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On the basicity of one trigonometric system in Orlicz spaces

Bilal Bilalov, Yonca Sezer, Umit Ildiz, Tural Hagverdi*

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Abstract. *In this article it is considered the trigonometric system, which is the collection of eigenfunction of the ordinary differential operator second order with nonlocal boundary condition. It is considered the Orlicz space on the segment* $(0, 2\pi)$. *It is established that if the Boyd indexes of this space belong to the interval* $(0, 1)$ *then the considered system forms a basis in this space. This system was used by several mathematics in the study of solvability and construction of solution of one second order degenerate elliptic equation with nonlocal boundary condition.*

Keywords. Orlicz space · trigonometric system · basicity

Mathematics Subject Classification (2010): 33B10, 46E30, 54D70

1 Introduction

The classical theory of solvability (in classical, strong or weak sense) of linear elliptic equations is well developed (see e.g. the monographs [1], [11]). Moreover there are some problems of mechanics and mathematical physics which do not fit to this theory. One of such problem is the following degenerate elliptic equation

* Corresponding author

B.T. Bilalov
Institute of Mathematics and Mechanics, The Ministry of Science and Education, Baku, Azerbaijan
Yildiz Technical University, Istanbul, Turkiye
Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
E-mail: b_bilalov@mail.ru
bilal.bilalov@yildiz.edu.tr

Y. Sezer

Yildiz Technical University, Istanbul, Turkiye
E-mail: ysezer@yildiz.edu.tr

U. Ildiz

Yildiz Technical University, Istanbul, Turkiye
E-mail: umitt.ildiz@gmail.com

T. Hagverdi
Institute of Mathematics and Mechanics, The Ministry of Science and Education, Baku, Azerbaijan
E-mail: turalhagverdi@gmail.com

$$\begin{aligned} y^m u_{xx} + u_{yy} &= 0\ , \ \ (x, y) \in (0, 2\pi) \times (0, +\infty), \\ u(x, 0) &= f(x)\ , \quad x \in (0, 2\pi), \\ u(0, y) &= u(2\pi, y)\ , \quad y \in (0, +\infty), \\ u_x(0, y) &= 0\ , \ \ y \in (0, +\infty), \end{aligned} \tag{1.1}$$

with $m \geq -2$, studied by Moiseev in [14]. When solving this problem in classical sense he applied the spectral method by using the fact that the corresponding to this problem trigonometric system

$$\{1;\, \cos nx;\, x \sin nx\}_{n\in\mathbb{N}}\,, \tag{1.2}$$

forms a Riesz basis in $L_2(0, 2\pi)$. Then the authors of the works [2]-[8], [10], [12], established the basisness of the system (1.2) in weighted Lebesgue and weighted grand Lebesgue spaces and using these facts to solve the problem (1.1) (in strong and weak sense) in corresponding Sobolev spaces generated by norm of these spaces.

Therefore in order to solve the problem (1.1) in other Sobolev spaces it needs to establish the basisness of system (1.2) in corresponding Banach function spaces. By this reason investigation of basicity properties (completeness, minimality, basisness) of the systems regarding various Banach function spaces has very science interest in view of theory of differential equation, spectral theory of differential operators and approximation theory (see e.g. works [8]-[17]).

2 Auxiliary Facts

First, let us take some standard notations. $\mathbb{N}$ will be the set of natural numbers, $\mathbb{R}$ will be the set of real numbers, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ and δ_{ij} is the Kronecker delta symbol. $C_0^\infty(0, 2\pi)$ is the set of all infinitely differentiable functions on $(0, 2\pi)$ with compact support in $(0, 2\pi)$. $L[M]$ denotes the linear span of the set M and c denotes constant (maybe difference in various places).

Moreover, we will use the following notions of basis theory.

Definition 2.1 *Let X be a Banach space on field K and X^* be the dual space of X. For the $\{x_n\}_{n\in\mathbb{N}} \subset X$ system to be a basis in the X space, there is only one $\{a_n\}_{n\in N} \subset K$ sequence that*

$$x = \sum_{n=1}^{\infty} a_n x_n,$$

for $\forall x \in X$. The $\{a_n\}_{n\in\mathbb{N}}$ sequence is the sequence of biorthogonal coefficients of the x element with respect to the $\{x_n\}_{n\in\mathbb{N}}$ system.

