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Boundedness of Gegenbauer fractional maximal commutator on generalized Gegenbauer-Morrey spaces

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Abstract. *In this paper, we obtain results of the Spanne-Guliyev, Adams-Guliyev, Adams-Gunavan and Gunawan-Guliyev type on the boundedness of the G-fractional maximal operator* M_G^α *on generalized Gegenbauer-Morrey (G-generalized Morrey) spaces. In addition, we characterize the boundedness of the kth order Gegenbauer fractional maximal commutator* $M_G^{b,\alpha,k}$ *on a generalized G-Morrey spaces* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$.

Keywords. G-fractional maximal operator, G-fractional maximal commutator, generalized G- Morrey spaces, BMO_G space.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35.

1 Introduction

The classical Morrey spaces were originally introduced by Morrey [34] to study the local behaviour of solutions to second-order elliptic partial differential equations. Later, various problems of harmonic analysis (HA) were studied in these spaces. As is known, such operators as: maximal functions, potentials and singular integrals are important object of HA, play a huge rol and have numerous applications in various HA problems: approximation theory, theory differential equation in various problems, physics and mechanics. On of the main problems of HA is the question of the boundedness of the above operators and their commutators in various functional spaces. Morrey spaces, generalized Morrey spaces, weighted Morrey spaces. Therefore, if is no coinicidence that a large number of work are devoted to this theory (see, for example [1-9, 11-14,16,17, 19-26, 33,35-40]). All this indicates the relevance of studuing various kinds of properties in these spaces.

Let $1 \leq p < \infty$, $0 \leq \lambda \leq n$. The classical Morrey space $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ is defined as follows

$$\mathcal{M}_{p,\lambda}(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \; \|f\|_{\mathcal{M}_{p,\lambda}} < \infty \right\},$$

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where

$$\|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{B} \Big(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_{B} |f(x)|^p dx \Big)^{\frac{1}{p}},$$

supremum is taken over all balls $B \subset \mathbb{R}^n$, and $|B|$ is Lebesgue measure. It is knowh that if $1 \le p < \infty$, then by $\lambda = 0$, $\mathcal{M}_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\lambda = n$, $\mathcal{M}_{p,n}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to zero on $\mathbb{R}^n$.

Denote by $W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL^p_{loc}(\mathbb{R}^n)$ with the finite norm

$$\|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n, t>0} \Big(t^{-\lambda} \left|\{y \in B(x,t) : |f(y)| > r\}\right| \Big)^{\frac{1}{p}}.$$

The fractional integral operator I_α, $0 < \alpha < n$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}. \tag{1.1}$$

For locally integralrable functions b, the commutator is defined as follows

$$[b, I_\alpha]f(x) := b(x) I_\alpha f(x) - I_\alpha(bf)(x). \tag{1.2}$$

On $\mathcal{M}_{p,\lambda}$ Morrey spaces classical theory of operators (1.1) is based on the Adams theorem [1] and the Spanne's theorem, published in the paper of Peetre [37], but classical $\mathcal{M}_{p,\lambda}$ theory of operators (1.2) is based on the theorem of Komori-Mizuhara [33] and the theorem of Shirai [39] which are given below.

Classical result Hardy-Littlewood -Sobolev's is as follows:

Theorem A. Let $1 < p < q < \infty$, $0 < \alpha < n$ and $1 \le p < n/\alpha$.

(i) If $1 < p < n/\alpha$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$$

is necessary and sufficient for the boundedness I_α from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

(ii) If $p = 1 < q < \infty$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{n}$$

is necessary and sufficient for the boundedness I_α from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$.

In [1] for the I_α in the Morrey space, Adams proved the following theorem.

Theorem B. (Adams [1]) Let $0 \le \alpha < n$, $0 \le \lambda < n$ and $1 \le p < (n-\lambda)/\alpha$

(i) If $1 < p < (n-\lambda)/\alpha$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

is necessary and sufficient for the boundedness I_α from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.

(ii) If $p = 1$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

is necessary and sufficient for the boundedness I_α from $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.

Theorema C. (Spanne [37]) Let $0 \leq \alpha < n$, $1 \leq p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. Then:

(i) If $p > 1$, then I_α is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\mu}(\mathbb{R}^n)$ if and only if $\lambda/p = \mu/q$.

(ii) If $p = 1$, then I_α is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\mu}(\mathbb{R}^n)$ if and only if $\lambda/p = \mu/q$.

Theorem D. (Komori-Mizuhara [33]) Let $0 \leq \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.

Then the following statements are equivalents:

(a) $b \in BMO(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$.

Theorem E. (Shirai [39]) Let $0 \leq \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $\lambda/p = \mu/q$.

Then the following conditions are equivalent:

(a) $b \in BMO(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\mu}(\mathbb{R}^n)$.

Let $f \in L^{\text{loc}}(\mathbb{R}^n)$. The fractional maximal operator M_α is defined for locally integrable functions in the form

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)| dy, \quad 0 \leq \alpha < n,$$

and its fractional maximal commutator generated by a locally integrable function b has the form

$$M_{b,\alpha}(f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |b(x) - b(y)||f(y)| dy.$$

From the pointwise estimates $M_\alpha f(x) \leq I_\alpha(|f|)(x)$ and $M_{b,\alpha}(f)(x) \leq [b, I_\alpha](|f|)(x)$ follows that Theorems A-E remains strength for the fractional maximal operator and its commutator.

Later these results were obtained for Gegenbauer-Morrey spaces and were reflected in [16,17,29,30].

In [27] we have introdused Riesz potential I_G^α generated by differential operator G in the following form

$$I_G^\alpha f(chx) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(cht) dr \right) A_{cht}^\lambda f(chx) sh^{2\lambda} t dt,$$

where

$$h_r(cht) = \int_1^\infty e^{-\nu(\nu+2\lambda)r} P_\nu^\lambda(cht)(\nu^2 - 1)^{\lambda - \frac{1}{2}} d\nu$$

$P_\nu^\lambda(cht)$ is the eigenfunction of G operator and

$$A_{cht}^\lambda f(chx) = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi f(chxcht - shxsht \cos\varphi)(\sin\varphi)^{2\lambda - 1} d\varphi,$$

is a generalized shift operator associated with Gegenbauer differential operator G [10]

$$G \equiv G_\lambda = \left(x^2 - 1\right)^{\frac{1}{2} - \lambda} \frac{d}{dx} \left(x^2 - 1\right)^{\lambda + \frac{1}{2}} \frac{d}{dx}, \;\; x \in (1, \infty), \;\; \lambda \in \left(0, \frac{1}{2}\right).$$

Denote by, $L_{p,\lambda}(\mathbb{R}_+)$, $1 \le p \le \infty$ the space of μ_λ - measurable functions ($\mu_\lambda(x) = sh^{2\lambda}x$) with finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left(\int_0^\infty |f(chx)|^p \, sh^{2\lambda} x dx \right)^{\frac{1}{p}}, \;\; 1 \le p < \infty,$$

$$\|f\|_{L_{\infty,\lambda}(\mathbb{R}_+)} = \|f\|_{L_\infty(\mathbb{R}_+)} = ess \sup_{x \in \mathbb{R}_+} |f(chx)| \, ,$$

and denote by $WL_{p,\lambda}(\mathbb{R}_+)$ the weak $L_{p,\lambda}(\mathbb{R}_+)$ space of μ-measurable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+)} = \sup_{r>0} r \left| \{x \in \mathbb{R}_+ : |f(chx)| > r\} \right|_\lambda^{\frac{1}{p}}$$

$$= \sup_{r>0} r \left(\int_{\{x \in \mathbb{R}_+ : |f(chx)| > r\}} sh^{2\lambda} x dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

In [15] for potential I_G^α the following theorem which is an analogue of Theorem A was proved.

Theorem F. Let $0 < \lambda < \frac{1}{2}, 0 < \alpha < 2\lambda + 1$ and $1 \le p < \frac{2\lambda+1}{\alpha}$.

(a) If $1 < p < \frac{2\lambda+1}{\alpha}$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1}$$

is necessary and sufficient for the boundedness for I_G^α from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$.

