Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **44** (1), 51-62 (2024).
https://doi.org/10.30546/2617-7900.44.1.2024.51

On Riesz property and equivalent basis property of the system of root vector functions of Dirac-type operator

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Received: 31.09.2023 / Revised: 28.01.2024 / Accepted: 15.02.2024

Abstract. *Dirac-type operator is considered on the finite interval $G = (a, b)$. It is assumed that its coefficient (potential) is a complex-valued matrix function summable on $G = (a, b)$. Riesz property criterion for a system of root vector functions is established and theorem on equivalent basis property in $L_p^2(G)$, $1 < p < \infty$, is proved.*

Keywords. Dirac-type operator · root vector function · Riesz property · equivalent basis property.

Mathematics Subject Classification (2010): 34L10, 34L20, 34B05, 34B24

1 Main concepts and statement of results

Riesz and basis properties of the systems of root vector functions of Dirac-type operator are studied in this work. Root vector functions are considered in generalized sense, i.e. regardless of boundary conditions (see [1]). With such a generalization, V.A. Il'in [1] found the necessary and sufficient conditions for unconditional basis property (Riesz basis property) of the systems of root vector functions of the operator $L = -d^2/dx^2 + q(x)$ for L_2. The work [1] served as a starting point for many mathematicians to study the Bessel, unconditional basis and basis properties of the systems of root vector functions of higher order differential operators.

For a Dirac operator with a potential from the class L_2, Bessel property and unconditional basis property criteria have been established in [2]. Componentwise uniform equiconvergence on a compact, uniform convergence, Riesz property of the systems of root vector functions of Dirac operator and unconditional basis property for Dirac-type operator have been considered in [3-7].

Basis property and other spectral properties of root vector functions of Dirac operator (with boundary conditions) have been treated in [8-16] and the references therein. In [8], the Riesz basis property for Dirac operator with a potential from the class L_2 and separated boundary conditions has been established. Dirac operator with a potential from the class L_2 and general regular conditions has been studied in [9], where the Riesz basis property

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of subspaces and, in case of strongly regular boundary conditions, the Riesz basis property have been proved. The case where the potential belongs to the class $L_p, p \geq 1$, has been considered in [10, 11], where the Riesz basis property (with strongly regular boundary conditions) and the Riesz basis property of subspaces (with regular boundary conditions) have been established. For Dirac-type operator with a potential from L_1 and strongly regular conditions, the Riesz basis property has been proved in [12].

Consider one-dimensional Dirac-type operator

$$Dy = B\frac{dy}{dx} + P(x)y, \quad y(x) = (y_1(x), y_2(x))^T,$$

where $B = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}, \quad b_2 < 0 < b_1, \ \ P(x) = diag(p_1(x), p_2(x)),$

and $p_1(x), p_2(x)$ are complex-valued summable functions on the arbitrary finite interval $G = (a, b)$ of the real axis.

Following [1], by the eigen vector function of the operator D corresponding to the complex eigenvalue λ, we will mean any complex-valued vector function $\overset{\circ}{u}(x)$ not identically zero, which is absolutely continuous on every closed subinterval of G and satisfies the equation $D\overset{\circ}{u} = \lambda \overset{\circ}{u}$ almost everywhere in G.

Similarly, by the associated vector function of degree $l, \ \ l \geq 1$, corresponding to the same λ and the same eigenfunction $\overset{\circ}{u}(x)$, we will mean any complex-valued vector function $\overset{l}{u}(x)$, which is absolutely continuous on every closed subinterval of G and satisfies the equation $D\overset{l}{u} = \lambda \overset{l}{u} + \overset{l-1}{u}$ almost everywhere in G.

Let $\{u_k(x)\}_{k=1}^{\infty}$ be an arbitrary system of root (eigen- and associated) vector functions of the operator D, and $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigenvalues. In the sequel we will assume that every vector function $u_k(x)$ belongs to the system $\{u_k(x)\}_{k=1}^{\infty}$ together with all corresponding associated functions of a lesser degree, and the lengths of the chains of root vector functions are uniformly bounded. This means, in particular, that every vector function $u_k(x)$ satisfies the equation

$$Du_k = \lambda_k u_k + \theta_k u_{k-1}$$

almost everywhere in G, where θ_k is equal to either 0 (in this case, $u_k(x)$ is an eigen vector function) or 1 (in this case, $u_k(x)$ is an associated vector function, $\lambda_k = \lambda_{k-1}$).

Let $L_p^2(G), \ p \geq 1$, be a space of two-component vector functions $f(x) = (f_1(x), f_2(x))^T$ with the norm

$$\|f\|_{p,2} = \left[\int_G \left(|f_1(x)|^2 + |f_2(x)|^2\right)^{p/2} dx\right]^{1/p}.$$

In case $p = \infty$, the norm in this space is defined by the equality $\|f\|_{\infty,2} = \sup\limits_{x \in \overline{G}} vrai\, |f(x)|$.