Definition 2.2 *$\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{x_n^*\}_{n\in\mathbb{N}} \subset X^*$ systems are called to be biorthogonal, if the condition*

$$x_n^*(x_k) = \delta_{nk}, \quad \forall n; k \in \mathbb{N},$$

is satisfied. Here δ_{nk} is the Kronecker delta symbol.

Definition 2.3 (Completeness) *Let X be a Banach space. If*

$$\overline{L[\{x_n\}_{n\in\mathbb{N}}]} = X,$$

then the system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called to be complete in X.

The completeness criterion for a system in Banach spaces is as follows.

Statement.[Completeness Criterion] Let X be a Banach space. The system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is complete in $X \Leftrightarrow f \in X^*$: $f(x_n) = 0,\ \forall n \in \mathbb{N} \Rightarrow f = 0$.

Definition 2.4 (Minimality) *If*

$$x_k \notin \overline{L[\{x_n\}_{n\in\mathbb{N}_k}]}, \quad \forall k \in \mathbb{N},$$

then the system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called to be minimal in X (here $\mathbb{N}_k = \mathbb{N} \setminus \{k\}$).

The minimality criterion for a system in Banach spaces is as follows.

Statement.[Minimality Criterion] The necessary and sufficient condition for a system to be minimal in Banach space is that the system has a biorthogonal system.

Statement.[Basicity Criterion] The system $\{x_n\}_{n\in\mathbb{N}}$ forms a basis in the X Banach space if and only if, if the following assertions hold:

1 The $\{x_n\}_{n\in\mathbb{N}}$ system is complete in X;
2 The $\{x_n\}_{n\in\mathbb{N}}$ system is minimal in X;
3 $P_m(x) = \sum_{k=1}^{m} x_k^*(x)x_k$ projectors are uniformly bounded ($\forall m \in \mathbb{N}$), i.e., there exists $C > 0$ such that

$$\|P_m(x)\|_X \le C\|x\|_X, \quad \forall x \in X.$$

Here the systems $\{x_n^*\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ are biorthogonal.

Let us give necessary concepts and facts related to Orlicz space.

Definition 2.5 *Continuous convex function $M : \mathbb{R} \to \mathbb{R}$ is called N-function if it is even and satisfies the condition*

$$\lim_{x\to 0} \frac{M(u)}{u} = 0; \ \lim_{x\to\infty} \frac{M(u)}{u} = \infty\ .$$

Definition 2.6 *Let M be a N-function. So the following function is called N-function complement to M:*

$$M^*(v) = \max_{u\ge 0}[u(v) - M(u)]$$

Function M^* can be described as follows. Let $p : \mathbb{R}_+ \to \mathbb{R}_+ = [0; +\infty)$ be right continuous for $t \ge 0$, non-decreasing function that satisfies the condition $p(0) = 0,\ p(\infty) = \lim_{t\to 0} p(t) = \infty$. Let us define

$$q(s) = \sup_{p(t)\le s} t\ , \quad s \ge 0.$$

The function q has the same properties as the function p, in fact for $s > 0$ it is positive, for $s \ge 0$ it is right continuous, non-decreasing and satisfies the conditions

$$p(0) = 0,\ p(\infty) = \lim_{t\to 0} p(t) = \infty.$$

M and M^* can be represented as follows

$$M(u) = \int_0^{|u|} p(t)dt, \quad M^*(v) = \int_0^{|v|} q(s)ds.$$

These N-functions are complement to each other.