(b) If $p = 1$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1}$$

is necessary and sufficient for the boundedness of I_G^α from $L_{1,\lambda}(\mathbb{R}_+)$ to $WL_{q,\lambda}(\mathbb{R}_+)$.

In [16] was introduced the concept of Gegenbauer-Morrey space (G-Morrey space) associated with differential operator G on the set of locally integrable functions with the finite norm

$$\|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} = \sup_{x \in \mathbb{R}_+, r>0} \left(r^{-\nu} \int_0^r A_{cht}^\lambda |f(chx)|^p \, sh^{2\lambda} t dt \right)^{\frac{1}{p}}, \; 0 \le \nu \le 2\lambda + 1,$$

and also weak $WL_{p,\lambda,\nu}(\mathbb{R}_+)$ space with the finite norm

$$\|f\|_{WL_{p,\lambda,\nu}(\mathbb{R}_+)} = \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t>0} \left(t^{-\nu} \left| \left\{ y \in (0,t) : A_{chy}^\lambda |f(chx)| > r \right\} \right|_\lambda \right)^{\frac{1}{p}}$$

$$= \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t>0} \left(t^{-\nu} \int\limits_{\left\{ y \in (0,t) : A_{chy}^\lambda |f(chx)| > r \right\}} sh^{2\lambda} y dy \right)^{\frac{1}{p}}.$$

For potential I_G^α on G- Morrey space we have the following theorem ([16, §2 Theorem 2.1]), which is an analogue of Theorem B.

Theorem G. Let $0 < \alpha < 2\lambda + 1, \; 0 < \nu < 2\lambda + 1 - \alpha p$ and $1 \le p < \frac{2\lambda+1}{\alpha}$

(i) If $1 < p < \frac{2\lambda+1-\nu}{\alpha}$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1 - \nu}$$

is necessary and sufficient for the boundedness I_G^α from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$.

(ii) If $p = 1 < \frac{2\lambda+1-\nu}{\alpha}$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1 - \nu}$$

is necessary and sufficient for the boundedness I_G^α from $L_{1,\lambda,\nu}(\mathbb{R}_+)$ to $WL_{q,\lambda,\nu}(\mathbb{R}_+)$.

In [28] Muckenhoupt type weighted class $A_p^\lambda(\mathbb{R}_+)$ associated and with differential operator G was introduced and in this class for operators M_G^α and I_G^α , on conditions $0 < \alpha < 2\lambda + 1$, $1 < p < \frac{2\lambda+1}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$, the strong and weak types weighted $(L_{p,\lambda,\omega}, L_{q,\lambda,\omega})$ inequalities ([28, § 4.1, Theorems 4.1 and 4.2 and also § 4.2, Theorems 4.4 and 4.5]) were proved.

In this paper on pair (ω_1, ω_2) the necessary and sufficient conditions for the boundedness of fractional maximal operator M_G^α, $0 \le \alpha < \gamma$, from one generalized G- Morrey space $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to another $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$, $1 < p \le q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$ and from the space $\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)$th the weak space $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$, $1 < q < \infty$, $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$ were obtained. Also find necessary and sufficient conditions on the ω, which ensure the Adams type boundedness of the M_G^α from $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ for $1 < p < q < \infty$ and from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ for $1 = p < q < \infty$.

In the case $b \in BMO_G$ on pair (ω_1, ω_2), the necessary and sufficient conditions for the boundedness of the commutator $M_G^{b,\alpha}$ from $\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ by $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$ and also the necessary and sufficient conditions for the boundedness of commutator $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ for $1 < p < q < \infty$.

Later we will consider $\omega(x, r)$, $\omega_1(x, r)$ and $\omega_2(x, r)$ as nonnegative Lebesgue measurable functions on $\mathbb{R}_+ = (0, \infty)$.

Note that all result obtained in the paper are the future development of Gegenbauer harmonic analysis theory, foundations of which were laid in [32]. This theory was later developed in different directions: approximation and embedding theory, transformation theory, theory of singulars integrals, maximal functions theory, theory of potentials and its commutator.

2 Definitions, notations and auxiliary results

Throughout the paper, we will denote by shx, chx the hyperbolic functions.

In what follows, the expression $A \lesssim B$ mean that there exists a constant C such that $0 < A \le CB$, where C may depend on some inessential parameters. If $A \lesssim B$ and $B \lesssim A$ then we write $A \approx B$ and saoy that A and B are equivalent.

Let $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$. Denote by $H_r = (0, r)$ and $r \in (0, \infty)$.

Later we will need the following relation (see [15,§1, formula (1.2)])

$$|H_r|_\lambda = \int_0^r sh^{2\lambda} tdt \approx \left(sh\frac{r}{2}\right)^\gamma \tag{2.1}$$

where $0 < \lambda < \frac{1}{2}$ and

$$\gamma = \gamma_\lambda(r) = \begin{cases} 2\lambda + 1, & \text{if } 0 < r < 2, \\ 4\lambda, & \text{if } 2 \le r < \infty \end{cases}$$

and $|H_r|_\lambda$ is an absolutely continuous measure of the interval H_r.

According to formula (2.1) Gegenbauer maximal operator (G-maximal operator) M_G, fractional maximal operator M_G^α and Gegenbauer fractional integral J_G^α for any $x \in \mathbb{R}_+$ are defined as follows:

$$M_G f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda} \int\limits_{H_r} A_{cht}^\lambda |f(chx)|\, sh^{2\lambda} t dt,$$

$$M_G^\alpha f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_{H_r} A_{cht}^\lambda |f(chx)|\, sh^{2\lambda} t dt, \quad 0 \le \alpha < \gamma.$$

It's obvious that $M_G^0 f(chx) \equiv M_G f(chx)$.

$$J_G^\alpha f(chx) = \int_0^\infty |H_t|_\lambda^{\frac{\alpha}{\gamma}-1} A_{cht}^\lambda f(chx) sh^{2\lambda} t dt, \quad 0 < \alpha < \gamma,$$

where

$$|H_t|_\lambda^{\frac{\alpha}{\gamma(t)}-1} = \begin{cases} |H_t|_\lambda^{\frac{\alpha}{2\lambda+1}-1}, & 0 < t < 2,\ 0 < \alpha < 2\lambda + 1, \\ |H_t|_\lambda^{\frac{\alpha}{4\lambda}-1}, & 2 \le t < \infty, \quad 0 < \alpha < 4\lambda. \end{cases}$$

In this section we present some generalization of Gegenbauer-Morrey space (G- Morrey space).

In [29] by analogy to Nakai [36] introduced the concept of generalized G-Morrey space on the set locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ which the finite norm

$$\|f\|_{M_{p,\lambda,\omega}(\mathbb{R}_+)} = \sup_{x \in \mathbb{R}_+, r>0} \left(\frac{1}{\omega(r)} \int\limits_{H_r} A_{cht}^\lambda |f(chx)|^p\, sh^{2\lambda} t dt \right)^{\frac{1}{p}}$$

and weak G-Morrey space $WM_{p,\lambda,\omega}(\mathbb{R}_+)$ with the finite norm

$$\|f\|_{WM_{p,\lambda,\omega}(\mathbb{R}_+)} = \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t>0} \left(\frac{1}{\omega(r)} \left| \left\{ y \in H_t : A_{chy}^\lambda |f(cht)| > r \right\} \right|_\lambda \right)^{\frac{1}{p}},$$

where $\omega(r)$ nonneagtive Lebesgue measurable function on $\mathbb{R}_+$, $1 \le p < \infty$.

Note that, when $\omega(r) \equiv 1$ the space $M_{p,\lambda,\omega}(\mathbb{R}_+)$ goes above considered space $L_{p,\lambda}(\mathbb{R}_+)$, and Theorem F is a conseqense of the Theorem H given below.

