r} t^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,t))} = 0.$$

Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [9,17,18] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [2,11,20]).

We now present the definition of generalized Morrey spaces (including weak version) associated with Schrödinger operator, which introduced by Guliyev in [12].

Definition 1.1 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. We denote by $M_{p,\varphi}^{\alpha,V} = M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ the generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L_{loc}^p(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{x\in\mathbb{R}^n, r>0} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} r^{-n/p} \|f\|_{L_p(B(x,r))}.$$

Also $WM_{p,\varphi}^{\alpha,V} = WM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ we denote the weak generalized Morrey space associated with Schrödinger operator,the space of all functions $f \in WL_{loc}^p(\mathbb{R}^n)$ with

$$\|f\|_{WM_{p,\varphi}^{\alpha,V}} = \sup_{x\in\mathbb{R}^n, r>0} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} r^{-n/p} \|f\|_{WL_p(B(x,r))} < \infty.$$

Remark 1.1 (i) When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey in [16];

(ii) When $\varphi(x,r) = r^{(\lambda-n)/p}$, $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the Morrey space associated with Schrödinger operator $L_{p,\lambda}^{\alpha,V}(\mathbb{R}^n)$ studied by Tang and Dong in [22];

(iii) When $\alpha = 0$, $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ introduced by Guliyev, Mizuhara and Nakai in [9, 17, 18].

(iv) The generalized Morrey space associated with Schrödinger operator $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ was introduced by Guliyev in [12].

For brevity, in the sequel we use the notations

$$\mathfrak{A}_{p,\varphi}^{\alpha,V}(f;x,r) := \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} r^{-n/p}\, \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}_{\Phi,\varphi}^{W,\alpha,V}(f;x,r) := \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} r^{-n/p}\, \varphi(x,r)^{-1} \|f\|_{WL_p(B(x,r))}.$$

Definition 1.2 *The vanishing generalized Morrey space associated with Schrödinger operator $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f;x,r) = 0. \tag{1.2}$$

The vanishing weak generalized Morrey space associated with Schrödinger operator $VWM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi}^{W,\alpha,V}(f;x,r) = 0.$$

The vanishing spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and $VWM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM_{p,\varphi}^{\alpha,V}} \equiv \|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f;x,r),$$

$$\|f\|_{VWM_{p,\varphi}^{\alpha,V}} \equiv \|f\|_{WM_{p,\varphi}^{\alpha,V}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{W,p,\varphi}^{\alpha,V}(f;x,r),$$

respectively.

Given a function $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

and the fractional operator function M_β is defined by

$$M_\beta f(x) := \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\beta}{n}}} \int_{B(x,r)} |f(y)|dy, \quad 0 < \beta < n.$$

Definition 1.3 *Let $\mathcal{L} = -\triangle + V$ with $V \in RH_{n/2}$. A variant of Hardy-Littlewood maximal operator M_V^θ (see [3]) is defined by*

$$M_V^\theta f(x) := \sup_{r>0} \frac{1}{\Psi_\theta(B(x,r))|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

and a variant of fractional maximal operator $M_{\beta,V}^\theta$ (see [23]) is defined by

$$M_{\beta,V}^\theta f(x) := \sup_{r>0} \frac{1}{(\Psi_\theta(B(x,r))|B(x,r)|)^{1-\frac{\beta}{n}}} \int_{B(x,r)} |f(y)|dy, \quad 0 < \beta < n.$$

The fractional integral associated with $\mathcal{L}$ is defined by

$$\mathcal{I}_\beta f(x) = \mathcal{L}^{-\beta/2} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x) \frac{dt}{t^{-\beta/2+1}}$$

for $0 < \beta < n$. Let $b \in BMO_\theta(\rho)$. The commutator of $\mathcal{I}_\beta$ is defined by

$$[b, \mathcal{I}_\beta] f(x) = b(x)\mathcal{I}_\beta f(x) - \mathcal{I}_\beta(bf)(x).$$

We now present the definition of generalized Morrey spaces related to certain nonnegative potentials.

In this paper, we consider the boundedness of the fractional integral operator $M_{\beta,V}^\theta$ on the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$. When b belongs to the new BMO space $BMO_\theta(\rho)$, we also show that $[b, M_{\beta,V}^\theta]$ is bounded on $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $M_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$. Our main results are as follows.