Definition 2.7 *N-function M satisfies Δ_2-condition for large values of u, if $\exists k > 0$ and $\exists u_0 \ge 0$:*

$$M(2u) \le kM(u), \quad \forall u \ge u_0.$$

Δ_2-condition is equivalent to that, for $\forall l > 1,\ \exists k(l) > 0$ and $\exists u_0 \geq 0$:

$$M(lu) \leq k(l)M(u), \quad \forall u \geq u_0.$$

Now let us define the Orlicz space. Let M be some N-function, $G \subset \mathbb{R}$ be a (Lebesgue) measurable set with finite measure. Denote by $L_0(G)$ the set of all functions measurable in G. Let

$$\rho_M(u) = \int_G M[u(x)]\, dx,$$

and

$$L_M(G) = \{u \in L_0(G) :\ \rho_M(u) < +\infty\}.$$

$L_M(G)$ is called an Orlicz class.
Let M and M^* be complement for each other N-functions. Assume

$$L_M^*(G) = \{u \in L_0(G) :\ |(u;v)| < +\infty,\ \forall v \in L_{M^*}(G)\},$$

here

$$(u;v) = \int_G u(x)\overline{v(x)}\, dx.$$

$L_M^*(G)$ is called Orlicz space. According to the norm $\|.\|_M$:

$$\|u\|_M = \sup_{\rho_M^*(v)\leq 1} |(u;v)|,$$

$L_M^*(G)$ is a Banach space. It should be noted that in $L_M^*(G)$ we can define equivalent norm to $\|.\|_M$:

$$\|u\|_{(M)} = \inf\left\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

$\|u\|_{(M)}$ is called the Luxembourg norm.

Statement. If N-function M satisfies the Δ_2-condition, then $L_M^*(G) = L_M(G)$ and the closure of the set of bounded (including continuous) functions coincides with $L_M^*(G)$.

More information about these and other facts we can refer to monographs [18], [19].

Definition 2.8 *We will say that the function M satisfies the ∇_2-condition, if*

$$\lim_{u\to\infty} \inf \frac{M(2u)}{M(u)} > 2,\ i.e.\ \exists\lambda > 2 \textit{ and } \exists u_0 > 0 :\ M(2u) \geq \lambda M(u),\ \ \forall u \geq u_0.$$

Denote by $\Delta_2(\infty)$ $(\nabla_2(\infty))$ the set of all N-functions, satisfying the Δ_2-condition (the ∇_2-condition).

We will need the concepts of Boyd indices of Orlicz spaces. By $M^{-1}(.)$ we denote the inverse of N-function $M(.)$.

Assume

$$h(t) = \lim_{x\to\infty} \sup \frac{M^{-1}(x)}{M^{-1}(tx)}\ , \quad t > 0.$$

Define the following numbers

$$\alpha_M = -\lim_{t\to\infty} \frac{\log h(t)}{\log t}\ ; \qquad \beta_M = -\lim_{t\to 0^+} \frac{\log h(t)}{\log t}\ .$$

The numbers α_M and β_M are called upper and lower Boyd indices for the Orlicz space $L_M(0,2\pi)$, correspondingly. These numbers satisfy the following relations

$$0 \le \alpha_M \le \beta_M \le 1 \ ; \quad \alpha_M + \beta_{M^*} = 1 \ ; \quad \alpha_{M^*} + \beta_M = 1,$$

where $M,\ M^* \in N$ complementary each to other N-functions.

The Orlicz space $L_M(0,2\pi)$ is reflexive if and only if holds the relation $0 < \alpha_M \le \beta_M < 1$. Moreover, if for numbers $p,\ q \in [1, +\infty]$, hold the inequality

$$1 \le q < \frac{1}{\beta_M} \le \frac{1}{\alpha_M} < p \le +\infty, \tag{2.1}$$

then it is valid the following continuous embeddings

$$L_p(0,2\pi) \ \subset \ L_M(0,2\pi) \ \subset \ L_q(0,2\pi). \tag{2.2}$$

More information about these and other facts can be found in works [13]-[19].

The conjugate function $\widetilde{f}$ of function f from the Orlicz Space $L_M(0,2\pi)$.