Let $0 < \delta \le 1$. Suppose that $\omega(r)$ satisfies the conditions

$$r \le t \le 2r \Rightarrow \omega(t) \approx \omega(r), \tag{a}$$

$$\int\limits_r^\infty \frac{\omega(t)}{t^{\nu\delta+1}} dt \lesssim \begin{cases} r^{-(2\lambda+1)\delta} \omega(r), \ \nu = 2\lambda + 1, \ 0 < r < 2, \\ r^{-4\lambda\delta} \omega(r), \quad \nu = 4\lambda, \quad 2 \le r < \infty. \end{cases} \tag{b}$$

Theorem H. [29] *Let* $0 < \lambda < \frac{1}{2},\ 0 < \alpha < 2\lambda + 1, 1 \le p < \frac{\alpha}{2\lambda+1}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$. *Suppose that* ω *satisfies the conditions (a) and (b). Then:*

(i) *if* $p > 1$ and $f \in M_{p,\lambda,\omega}(\mathbb{R}_+)$, *then the inequality*

$$\|I_G^\alpha f\|_{M_{q,\lambda,\omega^{\frac{q}{p}}}} \lesssim \|f\|_{M_{p,\lambda,\omega}}$$

is true.

(ii) *if* $p = 1$ *and* $f \in M_{1,\lambda,\omega}(\mathbb{R}_+)$ *then the inequality*

$$\|I_G^\alpha f\|_{WM_{q,\lambda,\omega^q}} \lesssim \|f\|_{M_{1,\lambda,\omega}}$$

is true.

Theorem L. [29] *Let the conditions (a) and (b) be satisfied. Then:*

(i) for $f \in M_{p,\lambda,\omega}(\mathbb{R}_+)$ and $1 \leq q < p < \infty$

$$\left\|M_G^q f\right\|_{M_{p,\lambda,\omega}} \lesssim \|f\|_{M_{p,\lambda,\omega}},$$

(ii) for $f \in M_{p,\lambda,\omega}(\mathbb{R}_+)$, $1 \leq p < \infty$ and any $t > 0$

$$\left\|M_G^p f\right\|_{WM_{p,\lambda,\omega}} \lesssim \|f\|_{M_{p,\lambda,\omega}}.$$

In accordance with the formula (2.1), we give the following generalization of G-Morrey space.

Definition 2.1 *Let* $1 \leq p < \infty$. *Generalized G-Morrey space* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *associated with Gegenbauer differential operator G is defined as a set of locally integrable functions* $f(chx), x \in \mathbb{R}_+$ *with the finite norm*

$$\|f\|_{\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} = \sup_{x,r \in \mathbb{R}_+} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \left\|A_{chx}^\lambda f\right\|_{L_{p,\lambda}(H_r)}$$
$$= \max\Big\{ \sup_{\substack{x \in \mathbb{R}_+ \\ 0<r<2}} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{2\lambda+1}{p}} \left\|A_{chx}^\lambda f\right\|_{L_{p,\lambda}(H_r)},$$
$$\sup_{\substack{x \in \mathbb{R}_+ \\ 2 \leq r<\infty}} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{4\lambda}{p}} \left\|A_{chx}^\lambda f\right\|_{L_{p,\lambda}(H_r)} \Big\},$$

where

$$\left\|A_{chx}^\lambda f\right\|_{L_{p,\lambda}(H_r)} = \Big(\int_0^r A_{cht}^\lambda \left|f(chx)\right|^p sh^{2\lambda} t dt\Big)^{\frac{1}{p}}.$$

Also the weak generalized G-Morrey space $W\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ of locally integrable functions $f(chx), x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{W\mathcal{M}_{p,\lambda,\omega}(\mathbb{R}_+)} = \sup_{x,r \in \mathbb{R}_+} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \left\|A_{chx}^\lambda f\right\|_{WL_{p,\lambda}(H_r)}$$
$$= \max\Big\{ \sup_{\substack{x \in \mathbb{R}_+ \\ 0<r<2}} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{2\lambda+1}{p}} \left\|A_{chx}^\lambda f\right\|_{WL_{p,\lambda}(H_r)},$$
$$\sup_{\substack{x \in \mathbb{R}_+ \\ 2 \leq r<\infty}} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{4\lambda}{p}} \left\|A_{chx}^\lambda f\right\|_{WL_{p,\lambda}(H_r)} \Big\},$$

where

$$\left\|A_{chx}^{\lambda} f\right\|_{WL_{p,\lambda}(H_r)} = \sup_{t>0} \sup_{x,r\in\mathbb{R}_+} t \left|\left\{y \in H_r : A_{chy}^{\lambda} |f(chx)| > t\right\}\right|_{\lambda}^{\frac{1}{p}}.$$

We will prove some auxiliary statements, which we will need later.

Lemma 2.2. *Let $\omega(x,r)$ be a positive measurable function on $\mathbb{R}_+$.*

(i) *If*

$$\sup_{t<r<\infty} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \omega(x,r)^{-1} = \infty \tag{2.2}$$

is true for some $t > 0$ and any $x \in \mathbb{R}_+$, then $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+) = \Theta$.

(ii) *If*

$$\sup_{0<r<t} \omega(x,r)^{-1} = \infty \tag{2.3}$$

is true for some $t > 0$ and any $x \in \mathbb{R}_+$, then $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+) = \Theta$, where Θ is a all functions that are equivalent to zero on $\mathbb{R}_+$.

Proof. Suppose (2.2) is true and f is not equivalent to zero. Then $\sup_{x\in\mathbb{R}_+} \|f\|_{L_{p,\lambda}(H_t)} > 0$, hence

$$\|f\|_{\mathcal{M}_{p,\gamma,\omega}} \geq \sup_{x,r\in\mathbb{R}_+} \sup_{t<r<\infty} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_r)} \geq$$

$$\geq \sup_{x\in\mathbb{R}_+} \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_t)} \sup_{t<r<\infty} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}}.$$

Hence $\|f\|_{\mathcal{M}_{p,\lambda,\omega}} = \infty$.

(ii) Let $f \in \mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ and (2.2) be satisfied, then there are two possibilities:

Case 1: $\sup_{0<r<t} \omega(x,r)^{-1} = \infty$ for all $t > 0$.

Case 2: $\sup_{0<r<t} \omega(x,r)^{-1} < \infty$ for some $t \in (0,s)$.

For Case 1, by Lebesgue differentiation theorem (see [27], Corollary 2.1 from Theorem 2.2), for almost $x \in \mathbb{R}_+$

$$\lim_{r\to 0+} \frac{\left\|A_{chx}^{\lambda} f\chi_{H_r}\right\|_{L_{p,\lambda}}}{\left\|\chi_{H_r}\right\|_{L_{p,\lambda}}} = |f(chx)|, \tag{2.4}$$

where χ_{H_r} is a characteristic function of a set H_r. We require that $f(chx) = 0$ for all those x. Indeed, fix x and assume $|f(chx)| > 0$. Then from (2.1) and (2.4) there exists $t_0 > 0$ such that

$$\frac{\left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}}}{\left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}}} \gtrsim |f(chx)|,$$

for all $0 < r \leq t_0$. Consequently,

$$\|f\|_{\mathcal{M}_{p,\gamma,\omega}} \geq \sup_{0<r\leq t_0} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_r)}$$

$$\gtrsim \sup_{0<r\leq t_0} \omega(x,r)^{-1} |f(chx)|.$$

Hence $\|f\|_{\mathcal{M}_{p,\gamma,\omega}} = \infty$, so $f \notin \mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ and we have arrived at a contradiction.

Note that Case 2 implies that $\sup\limits_{t<r<s} \omega(x,r)^{-1} = \infty$, consequently

$$\sup_{t<r<\infty} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \geq \sup_{t<r<s} \omega(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}}$$

$$\geq \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{p}} \sup_{t<r<s} \omega(x,r)^{-1} = \infty,$$

which is the case in (i).

Denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(cht), t > 0$ with the finite norm

$$\|g\|_{L\infty,v(0,\infty)} = ess\sup_{t>0} v(cht)g(cht)$$

and

$$L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty).$$

Remark 2.3. *Denote by Ω_p^γ a set off all positive measurable functions ω on $\mathbb{R}_+$ such that for all $r > 0$*

$$\sup_{x\in\mathbb{R}_+} \left\| \frac{\left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}}}{\omega(x,r)} \right\|_{L_\infty(t,\infty)} < \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}_+} \left\|\omega(x,r)^{-1}\right\|_{L_\infty(0,t)} < \infty,$$

respectively. Lemma 2.1 shows, that it makes sense to consider only functions ω, from Ω_p^γ, which we will assume in what follows.