Obviously, for the vector functions $f(x) \in L_p^2(G)$, $g(x) \in L_q^2(G)$, $p^{-1} + q^{-1} = 1$, $p \geq 1$, the "scalar product"

$$(f, g) = \int_a^b \sum_{j=1}^{2} f_j(x)\overline{g_j(x)}dx$$

is defined.

Definition 1.1. *A system* $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_q^2(G),\ q \geq 2$, *is called a Riesz system, or a system which satisfies the Riesz property, if there exists a constant* $M = M(p)$ *such that the inequality*

$$\sum_{k=1}^{\infty} |(f, \varphi_k)|^q \leq M \, \|f\|_{p,2}^q$$

holds for an arbitrary function $f(x) \in L_p^2(G)$, *where* $p^{-1} + q^{-1} = 1$.

Definition 1.2. *A system* $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_p^2(G), p \geq 1$, *is called* p*-close to the system* $\{\psi_k(x)\}_{k=1}^{\infty} \subset L_p^2(G)$ *in* $L_p^2(G)$ *if the relation*

$$\sum_{k=1}^{\infty} \|\varphi_k - \psi_k\|_{p,2}^p < \infty$$

holds.

Definition 1.3. *Two sequences of elements in the Banach space* X *are called equivalent if there exists a bounded, linear and boundedly invertible operator in* X, *which maps one of these sequences into another.*

The following theorems are proved in this work.

Theorem 1.1 *(Criterion of Reizs property). Let* $P(x) \in L_1(G)$ *and there exist a constant* C_0 *such that*

$$|Im\lambda_k| \leq C_0, \quad k = 1, 2, ... \tag{1.1}$$

Then, for the system $\left\{u_k(x) \, \|u_k\|_{q,2}^{-1}\right\}_{k=1}^{\infty} \subset L_q^2(G)$ *to be Riesz, it is necessary and sufficient that there exists a constant* M_1 *such that the inequality*

$$\sum_{|Re\lambda_k - \nu| \leq 1} 1 \leq M_1 \tag{1.2}$$

holds for every real number ν.

Let D^* be a formal adjoint operator of D, i.e. $D^* = -B^* \frac{d}{dx} + P^*(x)$, where $P^*(x)$ is an adjoint matrix function of $P(x)$, and B^* is an adjoint matrix of B. Denote by $\{\upsilon_k(x)\}_{k=1}^{\infty}$ a biorthogonal adjoint system of $\{u_k(x)\}_{k=1}^{\infty}$ and assume that it consists of root vector functions of the operator D^*, i.e. $D^*\upsilon_k = \overline{\lambda_k}\upsilon_k + \theta_{k+1}\upsilon_{k+1}$.

Theorem 1.2 *(On equivalent basis property). Let* $1 < p \leq 2$, $P(x) \in L_1(G)$, *the lenghts of the chains of root vector functions be uniformly bounded, conditions* (1.1), (1.2) *be satisfied, there exist a constant* M_2 *such that*

$$\|u_k\|_{2,2} \, \|\upsilon_k\|_{2,2} \leq M_2, \quad k = 1, 2, ..., \tag{1.3}$$

and the system $\left\{u_k(x) \, \|u_k\|_{p,2}^{-1}\right\}_{k=1}^{\infty}$ *be* $p-$*close to some basis* $\{\psi_k(x)\}_{k=1}^{\infty}$ *in* $L_p^2(G)$. *Then the systems* $\left\{u_k(x) \, \|u_k\|_{p,2}^{-1}\right\}$ *and* $\left\{\upsilon_k(x) \, \|u_k\|_{p,2}\right\}_{k=1}^{\infty}$ *are the bases in* $L_p^2(G)$ *and* $L_q^2(G)$, *respectively, and these systems are equivalent to the basis* $\{\psi_k(x)\}_{k=1}^{\infty}$ *and its biorthogonal adjoint, respectively.*

Remark 1.1. *If in Theorem 1.2 the systems* $\{u_k(x)\}_{k=1}^{\infty}$ *and* $\{\upsilon_k(x)\}_{k=1}^{\infty}$ *are interchanged, then we get the basicity of the system* $\{u_k(x)\}_{k=1}^{\infty}$ *in* $L_p^2(G)$ *for* $p \geq 2$.

2 Auxiliary statements.

Statements below will be used to prove the above theorems.