Theorem 1.1 *Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfies the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \leq c_0 \varphi_2(x,r), \tag{1.3}$$

where c_0 does not depend on x and r. Then the operator $M_{\beta,V}^\theta$ is bounded on $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ for $p > 1$ and from $M_{1,\varphi_1}^{\alpha,V}$ to $WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}$. Moreover, for $p > 1$

$$\|M_{\beta,V}^\theta f\|_{M_{q,\varphi_2}^{\alpha,V}} \leq C \|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

and for $p = 1$

$$\|M_{\beta,V}^\theta f\|_{WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}} \leq C \|f\|_{M_{1,\varphi_1}^{\alpha,V}},$$

where C does not depend on f.

Theorem 1.2 *Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfies the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \leq c_0 \varphi_2(x,r), \tag{1.4}$$

where c_0 does not depend on x and r. If $b \in BMO_\theta(\rho)$, then the operator $[b, M^\theta_{\beta,V}]$ is bounded from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{q,\varphi_2}$ and

$$\|[b, M^\theta_{\beta,V}]f\|_{M^{\alpha,V}_{q,\varphi_2}} \leq C[b]_\theta \|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where C does not depend on f.

Theorem 1.3 *Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 \leq p < \infty$ and $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ satisfies the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t} < \infty$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x,t) \frac{dt}{t^{1-\beta}} \leq C_0 \varphi_2(x,r), \tag{1.5}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $M^\theta_{\beta,V}$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$ for $p > 1$ and from $VM^{\alpha,V}_{1,\varphi_1}$ to $VWM^{\alpha,V}_{\frac{n}{n-\beta},\varphi_2}$.

Theorem 1.4 *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$, $1 < p < \infty$, and $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ satisfies the conditions*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \varphi_1(x,t) t^\beta \leq c_0 \varphi_2(x,r), \tag{1.6}$$

where c_0 does not depend on x and r,

$$\lim_{r\to 0} \frac{\ln \frac{1}{r}}{\inf_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0 \tag{1.7}$$

and

$$c_\delta := \int_\delta^\infty \Big(1 + |\ln t|\Big) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t^{1-\beta}} < \infty \tag{1.8}$$

for every $\delta > 0$. Then the operator $[b, M^\theta_{\beta,V}]$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some Preliminaries

We would like to recall the important properties concerning the critical function.

Lemma 2.1 *[21] Let $V \in RH_{n/2}$. For the associated function ρ there exist C and $k_0 \geq 1$ such that*

$$C^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq C\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{1+k_0}} \tag{2.1}$$

for all $x, y \in \mathbb{R}^n$.

Lemma 2.2 *Suppose $x \in B(x_0, r)$. Then for $k \in N$ we have*

$$\frac{1}{\Big(1+\frac{2^k r}{\rho(x)}\Big)^N} \lesssim \frac{1}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}.$$

Proof. By (2.1) we have

$$\begin{aligned}\frac{1}{\Big(1+\frac{2^k r}{\rho(x)}\Big)^N} &\lesssim \frac{1}{\Bigg(1+\frac{2^k r}{\rho(x_0)\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{\frac{k_0}{k_0+1}}}\Bigg)^N} \\ &\lesssim \frac{\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{\frac{k_0 N}{k_0+1}}}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^N} \lesssim \frac{1}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}.\end{aligned}$$

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

Lemma 2.3 *[3] Let $1 \leq s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\Big(\frac{1}{|B|}\int_B |b(y)-b_B|^s dy\Big)^{1/s} \leq [b]_\theta \Big(1+\frac{r}{\rho(x)}\Big)^{\theta'}$$

for all $B = B(x, r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in (2.1).

Lemma 2.4 *[3] Let $1 \leq s < \infty$, $b \in BMO_\theta(\rho)$, and $B = B(x, r)$. Then*

$$\Big(\frac{1}{|2^k B|}\int_{2^k B} |b(y)-b_B|^s dy\Big)^{1/s} \leq [b]_\theta k \Big(1+\frac{2^k r}{\rho(x)}\Big)^{\theta'}$$

for all $k \in \mathbb{N}$, with θ' as in Lemma 2.3.

Let K_β be the kernel of $\mathcal{I}_\beta$. The following result give the estimate on the kernel $K_\beta(x, y)$.