Definition 2.9 *For any $f \in L_M(0,2\pi) \subset L_1(0,2\pi)$, the conjugate $\widetilde{f}$ of f is given by*

$$\widetilde{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \frac{f(x+t) - f(x-t)}{2\tan\frac{t}{2}}\,dt.$$

By $S_n[f]$, $n = 0, 1, ...;$ we denote the partial sum of Fourier series of function $f \in L_M(0,2\pi)$:

$$S_n[f](x) = \sum_{|k| \le n} c_k\, e^{ikx} = \frac{1}{\pi}\int_0^{2\pi} f(t)\, D_n(x-t)\,dt,$$

where

$$c_k = c_k(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-ikx}\,dx, \ \ k \in \mathbb{Z},$$

are Fourier coefficients of $f(.)$ and

$$D_n(x) = \frac{1}{2}\sum_{|k|\le n} e^{ikx} = \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{2\sin\frac{x}{2}}, \ \ n = 0, 1, ...;$$

is a Dirichlet kernel of order n.

We need the following Ryan's theorem from the monograph [19, p.196].

Theorem 2.1 **(Ryan)** *Let M be an N-function. Then the following are equivalent:*

(i) $L_M(0,2\pi)$ is reflexive;
(ii) There is a constant $C > 0$ such that for all $f \in L_M(0,2\pi)$:

$$\|\widetilde{f}\|_{L_M(0,2\pi)} \le C\, \|f\|_{L_M(0,2\pi)};$$

(iii) There is a constant $A > 0$ such that for all $n \ge 1$ and $f \in L_M(0,2\pi)$:

$$\|S_n[f]\|_{L_M(0,2\pi)} \le C\, \|f\|_{L_M(0,2\pi)}.$$

From these facts direct follows the following

Corollary 2.1 *For N-function M:*

$$\lim_{n\to\infty} \|S_n[f] - f\|_{L_M(0,2\pi)} = 0,$$

for all $f \in L_M(0,2\pi)$ if and only if $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$.

Also valid the following Ryan's theorem.

Theorem 2.2 (Ryan) *[19] Let M be N-function. If holds the part (iii) of Theorem 2.1 (Ryan), then $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$; so $L_M(0,2\pi)$ is reflexive.*

Taking into account the Theorems 2.1, 2.2 and Corollary 2.1, we arrive to the following conclusion.

Corollary 2.2 *Let M be N-function. Then the Boyd indices of Orlicz space $L_M(0,2\pi)$: α_M; $\beta_M \in (0,1)$ if and only if $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$.*

3 Main Results

Consider the following trigonometric system

$$y_0^c = 1;\ \ y_n^c(x) = \cos nx;\ \ y_n^s(x) = x \sin nx,\ n \in \mathbb{N}, \tag{3.1}$$

$$\vartheta_0^c(x) = \frac{2\pi - x}{2\pi^2};\ \ \vartheta_n^c(x) = \frac{2\pi - x}{\pi^2}\cos nx;\ \ \vartheta_n^s(x) = \frac{1}{\pi^2}\sin nx,\ n \in \mathbb{N}. \tag{3.2}$$

Let us prove the following

Lemma 3.1 *Let $L_M(0,2\pi)$ be Orlicz space with Boyd indices $\alpha_M,\ \beta_M \in (0,1)$. Then the system (3.1) is minimal in $L_M(0,2\pi)$.*

Proof. Consider the following functionals

$$e_0^c(f) = \frac{1}{2\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\,dx;$$
$$e_n^c(f) = \frac{1}{\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\cos nx\,dx;$$
$$e_n^s(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

In the work [14] it is established the following relations:

$$\begin{array}{rl} e_n^c(y_m^c) = & \delta_{nm};\quad \forall n,m \in \mathbb{N}; \\ e_n^c(y_m^s) = & 0,\ \forall n \in \mathbb{Z}_+;\ \forall m \in \mathbb{N}; \\ e_n^s(y_m^c) = & 0,\ \forall n \in \mathbb{N};\ \forall m \in \mathbb{Z}_+; \\ e_n^s(y_m^s) = & \delta_{nm};\quad \forall n,m \in \mathbb{Z}_+. \end{array} \tag{3.3}$$

Let us show that the functionals $\{e_n^c;\ e_n^s\}$ belong to the space $(L_M(0,2\pi))^*$. It is evident that $\exists p,q \in (1,+\infty)$, for which it is valid the inequality

$$1 \le q < \frac{1}{\beta_M} \le \frac{1}{\alpha_M} < p \le +\infty. \tag{3.4}$$

Then from the embeddings (2.2) follows the following estimates

$$\|f\|_{L_q(0,2\pi)} \le c\,\|f\|_{L_M(0,2\pi)};\ \ \forall f \in L_M(0,2\pi),$$

where $c > 0$ some constant.