Statement 2.1 *(see [7]). If the functions $p_1(x)$ and $p_2(x)$ belong to the class $L_1^{loc}(G)$ and the points $x-t, x, x+t$, lie in the interval G, then the following formulas are true for the root vector function $u_k(x)$:*

$$u_k(x \pm t) = \left[\cos \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} I \mp \sin \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} \frac{B}{\sqrt{|b_1 b_2|}}\right] u_k(x)$$

$$\pm B^{-1} \int_x^{x \pm t} \left(\sin \frac{\lambda_k (t-|\xi-x|)}{\sqrt{|b_1 b_2|}} \frac{B}{\sqrt{|b_1 b_2|}} \mp \cos \frac{\lambda_k (t-|\xi-x|)}{\sqrt{|b_1 b_2|}} I\right) \tag{2.1}$$

$$\times \left[P(\xi) u_k(\xi) - \theta_k u_{k-1}(\xi)\right] d\xi,$$

$$u_k(x-t) + u_k(x+t) = 2u_k(x) \cos \frac{\lambda_k t}{\sqrt{|b_1 b_2|}}$$

$$+B^{-1} \int_{x-t}^{x+t} \left(\sin \frac{\lambda_k (t-|\xi-x|)}{\sqrt{|b_1 b_2|}} \frac{B}{\sqrt{|b_1 b_2|}} - sign(\xi - x) \right. \tag{2.2}$$

$$\left. \times \cos \frac{\lambda_k (t-|\xi-x|)}{\sqrt{|b_1 b_2|}} I\right) \left[P(\xi) u_k(\xi) - \theta_k u_{k-1}(\xi)\right] d\xi,$$

where I is a unit matrix function.

Statement 2.2 *(see [7]). Let the functions $p_1(x)$ and $p_2(x)$ belong to the class $L_1(G)$. Then there exist the constants $C_i(n_k, G, b_1, b_2)$, $\ i = 1, 2$, independent of λ_k such that*

$$\|\theta_k u_{k-1}\|_{\infty,G} \le C_1(n_k, G, b_1, b_2)(1 + |Im\lambda_k|) \|u_k\|_{\infty,G}, \tag{2.3}$$

$$\|u_k\|_{\infty,G} \le C_2(n_k, G, b_1, b_2)(1 + |Im\lambda_k|)^{1/r} \|u_k\|_{r,G}, \tag{2.4}$$

where n_k is a degree of the root vector function $u_k(x)$, $\ r \ge 1$.

3 Proof of the Riesz property criterion.

In this section, we prove Theorem 1.1 (On the Riesz property of the systems of root vector functions of the operator D).

Necessity. Consider any real number ν. Introduce an index set $I_\nu = \{k : |Re\lambda_k - \nu| \le 1$, $|Im\lambda_k| \le C_0\}$, where C_0 is a constant appearing in the condition (1.1). Let's choose the positive numbers R and R^* such that $R \le R^*$ and the inequality $\omega(R) \le L^{-1}$ holds for every set $E \subset \overline{G}, mesG \le 2R^*$, where L is a positive number to be defined later and

$$\omega(R) = \sup_{E \subset \overline{G}} \left\{ \|P\|_{1,E} \right\}, \quad \|P\|_{1,E} = \int_E (|p_1(x)| + |p_2(x)|)\, dx.$$

Let $x \in \left[a, \frac{a+b}{2}\right]$. Let's write the mean value formula (2.2) for the points $x, x+t, x+2t$, where $t \in [0, R]$:

$$u_k(x) = 2u_k(x+t) \cos \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} - u_k(x+2t)$$

$$+B^{-1} \int_x^{x+2t} \left\{ \sin \frac{\lambda_k (t - |x+t-\xi|)}{\sqrt{|b_1 b_2|}} \frac{B}{\sqrt{|b_1 b_2|}} - sign(\xi - x - t) \right.$$

$$- \cos \frac{\lambda_k (t - |x+t-\xi|)}{\sqrt{|b_1 b_2|}} I \Bigg\} \left[P(\xi) u_k(\xi) - \theta_k u_{k-1}(\xi) \right] d\xi.$$

Add and subtract the function $2u_k(x+t) \cos \frac{\nu t}{\sqrt{|b_1 b_2|}}$ on the right-hand side of this equality and perform the operation $R^{-1} \int_0^R dt$. Then we get

$$u_k(x) = 2R^{-1} \int_0^R u_k(x+t) \cos \frac{\nu t}{\sqrt{|b_1 b_2|}} dt - R^{-1} \int_0^R u_k(x+2t) dt$$

$$+4R^{-1} \int_0^R u_k(x+t) \sin \frac{\lambda_k + \nu}{2\sqrt{|b_1 b_2|}} \sin \frac{\nu - \lambda_k}{2\sqrt{|b_1 b_2|}} dt$$

$$+R^{-1} B^{-1} \int_0^R \int_x^{x+2t} \left\{ \sin \frac{\lambda_k (t - |x+t-\xi|)}{\sqrt{|b_1 b_2|}} \frac{B}{\sqrt{|b_1 b_2|}} \right.$$

$$\left. - \, sign(\xi - x - t) \cos \frac{\lambda_k (t - |x+t-\xi|)}{\sqrt{|b_1 b_2|}} I \right\}$$

$$\times \left[P(\xi) u_k(\xi) - \theta_k u_{k-1}(\xi) \right] d\xi.$$

Using formula (2.1) in the third term, we get

$$u_k(x) = R^{-1} \int_G u_k(z) V(z) dz + 4R^{-1} \int_0^R \left(\cos \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} I \right.$$