Lemma 2.5 *[4] If $V \in RH_{n/2}$, then for every N, there exists a constant C such that*

$$|K_\beta(x,y)| \leq \frac{C}{\Big(1+\frac{|x-y|}{\rho(x)}\Big)^N}\frac{1}{|x-y|^{n-\beta}}. \tag{2.2}$$

Finally, we recall a relationship between essential supremum and essential infimum.

Lemma 2.6 *[25] Let f be a real-valued nonnegative function and measurable on E. Then*

$$\Big(\operatorname*{ess\,inf}_{x\in E} f(x)\Big)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

Lemma 2.7 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$.*

(i) If

$$\sup_{t<r<\infty} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty \quad \textit{for some } t>0 \textit{ and for all } x\in\mathbb{R}^n, \tag{2.3}$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} = \infty \quad \textit{for some } \tau>0 \textit{ and for all } x\in\mathbb{R}^n, \tag{2.4}$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n) = \Theta$.

Proof. (i) Let (2.3) be satisfied and f be not equivalent to zero. Then $\sup_{x\in\mathbb{R}^n} \|f\|_{L_p(B(x,t))} > 0$, hence

$$\begin{aligned}\|f\|_{M^{\alpha,V}_{p,\varphi}} &\ge \sup_{x\in\mathbb{R}^n} \sup_{t<r<\infty} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x,r))} \\ &\ge \sup_{x\in\mathbb{R}^n} \|f\|_{L_p(B(x,t))} \sup_{t<r<\infty} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} r^{-\frac{n}{p}}.\end{aligned}$$

Therefore $\|f\|_{M^{\alpha,V}_{p,\varphi}} = \infty$.

(ii) Let $f \in M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and (2.4) be satisfied. Then there are two possibilities:

Case 1: $\sup_{0<r<t} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} = \infty$ for all $t>0$.

Case 2: $\sup_{0<r<t} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} < \infty$ for some $t \in (0,\tau)$.

For Case 1, by Lebesgue differentiation theorem, for almost all $x \in \mathbb{R}^n$,

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L_p}}{\|\chi_{B(x,r)}\|_{L_p}} = |f(x)|. \tag{2.5}$$

We claim that $f(x) = 0$ for all those x. Indeed, fix x and assume $|f(x)| > 0$. Then by (2.5) there exists $t_0 > 0$ such that

$$r^{-\frac{n}{p}} \|f\|_{L_p(B(x,r))} \ge 2^{-1} v_n^{\frac{1}{p}} |f(x)|$$

for all $0 < r \le t_0$. Consequently,

$$\begin{aligned}\|f\|_{M^{\alpha,V}_{p,\varphi}} &\ge \sup_{0<r<t_0} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x,r))} \\ &\ge 2^{-1} v_n^{\frac{1}{p}} |f(x)| \sup_{0<r<t_0} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1}.\end{aligned}$$

Hence $\|f\|_{M^{\alpha,V}_{p,\varphi}} = \infty$, so $f \notin M_{p,\varphi}(\mathbb{R}^n)$ and we have arrived at a contradiction.

Note that Case 2 implies that $\sup\limits_{s<r<\tau} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x, r)^{-1} = \infty$, hence

$$\sup_{s<r<\infty} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x, r)^{-1} r^{-\frac{n}{p}} \geq \sup_{s<r<\tau} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x, r)^{-1} r^{-\frac{n}{p}} \geq \tau^{-\frac{n}{p}} \sup_{s<r<\tau} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x, r)^{-1} = \infty,$$

which is the case in (i).

Remark 2.1 We denote by $\Omega_p^{\alpha,V}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0, \infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \Big\| \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x, r)} \Big\|_{L_\infty(t,\infty)} < \infty, \text{ and } \sup_{x\in\mathbb{R}^n} \Big\| \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x, r)^{-1} \Big\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.7, we always assume that $\varphi \in \Omega_p^{\alpha,V}$.

Remark 2.2 We denote by $\Omega_{p,1}^{\alpha,V}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0, \infty)$ such that

$$\inf_{x\in\mathbb{R}^n} \inf_{r>\delta} \left(1 + \frac{r}{\rho(x)}\right)^{-\alpha} \varphi(x, r) > 0, \text{ for some } \delta > 0, \tag{2.6}$$

and

$$\lim_{r\to 0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{n/p}}{\varphi(x, r)} = 0,$$

For the non-triviality of the space $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_{p,1}^{\alpha,V}$.