A function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be almost increasing (resp. almost decreasing) if $\omega(r) \lesssim \omega(s)$ (resp $\omega(r) \gtrsim \omega(s)$) for $r \leq s$. Let $1 \leq p < \infty$. Denote by Φ_p^γ a set of all almost decreasing functions $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $t \in \mathbb{R}_+ \to \left(sh\frac{t}{2}\right)^{\frac{\gamma}{p}} \omega(t) \in \mathbb{R}_+$ is almost increasing.

Lemma 2.4. *Let $\omega \in \Phi_p^\gamma, 1 \leq p < \infty$, $H_0 = (0, r_0)$ and χ_{H_0} is the characteristic function of the characterictic function of the interval H_0, then $\chi_{H_0} \in \mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$.*

Moreover,

$$\frac{1}{\omega(r_0)} \leq \left\|\chi_{H_0}\right\|_{W\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} \leq \left\|\chi_{H_0}\right\|_{\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} \lesssim \frac{1}{\omega(r_0)}.$$

Proof. Let $\omega \in \Phi_p^\gamma, 1 \leq p < \infty$, $H_0 = (0, r_0)$ be any interval on $\mathbb{R}_+$. It is easy to see that

$$\left\|\chi_{H_0}\right\|_{W\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} = \sup_{x\in\mathbb{R}_+} \frac{1}{\omega(r)} \left(\frac{|H_r \cap H_0|_\lambda}{|H_r|_\lambda}\right)^{\frac{1}{p}}$$

$$\geq \frac{1}{\omega(r_0)} \left(\frac{|H_0 \cap H_0|_\lambda}{|H_0|_\lambda}\right)^{\frac{1}{p}} = \frac{1}{\omega(r_0)}.$$

Now, if $r \leq r_0$ then $\omega(r_0) \lesssim \omega(r)$ and

$$\frac{1}{\omega(r)} \left(\frac{|H_r \cap H_0|_\lambda}{|H_r|_\lambda}\right)^{\frac{1}{p}} \lesssim \frac{1}{\omega(r_0)} \left(\frac{|H_r \cap H_r|_\lambda}{|H_r|_\lambda}\right)^{\frac{1}{p}} \lesssim \frac{1}{\omega(r_0)}.$$

On the other hand, if $r_0 \leq r$ then by (2.1)

$$\omega(r_0) \left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}} \lesssim \omega(r) \left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}},$$

then

$$\frac{1}{\omega(r)}\left(\frac{|H_r\cap H_0|_\lambda}{|H_r|_\lambda}\right)^{\frac{1}{p}} \lesssim \frac{(|H_0\cap H_0|_\lambda)^{\frac{1}{p}}}{\omega(r)\left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}}} = \frac{(|H_0|_\lambda)^{\frac{1}{p}}}{\omega(r)\left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}}}$$

$$\lesssim \frac{\left(sh\frac{r_0}{2}\right)^{\frac{\gamma}{p}}}{\omega(r)\left(sh\frac{r}{2}\right)^{\frac{\gamma}{p}}} \lesssim \frac{1}{\omega(r_0)}.$$

This completes the proof.

Lemma 2.5. [32] *If $f\in L^{loc}_{1,\lambda}(\mathbb{R}_+)$. $1\le p\le\infty$. Then for any $0\le t<\infty$ the following inequality*

$$\left\|A^\lambda_{chx}f\right\|_{L_{p,\lambda}(\mathbb{R}_+)} \le \|f\|_{L_{p,\lambda}(\mathbb{R}_+)}$$

holds.

Denote

$$M^\alpha_{G_1}f(chx) = \sup_{r\in(0,2)}\frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{2\lambda+1}}}\int_{H_r} A^\lambda_{cht}\,|f(chx)|\,sh^{2\lambda}tdt.$$

Lemma 2.6. (1) *Let $1\le p<\infty, 0\le\alpha<\frac{2\lambda+1}{p}, \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$. Then for $p>1$ and any interval $H_r=(0,r), 0<r<2$ the following inequality*

$$\left\|A^\lambda_{chx}M^\alpha_{G_1}f\right\|_{L_{q,\lambda}(H_r)} \lesssim \left\|A^\lambda_{chx}f\right\|_{L_{q,\lambda}(H_r)}$$

$$+\left(sh\frac{r}{2}\right)^{\frac{2\lambda+1}{q}}\sup_{r<s<2}\left(sh\frac{s}{2}\right)^{\alpha-2\lambda-1}\left\|A^\lambda_{chx}f\right\|_{L_{q,\lambda}(H_s)} \tag{2.5}$$

holds for all $f\in L^{loc}_{p,\lambda}(R_+)$.

Moreover, for $p=1$ the following inequality

$$\left\|A^\lambda_{chx}M^\alpha_{G_1}f\right\|_{WL_{q,\lambda}(H_r)} \lesssim \left\|A^\lambda_{chx}f\right\|_{L_{1,\lambda}(H_r)}$$

$$+\left(sh\frac{r}{2}\right)^{\frac{2\lambda+1}{q}}\sup_{r<s<2}\left(sh\frac{s}{2}\right)^{\alpha-2\lambda-1}\left\|A^\lambda_{chx}f\right\|_{L_{1,\lambda}(H_s)}, \tag{2.6}$$

holds for all $f\in L^{loc}_{1,\lambda}(\mathbb{R}_+)$.

Proof. Let $1<p<q<\infty$. For the interval $H_r=(0,2)$, where $r\in(0,2)$ let $f=f_1+f_2$, where $f_1=f\chi_{H_r}$ and $f_2=f\chi_{(H_r)^c}=f\chi_{(r,2)}$, then

$$\left\|M^\alpha_{G_1}f\right\|_{L_{q,\lambda}(H_r)} \lesssim \left\|M^\alpha_{G_1}f_1\right\|_{L_{q,\lambda}(H_r)} + \left\|M^\alpha_{G_1}f_2\right\|_{L_{q,\lambda}(H_r)}. \tag{2.7}$$

From continuity of the operator $M^\alpha_G: L_{p,\lambda}(\mathbb{R}_+)\to L_{q,\lambda}(\mathbb{R}_+)$, (see [18, Corollary 5.6]) follows that

$$\left\|M^\alpha_{G_1}f_1\right\|_{L_{q,\lambda}(H_r)} \lesssim \|f_1\|_{L_{p,\lambda}(H_r)} \lesssim \|f\|_{L_{p,\lambda}(H_r)}. \tag{2.8}$$

If $H_s\cap(H_r)^c\ne\varnothing$, then $s>r$ and, we have

$$
\begin{aligned}
M_{G_2}^{\alpha} f_1(chx) &= \sup_{s>0} |H_s|_{\lambda}^{\frac{\alpha}{\gamma}-1} \int\limits_{H_s \cap (H_r)^c} A_{cht}^{\lambda} |f_1(chx)|\, sh^{2\lambda} t dt \\
&\leq \sup_{r<s<2} |H_s|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{r}^{s} A_{cht}^{\lambda} |f_1(chx)|\, sh^{2\lambda} t dt \\
&\lesssim \sup_{r<s<2} |H_s|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{0}^{s} A_{cht}^{\lambda} |f(chx)|\, sh^{2\lambda} t dt \\
&= \sup_{r<s<2} |H_s|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H_s} A_{cht}^{\lambda} |f(chx)|\, sh^{2\lambda} t dt. \qquad (2.9)
\end{aligned}
$$

From (2.9) and Lemma 2.5, we get

$$
\left\| M_{G_2}^{\alpha} f_1 \right\|_{L_{q,\lambda}(H_r)} \lesssim |H_r|_{\lambda}^{\frac{1}{q}} \sup_{r<s<2} |H_s|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \|f\|_{L_{p,\lambda}(H_s)}. \qquad (2.10)
$$

Taking into account (2.8) and (2.10) in (2.7), we obtain

$$
\left\| M_{G_1}^{\alpha} f \right\|_{L_{q,\lambda}(H_r)} \lesssim \|f\|_{L_{p,\lambda}(H_s)} + |H_r|_{\lambda}^{\frac{1}{q}} \sup_{r<s<2} |H_s|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \|f\|_{L_{p,\lambda}(H_s)}
$$

Using equality (see[16], proof of Theorem 1.4) $M_G A_{cht}^{\lambda} f(chx) = A_{cht}^{\lambda} M_G f(chx)$, and (2.1), we have (2.5).