$$\left. - \sin \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} \frac{B}{\sqrt{|b_1 b_2|}} \right) \sin \frac{(\lambda_k + \nu) t}{2\sqrt{|b_1 b_2|}} \sin \frac{(\nu - \lambda_k) t}{2\sqrt{|b_1 b_2|}} dt u_k(x)$$

$$+4R^{-1} B^{-1} \int_0^R \int_x^{x+t} \left\{ \sin \frac{\lambda_k (t - |\xi - x|)}{\sqrt{|b_1 b_2|}} \frac{B}{\sqrt{|b_1 b_2|}} + \cos \frac{\lambda_k (t - |\xi - x|)}{\sqrt{|b_1 b_2|}} I \right\}$$

$$\times \left[P(\xi) u_k(\xi) - \theta_k u_{k-1}(\xi) \right] \sin \frac{(\nu + \lambda_k) t}{2\sqrt{|b_1 b_2|}} \sin \frac{(\nu - \lambda_k) t}{2\sqrt{|b_1 b_2|}} dt$$

$$+R^{-1} B^{-1} \int_0^R \int_x^{x+2t} \left\{ \sin \frac{\lambda_k (t - |x+t-\xi|)}{\sqrt{|b_1 b_2|}} \frac{B}{\sqrt{|b_1 b_2|}} + \cos \frac{\lambda_k (t - |x+t-\xi|)}{\sqrt{|b_1 b_2|}} I \right\}$$

$$\times \left[P(\xi) u_k(\xi) - \theta_k u_{k-1}(\xi) \right] d\xi dt = R^{-1} \int_G u_k(z) V(z) dz + J_1 + J_2 + J_3, \qquad (3.1)$$

where $V(z) = 2 \cos \frac{\nu (x-z)}{\sqrt{|b_1 b_2|}} - \frac{1}{2}$ for $x \le z \le x+R,$ $V(z) = -\frac{1}{2}$ for $x+R < z \le x+2R,$ and $V(z) = 0$ for $z \notin [x, x+2R].$

Let $k \in I_\nu$. Let's estimate the integrals $J_i, \ i = \overline{1,3}$. Using the inequalities

$$|\sin z| \le 2, \quad |\cos z| \le 2, \quad |\sin z| \le 2\,|z|\,, \qquad (3.2)$$

which hold for $|Imz| \le 1$, we obtain

$$|J_1| \le 8R \left(\frac{2}{\sqrt{|b_1 b_2|}} + \frac{|b_1| + |b_2|}{|b_1 b_2|} \right) |\nu - \lambda_k| \; |u_k(x)| \le 8R \left(\frac{2}{\sqrt{|b_1 b_2|}} + \frac{|b_1| + |b_2|}{|b_1 b_2|} \right)$$

$$\times(1+|Im\lambda_k|)\,|u_k(x)| \leq 8R\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)(1+C_0)\,\|u_k\|_{\infty,2}\,.$$

Applying the inequalities (3.2) and the Holder inequality for $p=1, q=\infty$, we find

$$|J_2| \leq 32\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\left(\omega(R)\,\|u_k\|_{\infty,2}+\frac{R}{2}\,\|\theta_k u_{k-1}\|_{\infty,2}\right);$$

$$|J_3| \leq 2\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\left(\omega(R)\,\|u_k\|_{\infty,2}+R\,\|\theta_k u_{k-1}\|_{\infty,2}\right).$$

Considering these estimates in the equality (3.1), we obtain

$$|u_k(x)| \leq R^{-1}\left|\int_G u_k(z)V(z)dz\right|+8\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)$$

$$\times(R(1+C_0)+5\omega(R))\,\|u_k\|_{\infty,2}+18R\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\|\theta_k u_{k-1}\|_{\infty,2}\,. \quad (3.3)$$

The inequality (3.3) can be proved similarly in case $x\in\left[\frac{a+b}{2},b\right]$. In this case, $V(z)=-\frac{1}{2}$ for $x-2R\leq z<x-R$, $V(z)=2\cos\frac{\nu(x-z)}{\sqrt{|b_1b_2|}}-\frac{1}{2}$ for $x-R\leq z\leq x$, and $V(z)=0$ for $z\notin[x-2R,x]$.

Consequently, the inequality (3.3) is true for every $x\in\overline{G}$.