3 Proof of Theorem 1.1

We first prove the following conclusions

Theorem 3.1 *Let $V \in RH_{n/2}$. If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ then the inequality*

$$\|M_{\beta,V}^{\theta} f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \sup_{2r<t<\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}$$

holds for any $f \in L_{loc}^p(\mathbb{R}^n)$. Moreover, for $p = 1$ the inequality

$$\|M_{\beta,V}^{\theta} f\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \sup_{2r<t<\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}}$$

holds for any $f \in L_{loc}^1(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ and $\lambda B = B(x_0, \lambda r)$ for any $\lambda > 0$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0, 2r)$. Then

$$\|M^{\theta}_{\beta,V} f\|_{L_q(B(x_0,r))} \le \|M^{\theta}_{\beta,V}(f_1)\|_{L_q(B(x_0,r))} + \|M^{\theta}_{\beta,V}(f_2)\|_{L_q(B(x_0,r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of $\mathcal{I}_\beta$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that

$$\begin{aligned}\|M^{\theta}_{\beta,V}(f_1)\|_{L_q(B(x_0,r))} &\lesssim \|f\|_{L_p(B(x_0,2r))} \\ &\lesssim r^{\frac{n}{q}} \|f\|_{L_p(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}} \\ &\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \end{aligned} \tag{3.1}$$

To estimate $\|M^{\theta}_{\beta,V}(f_2)\|_{L_p(B(x_0,r))}$, obverse that $x \in B$, $y \in (2B)^c$ implies $|x - y| \approx |x_0 - y|$. Then by (2.2) we have

$$\begin{aligned}\sup_{x\in B} |M^{\theta}_{\beta,V}(f_2)(x)| &\le \int_{(2B)^c} |K_\beta(x,y) f(y)| dy \\ &\lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-\beta}} dy \\ &\lesssim \sum_{k=1}^{\infty} (2^{k+1} r)^{-n+\beta} \int_{2^{k+1}B} |f(y)| dy.\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}\sup_{x\in B} |M^{\theta}_{\beta,V}(f_2)(x)| &\lesssim \sum_{k=1}^{\infty} \|f\|_{L_p(2^{k+1}B)} (2^{k+1} r)^{-1-\frac{n}{p}+\beta} \int_{2^k r}^{2^{k+1} r} dt \\ &\lesssim \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1} r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\ &\lesssim \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.\end{aligned} \tag{3.2}$$

Then

$$\|M^{\theta}_{\beta,V}(f_2)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{3.3}$$

holds for $1 \le p < n/\beta$. Therefore, by (3.1) and (3.3) we get

$$\|M^{\theta}_{\beta,V} f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{3.4}$$

holds for $1 \le p < n/\beta$.

When $p = 1$, by the boundedness of $M^{\theta}_{\beta,V}$ from $L_1(\mathbb{R}^n)$ to $WL_{\frac{n}{n-\beta}}(\mathbb{R}^n)$, we get

$$\|M^{\theta}_{\beta,V}(f_1)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim \|f\|_{L_1(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

By (3.3) we have

$$\|M_{\beta,V}^{\theta}(f_2)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \leq \|M_{\beta,V}^{\theta}(f_2)\|_{L_{\frac{n}{n-\beta}}(B(x_0,2r))} \lesssim r^{n-\beta}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}}\frac{dt}{t}.$$

Then

$$\|M_{\beta,V}^{\theta}f\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}}\frac{dt}{t}.$$

Proof of Theorem 1.1 From Lemma 2.6, we have

$$\frac{1}{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}} = \operatorname*{ess\,sup}_{t<s<\infty}\frac{1}{\varphi_1(x,s)s^{\frac{n}{p}}}.$$

Note the fact that $\|f\|_{L_p(B(x_0,t))}$ is a nondecresing function of t, and $f \in M_{p,\varphi_1}^{\alpha,V}$, then

$$\begin{aligned}
&\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}\\
&\lesssim \operatorname*{ess\,sup}_{t<s<\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x,s)s^{\frac{n}{p}}}\\
&\lesssim \sup_{0<s<\infty}\frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,s))}}{\varphi_1(x,s)s^{\frac{n}{p}}}\\
&\lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}.
\end{aligned}$$