Indeed

$$
\left\| A_{chx}^{\lambda} M_{G_1}^{\alpha} f \right\|_{L_{q,\lambda}(H_r)} \lesssim \left\| A_{chx}^{\lambda} f \right\|_{L_{p,\lambda}(H_r)} + \left(sh\frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \sup_{r<s<2} \left(sh\frac{s}{2} \right)^{\alpha-2\lambda-1} \left\| A_{chx}^{\lambda} f \right\|_{L_{p,\lambda}(H_s)}.
$$

Let $p = 1$. Obviously that for interval H_r

$$
\left\| M_{G_1}^{\alpha} f_1 \right\|_{WL_{q,\lambda}(H_r)} \lesssim \left\| M_{G_1}^{\alpha} f_1 \right\|_{WL_{q,\lambda}(H_r)} + \left\| M_{G_1}^{\alpha} f_2 \right\|_{WL_{q,\lambda}(H_r)}.
$$

From continuity of the operator $M_G^{\alpha} : L_{1,\lambda}(\mathbb{R}_+) \to WL_{q,\lambda}(\mathbb{R}_+)$ (see[18, Corollary 5.6]) we get

$$
\left\| M_{G_1}^{\alpha} f_1 \right\|_{WL_{q,\lambda}(H_r)} \lesssim \|f\|_{L_{1,\lambda}(H_r)}. \qquad (2.11)
$$

Similar to the previous from (2.10) and (2.11), we obtain (2.6). The proof of the Lemma is complete.

Denote

$$
M_{G_2}^{\alpha} f(chx) = \sup_{r\in[2,\infty)} \frac{1}{|H_r|_{\lambda}^{1-\frac{\alpha}{4\lambda}}} \int_{H_r} A_{cht}^{\lambda} |f(chx)|\, sh^{2\lambda} t dt.
$$

Lemma 2.6. (2). *Let* $1 \leq p < \infty, 0 \leq \alpha < \frac{4\lambda}{p}, \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{4\lambda}$. *Then for* $p > 1$ *and any interval* $H_r = (0, r), r \in [2, \infty)$ *the following inequality*

$$
\left\| A_{chx}^{\lambda} M_{G_2}^{\alpha} f \right\|_{L_{q,\lambda}(H_r)} \lesssim \left\| A_{chx}^{\lambda} f \right\|_{L_{p,\lambda}(H_r)}
$$

$$+\left(sh\frac{r}{2}\right)^{\frac{4\lambda}{q}}\sup_{s>r\geq 2}\left(sh\frac{s}{2}\right)^{\alpha-4\lambda}\left\|A^{\lambda}_{chx}f\right\|_{L_{p,\lambda}(H_s)} \tag{2.12}$$

holds for all $f\in L^{loc}_{p,\lambda}(\mathbb{R}_+)$.

Moreover, for $p=1$ the following inequality

$$\left\|A^{\lambda}_{chx}M^{\alpha}_{G_2}f\right\|_{WL_{q,\lambda}(H_r)}\lesssim\left\|A^{\lambda}_{chx}f\right\|_{L_{1,\lambda}(H_r)}$$

$$+\left(sh\frac{r}{2}\right)^{\frac{4\lambda}{q}}\sup_{s>r\geq 2}\left(sh\frac{s}{2}\right)^{\alpha-4\lambda}\left\|A^{\lambda}_{chx}f\right\|_{L_{1,\lambda}(H_s)}, \tag{2.13}$$

holds for all $f\in L^{loc}_{1,\lambda}(\mathbb{R}_+)$.

Proof. Let $1<p<q<\infty$. For the interval $H_r=(0,r), r\in[2,\infty)$ let $f=f_1+f_2$, where $f_1=f\chi_{H_r}$ and $f_2=f\chi_{(H_r)^c}=f\chi_{(r,\infty)}$, then

$$\left\|M^{\alpha}_{G_2}f\right\|_{L_{q,\lambda}(H_r)}\leq\left\|M^{\alpha}_{G_2}f_1\right\|_{L_{q,\lambda}(H_r)}+\left\|M^{\alpha}_{G_2}f_2\right\|_{L_{q,\lambda}(H_r)}. \tag{2.14}$$

From continuity of the operator $M^{\alpha}_{G}:L_{p,\lambda}(\mathbb{R}_+)\to L_{q,\lambda}(\mathbb{R}_+)$, follows that

$$\left\|M^{\alpha}_{G_2}f\right\|_{L_{q,\lambda}(H_r)}\lesssim\|f\|_{L_{p,\lambda}(H_r)}. \tag{2.15}$$

If $H_s\cap(H_r)^c\neq\varnothing$, then $s>r$ and, we get

$$M^{\alpha}_{G_2}f_2(chx)=\sup_{s>0}|H_s|_{\lambda}^{\frac{\alpha}{4\lambda}-1}\int\limits_{H_s\cap(H_r)^c}A^{\lambda}_{cht}\left|f_2(chx)\right|sh^{2\lambda}tdt$$

$$\leq\sup_{2\leq r<s<\infty}|H_s|_{\lambda}^{\frac{\alpha}{4\lambda}-1}\int\limits_r^s A^{\lambda}_{cht}\left|f_2(chx)\right|sh^{2\lambda}tdt$$

$$\lesssim\sup_{2\leq r<s<\infty}|H_s|_{\lambda}^{\frac{\alpha}{4\lambda}-1}\int\limits_0^s A^{\lambda}_{cht}\left|f(chx)\right|sh^{2\lambda}tdt=$$

$$=\sup_{2\leq r<s<\infty}|H_s|_{\lambda}^{\frac{\alpha}{4\lambda}-1}\int\limits_{H_s}A^{\lambda}_{cht}\left|f(chx)\right|sh^{2\lambda}tdt. \tag{2.16}$$

By Hölder's inequality from (2.16) and (2.1), we have

$$\left\|M^{\alpha}_{G_2}f_2\right\|_{L_{q,\lambda}(H_r)}\lesssim|H_r|_{\lambda}^{\frac{1}{q}}\sup_{2\leq r<s<\infty}|H_s|_{\lambda}^{\frac{\alpha}{4\lambda}-1}\|f\|_{L_{p,\lambda}(H_s)}. \tag{2.17}$$

Taking into account (2.15) and (2.17) in (2.14), we obtain

$$\left\|M^{\alpha}_{G_2}f\right\|_{L_{q,\lambda}(H_r)}\lesssim\|f\|_{L_{p,\lambda}(H_r)}+|H_r|_{\lambda}^{\frac{1}{q}}\sup_{2\leq r<s<\infty}|H_s|_{\lambda}^{\frac{\alpha}{4\lambda}-1}\|f\|_{L_{p,\lambda}(H_s)}.$$

Arguing as above, we get (2.12).

Now let $p=1$. Clear that

$$\left\|M^{\alpha}_{G_2}f\right\|_{WL_{q,\lambda}(H_r)}\lesssim\left\|M^{\alpha}_{G_2}f_1\right\|_{WL_{q,\lambda}(H_r)}+\left\|M^{\alpha}_{G_2}f_2\right\|_{WL_{q,\lambda}(H_r)}.$$

From continuity of the operator $M_G^\alpha : L_{1,\lambda}(\mathbb{R}_+) \to WL_{q,\lambda}(\mathbb{R}_+)$, we have

$$\left\| M_{G_2}^\alpha f_1 \right\|_{WL_{q,\lambda}(H_r)} \lesssim \|f\|_{L_{1,\lambda}(H_r)}. \tag{2.18}$$

From (2.17) and (2.18), we obtain (2.13).