Applying the estimates (2.3), (2.4) and taking into account the relation $1+|Im\lambda_k|\leq 1+C_0$, from (3.3) we obtain

$$|u_k(x)| \leq R^{-1}\left|\int_G u_k(z)V(z)dz\right|$$

$$+8\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\Big\{5\omega(R)C_2(n_k,G,b_1,b_2)(1+C_0)^{1/q}$$

$$+RC_2(n_k,G,b_1,b_2)(1+C_0)^{1+1/q}$$

$$+18RC_1(n_k,G,b_1,b_2\,)C_2(n_k,G,b_1,b_2)\theta_k 1+C_0)^{1+1/q}\Big\}\,\|u_k\|_{q,2}\,.$$

Due to the uniform boundedness of the lengths of the chains, we have

$$40\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)C_2(n_k,G,b_1,b_2)\leq\gamma_1=const,$$

$$144\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)C_2(n_k,G,b_1,b_2)C_2(n_k,G,b_1,b_2)\leq\gamma_2=const.$$

Consequently,

$$|u_k(x)| \leq R^{-1}\left|\int_G u_k(z)V(z)dz\right|+\Big\{\omega(R)\gamma_1(1+C_0)^{1/q}+\gamma_1R(1+C_0)^{1+1/q}$$

$$+\,R\gamma_2\theta_k(1+C_0)^{1+1/q}\Big\}\,\|u_k\|_{q,2}\,.$$

Multiplying both sides of this inequality by $\|u_k\|_{q,2}^{-1}$, raising to a degree q and applying the inequality $\left|\sum_{i=1}^{n} a_i\right|^q \leq n^{q-1} \sum_{i=1}^{n} |a_i|^q$, we find

$$|u_k(x)|^q \|u_k\|_{q,2}^{-q} \leq 3^{q-1} R^{-q} \left\{ \left|\int_G u_k^1(z) V(z) dz\right|^q + \left|\int_G u_k^2(z) V(z) dz\right|^q \right\}$$

$$\times \|u_k\|_{q,2}^{-q} + 3^{q-1} \left\{ \gamma_1 L^{-1} (1+C_0)^{1/q} + R\gamma_1 (1+C_0)^{1+1/q} + R\gamma_2 \theta_k (1+C_0)^{1+1/q} \right\}^q ,$$

where $u_k(x) = \left(u_k^1(x), u_k^2(x)\right)^T$.

By virtue of Riesz inequality and $\|V\|_p^q \leq 3^q R^{q/p}$, we obtain

$$\sum_{k \in J} |u_k(x)|^q \|u_k\|_{q,2}^{-q} \leq 2 \cdot 3^{2q-1} M R^{q\left(\frac{1}{p}-1\right)}$$

$$+3^{q-1} \left\{ \gamma_1 L^{-1} (1+C_0)^{1/q} + R\gamma_1 (1+C_0)^{1+1/q} + R\gamma_2 \theta_k (1+C_0)^{1+1/q} \right\} \sum_{k \in J} 1,$$

where $J \subset I_\nu$ is an arbitrary finite set of indices k, which correspond to the root functions $u_k(x)$. Integrating this inequality over $x \in G$ and choosing R^* (consequently, the number L^{-1} too) small enough to have an estimate

$$3^{q-1} \left\{ \gamma_1 L^{-1} (1+C_0)^{1/q} + R\gamma_1 (1+C_0)^{1+1/q} + R\gamma_2 \theta_k (1+C_0)^{1+1/q} \right\}^q < \frac{1}{2mesG},$$

we arrive at the inequality

$$\sum_{k \in J} 1 \leq 4 \cdot 3^{2q-1} M R^{-1} mesG,$$

which, due to the arbitrariness of the finite set J and the uniform boundedness of the chains of root vector functions, implies the necessity of the inequality (1.2).

Sufficiency. For simplicity we consider $G = (0, 2\pi)$. Note that in this case it suffices for us to establish the Bessel property of the system $\left\{u_k(x) \|u_k\|_{2,2}^{-1}\right\}_{k=1}^{\infty}$ in $L_2^2(0, 2\pi)$. In fact, due to the estimate (2.4) and the condition (1.1), for every vector function $f(x) \in L_2^2(0, 2\pi)$ we have

$$\sup_k \left| \int_0^{2\pi} \left(f(x), u_k(x) \|u_k\|_{2,2}^{-1} \right) dx \right| \leq const \|f\|_{1,2} .$$

Therefore, by Riesz-Thorin interpolation theorem, (see, e.g., [17, p.144]), the system $\left\{u_k(x) \|u_k\|_{2,2}^{-1}\right\}_{k=1}^{\infty}$ is Riesz.

On the other hand, by (2.4) and (1.1), (1.2), we have

$$\|u_k\|_{2,2} \|u_k\|_{q,2}^{-1} \leq (2\pi)^{\frac{1}{2}} \|u_k\|_{\infty,2} \|u_k\|_{q,2}^{-1} \leq const, \quad k = 1, 2, \dots .$$

Consequently, for every $f(x) \in L_p^2(0, 2\pi)$, $1 < p \leq 2$, the estimate

$$\sum_{k=1}^{\infty} \left| \left(f, u_k \|u_k\|_{2,2}^{-1} \right) \right|^q = \sum_{k=1}^{\infty} \|u_k\|_{2,2}^q \|u_k\|_{q,2}^{-q} \left| \left(f, u_k \|u_k\|_{2,2}^{-1} \right) \right|^q$$

$$\leq const \sum_{k=1}^{\infty} \left| \left(f, u_k \|u_k\|_{2,2}^{-1} \right) \right|^q \leq M_3 \|f\|_{p,2}^q$$

is true.