Since $\alpha \geq 0$, and (φ_1,φ_2) satisfies the condition (1.3), then

$$\begin{aligned}
&\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\
&=\int_{2r}^{\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}\int_{2r}^{\infty}\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\int_{r}^{\infty}\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\varphi_2(x_0,r).
\end{aligned}\tag{3.5}$$

Then by Theorem 3.1 we get

$$\begin{aligned}
&\|M^{\theta}_{\beta,V}f\|_{M^{\alpha,V}_{q,\varphi_2}}\\
&\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha}\varphi_2(x_0,r)^{-1}r^{-n/q}\|\mathcal{I}_\beta f\|_{L_p(B(x_0,r))}\\
&\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha}\varphi_2(x_0,r)^{-1}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.
\end{aligned}$$

Let $q=\frac{n}{n-\beta}$, similar to the estimates of (3.5) we have

$$\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}}\frac{dt}{t}\lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}}\Big(1+\frac{r}{\rho(x_0)}\Big)^{-\alpha}\varphi_2(x_0,r).$$

Thus by Theorem 3.1 we get

$$\begin{aligned}
&\|M^{\theta}_{\beta,V}f\|_{WM^{\alpha,V}_{\frac{n}{n-\beta},\varphi_2}}\\
&\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha}\varphi_2(x_0,r)^{-1}r^{\beta-n}\|\mathcal{I}_\beta f\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))}\\
&\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha}\varphi_2(x_0,r)^{-1}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}}\frac{dt}{t}\\
&\lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}}.
\end{aligned}$$

4 Proof of Theorem 1.2

As the proof of Theorem 1.1, it suffices to prove the following result.

Theorem 4.1 *Let $V\in RH_{n/2}$, $b\in BMO_\theta(\rho)$. If $1<p<n/\beta$, $1/q=1/p-\beta/n$ then the inequality*

$$\|[b,M^{\theta}_{\beta,V}f]\|_{L_q(B(x_0,r))}\lesssim [b]_\theta r^{\frac{n}{q}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t} \tag{4.1}$$

holds for any $f\in L^p_{loc}(\mathbb{R}^n)$.

Proof. We write f as $f=f_1+f_2$, where $f_1(y)=f(y)\chi_{B(x_0,2r)(y)}$. Then

$$\|[b,M^{\theta}_{\beta,V}]f\|_{L_q(B(x_0,r))}\leq \|[b,M^{\theta}_{\beta,V}](f_1)\|_{L_q(B(x_0,r))}+\|[b,M^{\theta}_{\beta,V}](f_2)\|_{L_q(B(x_0,r))}.$$

By the boundedness of $[b,M^{\theta}_{\beta,V}]$ on $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ and (3.1) we get

$$\begin{aligned}
\|[b,M^{\theta}_{\beta,V}](f_1)\|_{L_q(B(x_0,r))}&\lesssim [b]_\theta\|f\|_{L_p(B(x_0,2r))}\\
&\lesssim [b]_\theta r^{\frac{n}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\
&\lesssim [b]_\theta r^{\frac{n}{q}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.
\end{aligned} \tag{4.2}$$

We now turn to deal with the term $\|[b, M^{\theta}_{\beta,V}](f_2)\|_{L_q(B(x_0,r))}$. For any given $x \in B(x_0, r)$ we have

$$|[b, M^{\theta}_{\beta,V}]f_2(x)| \leq |b(x) - b_{2B}|\,|\mathcal{I}_{\beta}(f_2)(x)| + |\mathcal{I}_{\beta}((b - b_{2B})f_2)(x)|.$$

By (2.2), Lemma 2.2 and (3.2) we have

$$\begin{aligned}\sup_{x\in B}|M^{\theta}_{\beta,V}(f_2)(x)| &\lesssim \int_{(2B)^c} \frac{1}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{|f(y)|}{|x_0-y|^{n-\beta}}dy \\ &\lesssim \frac{1}{\left(1+\frac{2r}{\rho(x)}\right)^N} \sum_{k=1}^{\infty}(2^{k+1}r)^{-n+\beta}\int_{2^{k+1}B}|f(y)|dy \\ &\lesssim \frac{1}{\left(1+\frac{2r}{\rho(x_0)}\right)^{N/(k_0+1)}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.\end{aligned}$$