Lemma 2.7. *Let* $1 \le p < \infty$, *and* $0 \le \alpha < \frac{\gamma}{p}$.

$$\text{(a) } Let\ \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma},\ \ \gamma = 2\lambda + 1,\ \text{ if }\ r \in (0,2),$$
$$\text{(b) Let } \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma},\ \ \gamma = 4\lambda,\ \text{ if }\ \ r \in [2,\infty).$$

Then for any interval $H_r = (0,r) \subset \mathbb{R}_+$ the following inequalities is valid:

$$\left\| A_{chx}^\lambda M_G^\alpha f \right\|_{L_{q,\lambda}(H_r)} \lesssim \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)} + \left(sh\frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{r}{2} \right)^{\alpha-\gamma} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}$$

for any $p > 1$ and all $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$ and

$$\left\| A_{chx}^\lambda M_G^\alpha f \right\|_{WL_{q,\lambda}(H_r)} \lesssim \left\| A_{chx}^\lambda f \right\|_{L_{1,\lambda}(H_r)} + \left(sh\frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{s}{2} \right)^{\alpha-\gamma} \left\| A_{chx}^\lambda f \right\|_{L_{1,\lambda}(H_s)}$$

for $p = 1$ and all $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$.

Proof. By the definition and (2.1) , we have

$$M_G^\alpha f(chx) = \sup_{s>r} |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int\limits_{H_s} A_{cht}^\lambda \left| f(chx) \right| sh^{2\lambda} t dt$$
$$= M_{G_1}^\alpha f(chx) + M_{G_2}^\alpha f(chx).$$

Taking into account (2.5),(2.6) also (2.12) and (2.13) we will receive our approval.

Let

$$\left(sh\frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}$$
$$= \max \left\{ \left(sh\frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \sup_{r<s<2} \left(sh\frac{s}{2} \right)^{-\frac{2\lambda+1}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}, \right.$$
$$\left. \left(sh\frac{r}{2} \right)^{\frac{4\lambda}{q}} \sup_{s>r\ge 2} \left(sh\frac{s}{2} \right)^{-\frac{4\lambda}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \right\}.$$

Lemma 2.8. *Suppose* $1 \le p < \infty$, *and let the conditions (2.19) holds.* *Then for any interval* $H_r = (0,r) \subset \mathbb{R}_+$ *the following inequalities is valid:*

$$\left\| A_{chx}^\lambda M_G^\alpha f \right\|_{L_{q,\lambda}(H_r)} \lesssim \left(sh\frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}$$

for any $p > 1$ *and all* $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$, *and*

$$\left\| A_{chx}^\lambda M_G^\alpha f \right\|_{WL_{q,\lambda}(H_r)} \lesssim \left(sh\frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}$$

for any $p = 1$ *and all* $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$.

Proof. Denote,

$$A := |H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_s} A_{cht}^\lambda \, |f(chx)| \, sh^{2\lambda} t dt \right),$$

$$B := \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)}.$$

Using the Hölder's inequality and (2.1), we get

$$A \lesssim |H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{\frac{1}{q}} \, |H_s|_\lambda^{\frac{\alpha}{\gamma}-1} \left(\int_{H_s} A_{cht}^\lambda \, |f(chx)|^p \, sh^{2\lambda} t dt \right)^{\frac{1}{p}} \right)$$

$$= |H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{-\frac{1}{q}} \right) \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \approx \left(sh\frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}.$$

On the other hand

$$|H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{-\frac{1}{q}} \left(\int_{H_s} A_{cht}^\lambda \, |f(chx)|^p \, sh^{2\lambda} t dt \right)^{\frac{1}{p}} \right)$$

$$\gtrsim |H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{-\frac{1}{q}} \right) \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)} \approx B.$$

By Lemma 2.7

$$\left\| A_{chx}^\lambda M_G^\alpha f \right\|_{L_{q,\lambda}(H_r)} \lesssim A + B.$$

From this it follows the approval of lemma.

3 Boundedness of fractional maximal operator M_G^α on the space $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$

In the following theorem (see [24, Theorem 3.2]) the Spanne-Guliyev result was obtained.

Theorem 3.1. *Suppose* $1 \le p < \infty$, $\omega_1 \in \Omega_p^\gamma$, $\omega_2 \in \Omega_q^\gamma$. *Moreover, let the conditions (2.19) holds, also the pair* (ω_1, ω_2) *satisfy the condition*

$$\sup_{s>r} \left(sh\frac{s}{2} \right)^\alpha \omega_1(x,s) \lesssim \omega_2(x,r). \tag{3.1}$$

Then M_G^α *is bounded from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ *for* $p > 1$ *and* $\mathcal{M}_G^\alpha$ *is bounded from* $\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ *for* $p = 1$.

Proof. Denote

$$E_\gamma = \begin{cases} (0,2), & if \ \gamma = 2\lambda + 1 \\ [2,\infty), & if \ \ \gamma = 4\lambda. \end{cases}$$

Then by Lemma 2.8, we have

$$\|M_G^\alpha f\|_{\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)} = \sup_{\substack{x \in \mathbb{R}_+ \\ r \in E_\gamma}} \omega_2(x,r)^{-1} \left(sh\frac{r}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G^\alpha \right\|_{L_{q,\gamma}(H_r)}$$

$$\lesssim \sup_{\substack{x\in\mathbb{R}_+\\ r\in E_\gamma}} \omega_2(x,r)^{-1} \sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left\| A^\lambda_{chx} f \right\|_{L_{p,\lambda}(H_s)}$$

$$\lesssim \sup_{\substack{x\in\mathbb{R}_+\\ r\in E_\gamma}} \omega_2(x,r)^{-1} \sup_{s>r} \left(sh\frac{s}{2}\right)^{\frac{\gamma}{p}-\frac{\gamma}{q}} \omega_1(x,s) \left(\frac{\omega_1(x,s)^{-1} \left\| A^\lambda_{chx} f \right\|_{L_{p,\lambda}(H_s)}}{\left(sh\frac{s}{2}\right)^{\frac{\gamma}{p}}} \right)$$

$$\lesssim \|f\|_{\mathcal{M}_{q,\gamma,\omega_1}(\mathbb{R}_+)} \sup_{x,r\in\mathbb{R}_+} \omega_2(x,r)^{-1} \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha} \omega_1(x,s)$$

$$\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)}, \text{ if } \ p\in(1,\infty).$$

If $p=1$, then

$$\|M^\alpha_G f\|_{W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)} \lesssim \sup_{\substack{x\in\mathbb{R}_+\\ r\in E_\gamma}} \omega_2(x,r)^{-1} \left(\sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left\| A^\lambda_{chx} f \right\|_{L_{1,\lambda}(H_s)} \right)$$

$$\lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)} \sup_{x,r\in\mathbb{R}_+} \omega_2(x,r)^{-1} \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha} \omega_1(x,s) \lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)}.$$

In the case, then $\alpha=0$ and $p=q$ from Theorem 3.1, we get the following corollary.

Corollary 3.2. *Let $1\le p<\infty$ and (ω_1,ω_2) satisfy the condition*

$$\sup_{s>r} \omega_1(x,s) \lesssim \omega_2(x,r). \tag{3.2}$$

Then M_G is bounded from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p>1$ and M_G is bounded from $\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p=1$.

Later we will need the following lemma.

Lemma 3.3. *The following inequality $t\le sht\le e^A t$, $A>0$ is true for any $t\in[0,A]\subset\mathbb{R}_+$.*

Proof. Consider the function $f(t)=sht-t$. From the inequality $f'(t)=cht-1\ge 0$ it follows that $f(t)$ is increasing on $\mathbb{R}_+$. Therefore, the functions $f(t)$ takes smallest value when $t=0$, and $f(t)=0$, hence, $f(t)\ge 0 \Leftrightarrow sht\ge t$. We will prove the right side of the inequality

$$sht\le e^A t \Longleftrightarrow \frac{e^t-e^{-t}}{2}\le e^A t \Longleftrightarrow e^{2t}\le 2e^A te^t+1.$$

We will find the minimum of the function $\varphi(t)=2e^Ate^t+1-e^{2t}$.

$$\varphi'(t)=2e^A(e^t+te^t)-2e^{2t}=2e^t(e^A+e^At-e^t)\ge 0 \Longleftrightarrow e^A(t+1)\ge e^t,$$

hence $\varphi(t)$ takes the smallest value at the point $t=0$ and $\varphi(0)=0$. Then $\varphi(t)\ge 0 \Longleftrightarrow 2e^Ate^t+1\ge e^{2t} \Longleftrightarrow sht\le e^At$, for any $A>0$.