So, we have to prove the Bessel property of the system $\left\{ u_k(x) \left\| u_k \right\|_{2,2}^{-1} \right\}_{k=1}^{\infty}$ in $L_2^2(0,2\pi)$. Considering the shift formula (2.1) for $u_k(x+t)$ as $x=0$ and then multiplying it scalarly by the vector function $f(t) = (f_1(t), f_2(t))^T \in L_2^2(0,2\pi)$, we conclude that to prove the Besselness of the system $\varphi_k(t) = u_k(t) \left\| u_k \right\|_{2,2}^{-1}, k = 1,2,...$ in $L_2^2(0,2\pi)$ it suffices to get the validity of the following inequalities:

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} \overline{f_i(t)} \cos \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt \right|^2 \left| \varphi_k^i(0) \right|^2 \leq C \left\| f \right\|_{2,2}^2, i = 1,2; \tag{3.4}$$

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} \overline{f_i(t)} \sin \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt \right|^2 \left| \varphi_k^{3-i}(0) \right|^2 \leq C \left\| f \right\|_{2,2}^2, i = 1,2; \tag{3.5}$$

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} \overline{f_1(t)} \int_0^t p_1(\xi) \varphi_k^1(\xi) \sin \frac{\lambda_k (t-\xi)}{\sqrt{|b_1 b_2|}} d\xi dt \right|^2 \leq C \left\| f \right\|_{2,2}^2, \tag{3.6}$$

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} \overline{f_1(t)} \int_0^t p_2(\xi) \varphi_k^2(\xi) \cos \frac{\lambda_k (t-\xi)}{\sqrt{|b_1 b_2|}} d\xi dt \right|^2 \leq C \left\| f \right\|_{2,2}^2, \tag{3.7}$$

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} \overline{f_2(t)} \int_0^t p_1(\xi) \varphi_k^1(\xi) \cos \frac{\lambda_k (t-\xi)}{\sqrt{|b_1 b_2|}} d\xi dt \right|^2 \leq C \left\| f \right\|_{2,2}^2, \tag{3.8}$$

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} \overline{f_2(t)} \int_0^t p_2(\xi) \varphi_k^2(\xi) \sin \frac{\lambda_k (t-\xi)}{\sqrt{|b_1 b_2|}} d\xi dt \right|^2 \leq C \left\| f \right\|_{2,2}^2, \tag{3.9}$$

$$\sum_{k=1}^{\infty} \left| \theta_k \int_0^{2\pi} \overline{f_i(t)} \int_0^t \frac{u_{k-1}^i(\xi)}{\left\| u_k \right\|_{2,2}} \sin \frac{\lambda_k (t-\xi)}{\sqrt{|b_1 b_2|}} d\xi dt \right|^2 \leq C \left\| f \right\|_{2,2}^2, i = 1,2; \tag{3.10}$$

$$\sum_{k=1}^{\infty} \left| \theta_k \int_0^{2\pi} \overline{f_i(t)} \int_0^t \frac{u_{k-1}^{3-i}(\xi)}{\left\| u_k \right\|_{2,2}} \cos \frac{\lambda_k (t-\xi)}{\sqrt{|b_1 b_2|}} d\xi dt \right|^2 \leq C \left\| f \right\|_{2,2}^2, \ i = 1,2; \tag{3.11}$$

where $\varphi_k^i(\xi) = u_k^i(\xi) \left\| u_k \right\|_{2,2}^{-1}$.

Let's prove the estimate (3.4). By the estimate (2.4) and the conditions (1.1),(1.2), we have

$$\left| \varphi_k^i(0) \right| = \left| u_k^i(0) \right| \left\| u_k \right\|_{2,2}^{-1} \leq \left\| u_k \right\|_{\infty,2} \left\| u_k \right\|_{2,2}^{-1}$$

$$\leq C_2(n_k, G, b_1, b_2)(1+C_0)^{1/2} \left\| u_k \right\|_{2,2} \left\| u_k \right\|_{2,2}^{-1} \leq C_2(n_k, G, b_1, b_2)(1+C_0)^{1/2} = const,$$

because the sequence $C_2(n_k, G, b_1, b_2)$ is bounded due to the condition (2.4). Therefore, for (3.4) to be valid it suffices that the inequality

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} \overline{f_i(t)} \cos \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt \right|^2 \leq C \left\| f \right\|_{2,2}^2, i = 1,2, \tag{3.12}$$

holds.

Under conditions (1.1) and (1.2) with $\nu \geq 1$, the validity of the inequality (3.12) has been proved in [1]. Hence it follows the validity of (3.12) for $Re\lambda_k \in (-\infty, +\infty)$, $|Im\lambda_k| \leq C_0$, because, by the condition of Theorem 1.1, the condition (1.2) holds for any $\nu \in (-\infty, +\infty)$. The inequality (3.5) can be proved in the same way.