Then by Lemma 2.3, and taking $N \geq (k_0+1)\theta$ we get

$$\begin{aligned}&\|(b(x) - b_{2B})M^{\theta}_{\beta,V}(f_2)\|_{L_q(B(x_0,r))} \\ &\lesssim [b]_{\theta} r^{\frac{n}{q}}\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta-N/(k_0+1)}\int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t} \\ &\lesssim [b]_{\theta} r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \qquad (4.3)\end{aligned}$$

Finally, let us estimate $\|M^{\theta}_{\beta,V}((b - b_{2B})f_2)\|_{L_q(B(x_0,r))}$. By (2.2), Lemma 2.2 and (3.2) we have

$$\begin{aligned}&\sup_{x\in\beta}|M^{\theta}_{\beta,V}((b-b_{2B})f_2)(x)| \\ &\lesssim \int_{(2B)^c} \frac{1}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{|b(y)-b_{2B}||f(y)|}{|x_0-y|^{n-\beta}}dy \\ &\lesssim \sum_{k=1}^{\infty}\frac{1}{(2^k r)^{n-\beta}\left(1+\frac{2^k r}{\rho(x)}\right)^N}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy \\ &\lesssim \sum_{k=1}^{\infty}\frac{1}{(2^k r)^{n-\beta}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy.\end{aligned}$$

Note that

$$\begin{aligned}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy &\lesssim \left(\int_{2^{k+1}B}|b(y)-b_{2B}|^{p'}\right)^{1/p'}\|f\|_{L_p(B(x_0,2^{k+1}r))} \\ &\lesssim [b]_{\theta} k\left(1+\frac{2^k r}{\rho(x_0)}\right)^{\theta'}(2^k r)^{\frac{n}{p'}}\|f\|_{L_p(B(x_0,2^{k+1}r))}.\end{aligned}$$

Then

$$\begin{aligned}\sup_{x\in B}|M^{\theta}_{\beta,V}((b-b_{2B})f_2)(x)| &\lesssim [b]_\theta\sum_{k=1}^{\infty}\frac{k(2^kr)^{-\frac{n}{p}+\beta}}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{N/(k_0+1)-\theta'}}\|f\|_{L_p(B(x_0,2^{k+1}r))}\\ &\lesssim [b]_\theta\sum_{k=1}^{\infty}k(2^kr)^{-\frac{n}{q}}\|f\|_{L_p(B(x_0,2^{k+1}r))}\\ &\lesssim [b]_\theta\sum_{k=1}^{\infty}k\int_{2^kr}^{2^{k+1}r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.\end{aligned}$$

Since $2^kr\le t\le 2^{k+1}r$, then $k\approx \ln\frac{t}{r}$. Thus

$$\begin{aligned}\sup_{x\in B}|M^{\theta}_{\beta,V}((b-b_{2B})f_2)(x)| &\lesssim [b]_\theta\sum_{k=1}^{\infty}k\int_{2^kr}^{2^{k+1}r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim [b]_\theta\sum_{k=1}^{\infty}\int_{2^kr}^{2^{k+1}r}\ln\frac{t}{r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim [b]_\theta\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.\end{aligned}$$

Then

$$\|M^{\theta}_{\beta,V}((b-b_{2B})f_2)\|_{L_q(B(x_0,r))}\lesssim [b]_\theta r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), the proof of Theorem 4.1 is completed.

Proof of Theorem 1.2. Since $f\in M^{\alpha,V}_{p,\varphi_1}$ and (φ_1,φ_2) satisfies the condition (1.4), by (3.5) we have

$$\begin{aligned}&\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &=\int_{2r}^{\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\int_{r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\varphi_2(x_0,r).\end{aligned} \tag{4.5}$$

Then from Theorem 4.1 we get

$$\begin{aligned}
&\|[b,\mathcal{I}_\beta]f\|_{M^{\alpha,V}_{q,\varphi_2}} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0}\Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha}\varphi_2(x_0,r)^{-1}r^{-n/q}\|[b,\mathcal{I}_\beta]f\|_{L_q(B(x_0,r))} \\
&\lesssim [b]_\theta \sup_{x_0\in\mathbb{R}^n, r>0}\Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha}\varphi_2(x_0,r)^{-1}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t} \\
&\lesssim [b]_\theta\|f\|_{M^{\alpha,V}_{p,\varphi_1}}.
\end{aligned}$$