Lemma 3.4. *Let $H_0=(0,r_0)$* then the estimate

$$\left(sh\frac{r_0}{2}\right)^{\alpha} \lesssim M^\alpha_G \chi_{H_0}(chx)$$

is true for any $x\in\mathbb{R}_+$.

Proof. We will choose c_0 so large, that the inequality $r \le c_0 r_0$ would hold. Then by Lemma 3.3, when $0 \le t < c_0$, the inequality $t \le sht \le e^{c_0}t$ is true and by according to (2.1), we get

$$|H_r|_\lambda^{1-\frac{\alpha}{\gamma}} = \left(\int_0^r sh^{2\lambda}tdt\right)^{1-\frac{\alpha}{\gamma}} \le \left(\int_0^{c_0 r_0} sh^{2\lambda}tdt\right)^{1-\frac{\alpha}{\gamma}}$$

$$\approx \left(sh\frac{c_0r_0}{2}\right)^{\gamma-\alpha} \lesssim \left(e^{c_0}\frac{c_0r_0}{2}\right)^{\gamma-\alpha} \lesssim (c_0e^{c_0})^{\gamma-\alpha}\left(sh\frac{r_0}{2}\right)^{\gamma-\alpha}.$$

On the other hand,

$$\int_{H_r} A_{cht}^\lambda \chi_{H_0}(chx) sh^{2\lambda}tdt \ge \int_{H_r\cap H_0} A_{cht}^\lambda \chi_{H_0}(chx) sh^{2\lambda}tdt$$

$$\ge \int_{H_0} sh^{2\lambda}tdt \approx \left(sh\frac{r_0}{2}\right)^\gamma.$$

Thus,

$$M_G^\alpha \chi_{H_0}(chx) = \sup_{r>0} |H_r|^{\frac{\alpha}{\gamma}-1} \int_{H_r} A_{cht}^\lambda \chi_{H_0}(chx) sh^{2\lambda}tdt \gtrsim \left(sh\frac{r_0}{2}\right)^\alpha.$$

The following theorem is an analogue of Theorem 4.3 from [25] and this is one of the main results of this paper, (Gunawan-Guliyev type result) (see [13, Theorem 2.3]).

Theorem 3.5. *Let* $0 \le \alpha < \gamma, p,\ q \in [1,\infty),\ \omega_1 \in \Omega_p^\gamma,\ \omega_2 \in \Omega_q^\gamma$.

(i) *If the conditions (2.19) holds and* $1 \le p < \frac{\gamma}{\alpha}$, *then the condition (3.1) is sufficient for the boundedness* M_G^α *from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ *for* $p = 1$. *Moreover, if* $1 < p < \frac{\gamma}{\alpha}$, *then the condition (3.1) is sufficient for the boundedness* M_G^α *from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

(ii) *If the function* $\omega_1 \in \Phi_p^\gamma$, *then the condition*

$$\left(sh\frac{r}{2}\right)^\alpha \omega_1(r) \lesssim \omega_2(r),\ r > 0 \tag{3.3}$$

is necessary for the boundedness M_G^α *from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ *for* $p = 1$, *and* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ *for* $p > 1$.

(iii) *Let the condition (2.19) holds and* $1 \le p < \frac{\gamma}{\alpha}$. *If* $\omega_1 \in \Omega_p^\gamma$, *then condition (3.3) is necessary and sufficient for the boundedness of* M_G^α *from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ *for* $p = 1$. *Moreover, if* $1 < p < \frac{\gamma}{\alpha}$, *then the condition (3.3) is necessary and sufficient for the boundedness* M_G^α *from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

Proof. The first part of the theorem follows from Theorem 3.1 when $s = r$, $\omega_1(x,r) = \omega_1(r)$ and $\omega_2(x,r) = \omega_2(r)$.

We will prove the second part of the theorem.

Let $H_0 = (0,r_0)$ and $x \in H_0$. By lemma 3.4, we have $\left(sh\frac{r}{2}\right)^\alpha \lesssim M_G^\alpha \chi_{H_0}(chx)$. Therefore, by Lemma 2.4 and 3.4, we obtain

$$\left(sh\frac{r_0}{2}\right)^\alpha \lesssim |H_0|_\lambda^{-\frac{1}{q}} \|M_G^\alpha \chi_{H_s}\|_{L_{q,\lambda}(H_0)} \lesssim \omega_2(r_0) \|M_G^\alpha \chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)}$$

$$\lesssim \omega_2(r_0) \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)} \lesssim \frac{\omega_2(r_0)}{\omega_1(r_0)}$$

or

$$\left(sh\frac{r_0}{2}\right)^{\alpha} \lesssim \frac{\omega_2(r_0)}{\omega_1(r_0)} \Leftrightarrow \left(sh\frac{r_0}{2}\right)^{\alpha} \omega_1(r_0) \lesssim \omega_2(r_0)$$

when $r_0 > 0$.

Since, this is true for any the $r_0 > 0$, then the second part is proved.

The third part follows from the first and second parts of the theorem.

Remark 3.6. *If we take* $\omega_1(r) = \left(sh\frac{r}{2}\right)^{\frac{\nu-\gamma}{p}}$ *and* $\omega_2(r) = \left(sh\frac{r}{2}\right)^{\frac{\mu-\gamma}{q}}$ *at Theorem 3.5 then the condition (3.3) are equivalent to* $0 < \nu < \gamma - \alpha p$ *and* $\frac{\nu}{p} = \frac{\mu}{q}$ *respectively.*

Therefore, we get the following analogue of Theorem C .

Corollary 3.7. Let $0 \le \alpha < \gamma$, $1 \le p < \frac{\gamma}{\alpha}$, $0 < \nu < \gamma - \alpha p$ and decides:

$$\text{(a) Let } \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma - \nu},\ \gamma = 2\lambda + 1, \text{ if } r \in (0,2),$$
$$\text{(b) Let } \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma - \nu}, \quad \gamma = 4\lambda, \text{ if } r \in [2,\infty). \tag{3.4}$$

Then for $p = 1$ M_G^{α} is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $WL_{q,\lambda,\mu}(\mathbb{R}_+)$ if and only if $\frac{\nu}{p} = \frac{\mu}{q}$.

Moreover, for $p > 1$ M_G^{α} is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$ if and only if $\frac{\nu}{p} = \frac{\mu}{q}$.

4 Adams-Guliyev and Adams-Gunavan type result

The following statement is an analogue of Theorem 4.5 from [25] and plays an important role in the proofs of our main results.

Theorem 4.1. *Let* $1 \le p < \infty$, $0 \le \alpha < \gamma$ *and* $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$. *Then the following inequality*

$$M_G^{\alpha} f(chx) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(chx) + \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left\| A_{chx}^{\lambda} f \right\|_{L_{p,\lambda}(H_s)}. \tag{4.1}$$

is true for all $x \in \mathbb{R}_+$.