Let's verify the inequalities (3.6)-(3.9). They all are proved similarly, so we will only prove (3.6). Denote

$$g_i(t,\xi) = \begin{cases} f_i(t+\xi), & 0 \le t \le 2\pi - \xi, \\ 0, & 2\pi - \xi < t \le 2\pi, \end{cases}$$

where $\xi \in [0, 2\pi], i = 1.2$. Then, by the estimate (2.4) for $r = 2$ and the conditions (1.1),(1.2), we obtain

$$T_k = \left| \int_0^{2\pi} \overline{f_1(t)} \int_0^t p_1(\xi)\varphi_k^1(\xi) \sin\frac{\lambda_k(t-\xi)}{\sqrt{|b_1 b_2|}} d\xi dt \right|^2$$

$$= \int_0^{2\pi} \overline{f_1(t)} \int_0^t p_1(\xi)\varphi_k^1(\xi) \sin\frac{\lambda_k(t-\xi)}{\sqrt{|b_1 b_2|}} d\xi dt \times$$

$$\times \int_0^{2\pi} f_1(t) \int_0^t \overline{p_1(\xi)\varphi_k^1(\xi) \sin\frac{\lambda_k(t-\xi)}{\sqrt{|b_1 b_2|}}} d\xi dt$$

$$= \int_0^{2\pi} p_1(\xi)\varphi_k^1(\xi) \int_0^{2\pi} \overline{g_1(t,\xi)} \sin\frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt d\tau \times$$

$$\times \int_0^{2\pi} \overline{p(\tau)\varphi_k^1(\tau)} \int_0^{2\pi} g_1(r,\tau) \overline{\sin\frac{\lambda_k r}{\sqrt{|b_1 b_2|}}} dr d\tau$$

$$= \int_0^{2\pi}\int_0^{2\pi} p_1(\xi)\overline{p_1(\tau)}\varphi_k^1(\xi)\overline{\varphi_k^1(\tau)} \int_0^{2\pi} \overline{g_1(t,\xi)} \sin\frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt \times$$

$$\times \int_0^{2\pi} g_1(r,\tau) \overline{\sin\frac{\lambda_k r}{\sqrt{|b_1 b_2|}}} dr d\xi d\tau$$

$$\le C_2^2(n_k, G, b_1, b_2)(1 + C_0) \int_0^{2\pi}\int_0^{2\pi} |p_1(\xi)|\,|p_1(\tau)| \left| \int_0^{2\pi} \overline{g_1(t,\xi)} \sin\frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt \right|$$

$$\times \left| \int_0^{2\pi} g_1(r,\tau) \overline{\sin\frac{\lambda_k r}{\sqrt{|b_1 b_2|}}} dr \right| d\xi d\tau$$

$$\le const \int_0^{2\pi}\int_0^{2\pi} |p_1(\xi)|\,|p_1(\tau)| \left| \int_0^{2\pi} \overline{g_1(t,\xi)} \sin\frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt \right|$$

$$\times \left| \int_0^{2\pi} \overline{g_1(r,\tau)} \sin\frac{\lambda_k r}{\sqrt{|b_1 b_2|}} dr \right| d\xi d\tau.$$

Then, for arbitrary positive integer N we obtain

$$\sum_{k=1}^N T_k \le const \int_0^{2\pi}\int_0^{2\pi} |p_1(\xi)|\,|p_1(\tau)|$$

$$\times \left(\sum_{k=1}^N \left| \int_0^{2\pi} \overline{g_1(t,\xi)} \sin\frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt \right| \left| \int_0^{2\pi} \overline{g_1(r,\tau)} \sin\frac{\lambda_k r}{\sqrt{|b_1 b_2|}} dr \right| \right) d\xi d\tau$$

$$\leq const \int_0^{2\pi} \int_0^{2\pi} |p_1(\xi)|\, |p_1(\tau)|\, \|g_1(\cdot,\xi)\|_2\, \|g_1(\cdot,\tau)\|_2\, d\xi d\tau.$$

As the inequality $\|g_1(\cdot,\xi)\|_2 \leq \|f_1\|_2$ holds for every fixed $\xi \in [0, 2\pi]$, we get

$$\sum_{k=1}^{N} T_k \leq const\, \|p_1\|_1^2\, \|f_1\|_2^2 \leq const\, \|f\|_{2,2}^2 .$$

Hence, due to the arbitrariness of the number N, we get the validity of the inequality (3.6).