Using these relations and applying the Hlder inequality we have

$$\begin{aligned}|e_n^c(f)| &\le c\int_0^{2\pi} |f(x)|\,|2\pi - x|\,|\cos nx|\,dx\\ &\le c\left(\int_0^{2\pi} |f|^q\,dx\right)^{\frac{1}{q}} \le\ c\,\|f\|_{L_M(0,2\pi)},\ \ \forall n \in \mathbb{Z}_+.\end{aligned}$$

Also

$$|e_n^s(f)| \le\ c\int_0^{2\pi} |f|\,dx\ \le\ c\left(\int_0^{2\pi} |f|^q\,dx\right)^{\frac{1}{q}} \le\ c\,\|f\|_{L_M(0,2\pi)},\ \ \forall n \in \mathbb{N},$$

where c denote constants. From here immediately follows that

$$\{e_n^c;\ e_n^s\} \subset (L_M(0,2\pi))^*\,.$$

Then based on minimality criterion from relations (3.3) we have the minimality of system (3.1) in $L_M(0,2\pi)$.

The lemma is proved.

Then let us prove the completeness of system in $L_M(0,2\pi)$. It is valid the following

Lemma 3.2 *Let the Boyd indices of Orlicz space* $L_M(0,2\pi)$ *belong to interval* $(0,1)$*, i.e.* $\alpha_M,\ \beta_M \in (0,1)$. *Then the system (3.1) is complete in* $L_M(0,2\pi)$.

Proof. From Corollary 2.2 follows that $M \in \varDelta_2(\infty) \cap \nabla_2(\infty)$ and in result from known facts (see e.g. the monographs [18,19]) follows that the class $C_0^\infty(0,2\pi)$ is dense in $L_M(0,2\pi)$. Let $f \in L_M(0,2\pi)$ is an arbitrary function. Take $\forall\varepsilon > 0$. Then $\exists\, g \in C_0^\infty(0,2\pi)$, such that $\|f - g\|_{L_M(0,2\pi)} < \varepsilon$. Let us consider the biorthogonal series of g on the system (3.1):

$$\widetilde{S_n}[g](x) = \sum_{k=0}^{n} e_k^c(g)\ y_k^c(x) + \sum_{k=0}^{n} e_k^s(g)\ y_k^s(x),\ \ n \in \mathbb{N}.$$

Consider the biorthogonal coefficients $\{e_n^c;\ e_n^s\}$:

$$\begin{aligned}e_k^c(g) &= c\int_0^{2\pi} g(x)(2\pi - x)\cos kx\,dx\\ &= \int_0^{2\pi} \widetilde{g}(x)\cos kx\,dx,\ \ k \in \mathbb{Z}_+,\end{aligned}$$

where $\widetilde{g}(x) = c\,g(x)(2\pi - x)$ and c some constant. It is evident that $\widetilde{g} \in C_0^\infty(0,2\pi)$ and in result $\widetilde{g}^{(n)}(0) = \widetilde{g}^{(n)}(2\pi) = 0,\ \ \forall n \in \mathbb{Z}_+$. Integrating by parts two times and taking into account these relations we have

$$
\begin{aligned}
e_k^c(g) &= \frac{1}{k}\int_0^{2\pi} \widetilde{g}(x)\; d\sin kx \\
&= -\frac{1}{k}\int_0^{2\pi} \widetilde{g}^{(1)}(x)\; \sin kx\; dx \\
&= -\frac{1}{k^2}\int_0^{2\pi} \widetilde{g}^{(2)}(x)\; \cos kx\; dx,
\end{aligned}
$$