5 Proof of Theorem 1.3

The statement is derived from the estimate (3.4). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 1.1. So we only have to prove that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;x,r)=0 \;\;\Rightarrow\;\; \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{q,\varphi_2}(M^{\theta}_{\beta,V}f;x,r)=0 \tag{5.1}$$

and

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f;x,r)=0 \;\;\Rightarrow\;\; \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{W,\alpha,V}_{n/(n-\beta),\varphi_2}(M^{\theta}_{\beta,V}f;x,r)=0. \tag{5.2}$$

To show that $\sup_{x\in\mathbb{R}^n}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|M^{\theta}_{\beta,V}f\|_{L_q(B(x,r))}<\varepsilon$ for small r, we split the right-hand side of (3.4):

$$\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|M^{\theta}_{\beta,V}f\|_{L_q(B(x,r))}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{5.3}$$

where $\delta_0>0$ (we may take $\delta_0>1$), and

$$I_{\delta_0}(x,r):=\frac{\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}}{\varphi_2(x,r)}\int_r^{\delta_0}t^{-\frac{n}{q}-1}\|f\|_{L_p(B(x,t))}dt$$

and

$$J_{\delta_0}(x,r):=\frac{\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}}{\varphi_2(x,r)}\int_{\delta_0}^{\infty}t^{-\frac{n}{q}-1}\|f\|_{L_p(B(x,t))}dt$$

and it is supposed that $r<\delta_0$. We use the fact that $f\in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\Big(1+\frac{t}{\rho(x)}\Big)^{\alpha}\varphi_1(x,t)^{-1}t^{-n/p}\|f\|_{L_p(B(x,t))}<\frac{\varepsilon}{2CC_0}$$

where C and C_0 are constants from (1.5) and (5.3). This allows to estimate the first term uniformly in $r\in(0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\;\; 0<r<\delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (2.6) we have

$$J_{\delta_0}(x,r) \le c_{\sigma_0} \frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_1(x,r)} \|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{σ_0} is the constant from (1.2). Then, by (2.6) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (5.1).

The proof of (5.2) is similar to the proof of (5.1).

6 Proof of Theorem 1.4

The norm inequality having already been provided by Theorem 1.2, we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_1(x,t)^{-1}t^{-n/p}\|f\|_{L_p(B(x,t))}=0$$
$$\Longrightarrow \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_2(x,t)^{-1}t^{-n/p}\|[b,M^\theta_{\beta,V}f]\|_{L_q(B(x,t))}=0.$$

To check that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_2(x,t)^{-1}t^{-n/p}\|[b,M^\theta_{\beta,V}f]\|_{L_q(B(x,t))}<\varepsilon \quad \text{for small } r,$$

we use the estimate (4.1):

$$\varphi_2(x,t)^{-1}t^{-n/p}\|[b,M^\theta_{\beta,V}f]\|_{L_q(B(x,t))} \lesssim \frac{[b]_\theta}{\varphi_2(x,r)}\int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.$$

We take $r<\delta_0$ where δ_0 will be chosen small enough and split the integration:

$$\left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_2(x,t)^{-1}t^{-n/p}\|[b,M^\theta_{\beta,V}f]\|_{L_q(B(x,t))} \le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \quad (6.1)$$

where

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{t}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{t}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\int_{\delta_0}^\infty\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_1(x,t)^{-1}t^{-n/p}\|f\|_{L_p(B(x,t))}<\frac{\varepsilon}{2CC_0}, \quad t\le\delta_0,$$

where C and C_0 are constants from (6.1) and (1.6), which yields the estimate of the first term uniform in $r \in (0, \delta_0)$: $\sup\limits_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \;\; 0 < r < \delta_0$.

For the second term, writing $1 + \ln\frac{t}{r} \le 1 + |\ln\, t| + \ln\,\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \le \frac{c_{\delta_0} + \widetilde{c_{\delta_0}}\,\ln\frac{1}{r}}{\varphi_2(x,r)} \|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (1.8) with $\delta = \delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (1.7) we can choose small r such that $\sup_{x\in\mathbb{R}^n} J_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, which completes the proof.

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