Now let's prove (3.10). By (2.3), (2.4) and (1.1), (1.2), we have

$$\theta_k \left|u_{k-1}^i(\xi)\right| \|u_k\|_{2,2}^{-1} \leq \theta_k C_1(n_k, G, b_1, b_2) C_2(n_k, G, b_1, b_2)(1 + C_0)^{\frac{2}{3}}$$

$$\times \|u_k\|_{2,2}\, \|u_k\|_{2,2}^{-1} \leq C = const$$

After changing the order of integration, the left-hand side of the inequality (3.10) is majorized from above by the series

$$C \sum_{k=1}^{\infty} \int_0^{2\pi} \left| \int_0^{2\pi} \overline{g_i(t,\xi)} \sin \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} dt \right|^2 d\xi.$$

This series converges due to the Bessel property of the system $\left\{ \sin \frac{\lambda_k t}{\sqrt{|b_1 b_2|}} \right\}_{k=1}^{\infty}$, and its sum is bounded from above by $const\, \|f\|_{2,2}^2$.

The inequality (3.10) is proved. The inequality (3.11) is proved similarly.

4 Proof of the Theorem 1.2.

As the system $\{\upsilon_k(x)\}_{k=1}^{\infty}$ consists of root vector functions of the operator D^* (formal adjoint of D), by Theorem 1.1, the conditions (1.1) and (1.2) provide the Riesz property of the system $\left\{\upsilon_k(x)\, \|\upsilon_k\|_{q,2}^{-1}\right\}_{k=1}^{\infty}$ in $L_p^2(G),\ 1 < p \leq 2, p^{-1} + q^{-1} = 1$, i.e.

$$\sum_{k=1}^{\infty} \left| \left(f, \upsilon_k(x)\, \|\upsilon_k\|_{q,2}^{-1}\right) \right|^q \leq M\, \|f\|_{p,2}^q \tag{4.1}$$

for every vector function $f(x) \in L_p^2(G)$.

The inequality (4.1), the condition (1.3) and the p-closeness of the systems $\left\{u_k(x)\, \|u_k\|_{p,2}^{-1}\right\}_{k=1}^{\infty}$ and $\{\psi_k(x)\}_{k=1}^{\infty}$ in $L_p^2(G)$ imply that the series $\sum_{k=1}^{\infty} \tilde{f}_k\, \|u_k\|_{p,2}\, \|\upsilon_k\|_{q,2} \left(u_k\, \|u_k\|_{p,2}^{-1} - \psi_k(x)\right)$ converges in $L_p^2(G)$ for every $f(x) \in L_p^2(G)$, where $\tilde{f}_k = \left(f, \upsilon_k\, \|\upsilon_k\|_{q,2}^{-1}\right)$. Denote the sum of this series by Kf. As the sequence $K_n f = \sum_{k=1}^{n} \tilde{f}_k\, \|u_k\|_{p,2}\, \|\upsilon_k\|_{q,2} \left(u_k(x)\, \|u_k\|_{p,2}^{-1} - \psi_k(x)\right)$ is fundamental in $L_p^2(G)$, the linear operator K acts in $L_p^2(G)$, i.e. $Kf \in L_p^2(G)$ for $f(x) \in L_p^2(G)$. Obviously, $\|Kf - K_n f\|_{p,2} = o(1)\, \|f\|_{p,2}$, i.e. the sequence of finite dimensional operators $\{K_n\}$ converges to the operator K. Consequently, this operator is compact in $L_p^2(G)$. Besides, $Ku_k\, \|u_k\|_{p,2}^{-1} - \psi_k$, i.e. $(E - K)u_k\, \|u_k\|_{p,2}^{-1} = \psi_k,\ k \in N$, where E is a unit operator.

Let's show that the operator $E - K$ is continuously invertible. The compactness of the operator K and the Fredholm alternative imply that if the operator $E - K$ in non-invertible, then there exists a non-zero element $g \in L_q^2(G)$ such that $(E-K)^* g = 0$. The element g satisfies the relation

$$(g, \psi_k) = \left(g, (E-K)u_k \, \|u_k\|_{p,2}^{-1}\right) = \left((E-K)^* g, u_k \, \|u_k\|_{p,2}^{-1}\right) = 0, \quad k \in N.$$

Hence, due to the basicity of the system $\{\psi_k(x)\}_{k=1}^{\infty}$ for $L_p^2(G)$, it follows that the element g is equal to 0. The obtained contradiction proves the invertibility of the operator $E - K$. Consequently, the system $\left\{u_k(x) \, \|u_k\|_{p,2}^{-1}\right\}_{k=1}^{\infty}$ is a basis for $L_p^2(G)$, and it is equivalent to the basis $\{\psi_k(x)\}_{k=1}^{\infty}$. If we denote by $\{z_k(x)\}_{k=1}^{\infty}$ a biorthogonal adjoint system of $\{\psi_k(x)\}_{k=1}^{\infty}$, then $\upsilon_k(x) \, \|u_k\|_{p,2} = (E-K)^* z_k(x)$. This means that the system $\left\{\upsilon_k(x) \, \|u_k\|_{p,2}\right\}_{k=1}^{\infty}$ is a basis in $L_q^2(G)$ equivalent to the basis $\{z_k(x)\}_{k=1}^{\infty}$.

Theorem 1.2 is proved.

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