and from here follows

$$|e_k^c(g)| \le \frac{c}{k^2}, \;\; \forall k \in \mathbb{N}.$$

Completely analogously we have the following estimate

$$|e_k^s(g)| \le \frac{c}{k^2}, \;\; \forall k \in \mathbb{N}.$$

From these estimates follows that the partial sums $\{\widetilde{S_n}[g]\}_{n\in\mathbb{N}}$ converges uniformly on $[0, 2\pi]$. From the results of the work [14] follows that the system (3.1) forms a basis in $L_2(0, 2\pi)$ and in result it is evident that the limit of sums $\{\widetilde{S_n}[g]\}_{n\in\mathbb{N}}$ is $g(.)$. It is obvious that $\exists c > 0$:

$$\|f\|_{L_M(0,2\pi)} \;\le\; c\,\|f\|_{L_\infty(0,2\pi)}; \;\; \forall f \in C[0, 2\pi].$$

Then $\exists n_\varepsilon \in \mathbb{N}$, such that for $\forall n \ge n_\varepsilon$ it holds

$$\|\widetilde{S_n}[g] - g\|_{L_M(0,2\pi)} \;\le\; c\,\|\widetilde{S_n}[g] - g\|_{L_\infty(0,2\pi)} \;<\; \varepsilon.$$

We have

$$\|f - \widetilde{S_n}[g]\|_{L_M(0,2\pi)} \;\le\; \|f - g\|_{L_M(0,2\pi)} \;+\; \|\widetilde{S_n}[g] - g\|_{L_M(0,2\pi)} \;<\; 2\varepsilon, \;\; \forall n \ge n_\varepsilon.$$

From arbitrariness of $\varepsilon > 0$ follows completeness of system (3.1) in $L_M(0, 2\pi)$.

Lemma is proved.

So let us prove the main theorem of this work.

Theorem 3.1 *Let M be N-function and the Boyd indices of Orlicz space $L_M(0, 2\pi)$: $\alpha_M,\ \beta_M \in (0, 1)$. Then the system (3.1) forms a basis in $L_M(0, 2\pi)$.*

Proof. Taking into account the Lemmas 3.1 and 3.2 it is sufficient to prove that the projectors

$$P_n(f) \;=\; \sum_{k=0}^{n} e_k^c(f)\, y_k^c \;+ \sum_{k=1}^{n} e_k^s(f)\, y_k^s, \;\; \forall n \in \mathbb{N}.$$

uniformly bounded in $L_M(0, 2\pi)$. We have

$$\|P_n(f)\|_{L_M(0,2\pi)} \;\le\; \left\|\sum_{k=0}^{n} e_k^c(f)\, y_k^c\right\|_{L_M(0,2\pi)} + \left\|\sum_{k=1}^{n} e_k^s(f)\, y_k^s\right\|_{L_M(0,2\pi)} = I_n^{(1)} + I_n^{(2)}, \;\; n \in \mathbb{N}.$$

Let us estimate $\{I_n^{(1)}\}$. We have

$$e_k^c(f) = c_k^+(\widetilde{f}), \;\; \forall k \in \mathbb{Z}_+,$$

where $c_k^+(\widetilde{f})$ is Fourier coefficient

$$c_k^+(\widetilde{f}) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \widetilde{f}(x)\ \cos kx\, dx,$$

of function $\widetilde{f}(.)$:

$$\widetilde{f}(x) = c\,(2\pi - x) f(x).$$

Since the classical trigonometric system $\{1;\ \cos nx;\ \sin nx\}_{n\in\mathbb{N}}$ forms a basis in $L_M(0,2\pi)$ (it follows from Corollary 2.1), then from basicity criterion follows

$$I_n^{(1)} \ = \ \left\| \sum_{k=0}^{n} c_k^+(\widetilde{f})\ \cos kx \right\|_{L_M(0,2\pi)} \ \le\ c\,\|\widetilde{f}\|_{L_M(0,2\pi)} \ \le\ c\,\|f\|_{L_M(0,2\pi)},$$

where the constant $c > 0$ does depend on n and f. Completely analogously we can establish

$$I_n^{(2)} \ \le\ c\,\|f\|_{L_M(0,2\pi)},\ \ \forall n \in \mathbb{N}.$$

In result from the basicity criterion follows that the system (3.1) forms a basis in $L_M(0,2\pi)$.

The theorem is proved.

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