Proof. Let $1 \le p < \infty$ and $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$. We split the function f as $f = f_1 + f_2$, where $f_1(chx) = f\chi_{(H_r)}(chx)$ and $f_2(chx) = f\chi_{(H_r)^c}(chx)$. Then

$$M_G^{\alpha} f(chx) \le M_G^{\alpha} f_1(chx) + M_G^{\alpha} f_2(chx).$$

First let's chow

$$M_G^{\alpha} f_1(chx) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(chx). \tag{4.2}$$

Valid according to the formula (2.1), we can write

$$\int_0^r A_{cht}^{\lambda} |f_1(chx)|\, sh^{2\lambda} t dt \lesssim \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} \frac{A_{cht}^{\lambda} |f(chx)| \left(sh\frac{t}{2}\right)^{\gamma-\alpha} sh^{2\lambda} t}{\left(sh\frac{t}{2}\right)^{\gamma-\alpha}} dt$$

$$\lesssim \sum_{k=0}^{\infty} \left(sh\frac{r}{2^{k+1}}\right)^{\gamma-\alpha} \int_{2^{-k-1}r}^{2^{-k}r} \frac{A_{cht}^{\lambda} |f(chx)|\, sh^{2\lambda} t}{\left(sh\frac{t}{2}\right)^{\gamma-\alpha}} dt$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\gamma-\alpha} J_G^\alpha f(chx) \approx |H_r|_\lambda^{1-\frac{\alpha}{\gamma}} J_G^\alpha f(chx). \tag{4.3}$$

From (4.3), we have

$$M_G^\alpha f(chx) \lesssim J_G^\alpha f(chx). \tag{4.4}$$

We will consider $J_G^\alpha f(chx)$.

$$J_G^\alpha f(chx) \lesssim \left(\int_0^r + \int_r^\infty\right) \frac{A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t}{\left(sh\frac{t}{2}\right)^{\gamma-\alpha}} dt = J_1 + J_2. \tag{4.5}$$

Let's estimate J_1. Applying Lemma 3.3, we obtain

$$J_1 = \sum_{k=0}^\infty \int_{2^{-k-1}r}^{2^{-k}r} \frac{A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t}{(sh\frac{t}{2})^{\gamma-\alpha}} dt \lesssim \sum_{k=0}^\infty \frac{\left(sh\frac{r}{2^{k+1}}\right)^\alpha}{\left(sh\frac{r}{2^{k+2}}\right)^\gamma} \int_0^{2^{-k}r} A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t dt$$

$$\lesssim 2^\gamma \left(sh\frac{r}{2}\right)^\alpha \sum_{k=0}^\infty \frac{2^{-k\alpha}}{\left(\frac{r}{2^{k+1}}\right)^\gamma} \int_0^{2^{-k}r} A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t dt$$

$$\lesssim \left(sh\frac{r}{2}\right)^\alpha M_G f(chx) \sum_{k=0}^\infty 2^{-k\alpha} \lesssim \left(sh\frac{r}{2}\right)^\alpha M_G f(chx). \tag{4.6}$$

We will estimate J_2. By Hölder's inequality, we have

$$J_2 = \int_r^\infty \frac{A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t}{\left(sh\frac{t}{2}\right)^{\gamma-\alpha}} dt$$

$$\lesssim \left(\int_r^\infty \frac{A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t}{\left(sh\frac{t}{2}\right)^{(\gamma-\alpha)p'}} dt\right)^{\frac{1}{p'}} \left(\int_r^\infty \frac{A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t}{\left(sh\frac{t}{2}\right)^{(\gamma-\alpha)p}} dt\right)^{\frac{1}{p}} = J_{21}^{\frac{1}{p'}} J_{22}^{\frac{1}{p}}. \tag{4.7}$$

Consider J_{21}. Again use the Lemma 3.3, we get

$$J_{21} = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \frac{A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t}{\left(sh\frac{t}{2}\right)^{(\gamma-\alpha)q}} dt$$

$$\lesssim \sum_{k=0}^\infty \frac{\left(sh 2^k \frac{r}{2}\right)^{\gamma-(\gamma-\alpha)q}}{(sh 2^k r)^\gamma} \int_0^{2^{k+1}r} A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t dt$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\gamma-(\gamma-\alpha)p'} M_G f(chx) \sum_{k=0}^\infty \left(2^k\right)^{\gamma-(\gamma-\alpha)q} \lesssim \left(sh\frac{r}{2}\right)^{\gamma-(\gamma-\alpha)p'} M_G f(chx), \tag{4.8}$$

We will similarly the following estimate for J_{22}

$$J_{22} \lesssim \left(sh\frac{r}{2}\right)^{\gamma-(\gamma-\alpha)p} M_G f(chx). \tag{4.9}$$

Then from (4.8) and (4.9), we have

$$J_{2k} \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(chx), \quad k=1,2.$$

From this and (4.2) follows that

$$J_2 \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(chx), \;\; 0 < sh\frac{r}{2} < 1. \tag{4.10}$$

If $sh\frac{r}{2} \geq 1$, then by using (4.8) and (4.9) in (4.7), we obtain

$$J_2 \lesssim \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}-\gamma+\alpha+\frac{\gamma}{p}-\gamma+\alpha} M_G f(chx) = \left(sh\frac{r}{2}\right)^{2\alpha-\gamma} M_G f(chx)$$

$$= \frac{\left(sh\frac{r}{2}\right)^{\alpha}}{\left(sh\frac{r}{2}\right)^{\gamma-\alpha}} M_G f(chx) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(chx). \tag{4.11}$$

Thus, from (4.10) and (4.11), we have

$$J_2 \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(chx). \tag{4.12}$$

Using (4.6) and (4.12), we obtain (4.2). If $H_s \cap (H_r)^c \neq \emptyset$ then $s > r$ and by (2.9) and (2.16), we have

$$M_G^{\alpha} f_2(chx) \lesssim \sup_{s>r} |H_r|_{\lambda}^{\frac{\alpha}{\gamma}-1} \int\limits_{H_\lambda} A^{\lambda}_{cht} \, |f(chx)| \, sh^{2\lambda} t dt$$

(we used Hölder's inequality)

$$\lesssim \sup_{s>r} |H_s|_{\lambda}^{\frac{\alpha}{\gamma}-1} \left(\int\limits_{H_s} A^{\lambda}_{cht} \, |f(chx)|^p \, sh^{2\lambda} t dt \right)^{\frac{1}{p}} \left(\int\limits_{H_s} sh^{2\lambda} t dt \right)^{\frac{1}{q}}$$

$$= \sup_{s>r} \frac{|H_s|_{\lambda}^{\frac{1}{q}}}{|H_s|_{\lambda}^{1-\frac{\alpha}{\gamma}}} \left\| A^{\lambda}_{chx} f \right\|_{L_{p,\lambda}(H_s)} = \sup_{s>r} \frac{|H_s|_{\lambda}^{\frac{\alpha}{\gamma}}}{|H_s|_{\lambda}^{\frac{1}{p}}} \left\| A^{\lambda}_{chx} f \right\|_{L_{p,\lambda}(H_s)}$$

$$\approx \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left\| A^{\lambda}_{chx} f \right\|_{L_{p,\lambda}(H_s)}, \tag{4.13}$$

At the end we used (2.1).
We get (4.1) by bringing together (4.2) and (4.13).

In the following theorem we obtain Adams-Guliyev type result (see [24, Theorem 3.3]).
Theorem 4.2. *Let* $1 \leq p < q < \infty, 0 < \alpha < \frac{\gamma}{p}, \omega \in \Omega_p^{\gamma}$ *and satisfy the conditions:*

$$\sup_{s>r} \omega(x,s)^{\frac{1}{p}} \lesssim \omega(x,r)^{\frac{1}{p}} \tag{4.14}$$

and

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha} \omega(x,s)^{\frac{1}{p}} \lesssim \left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}} \tag{4.15}$$

where $x, r \in \mathbb{R}_+$.

The operator M_G^{α} *is bounded from* $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ *when* $p > 1$ *and from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ *when* $p = 1$.

Proof. Let $1 \leq p < q < \infty, 0 < \alpha < \frac{\gamma}{p}$ and $f \in \mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$. Let's put $f = f_1 + f_2$, where $f_1 = f\chi_{(H_r)}$, $f_2 = f\chi_{(H_r)^c}$.

Then

$$M_G^\alpha f(chx) = M_G^\alpha f_1(chx) + M_G^\alpha f_2(chx).$$

For $M_G^\alpha f_1$ estimate (4.2) is valid. For $M_G^\alpha f_2$ by Hölder's inequality and (2.1), we have

$$M_G^\alpha f_2(chx) \lesssim \sup_{s>r} |H_s|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_s} A_{cht}^\lambda |f(chx)| \, sh^{2\lambda} t dt$$

$$\lesssim \sup_{s>r} \left(sh\frac{s}{2}\right)^{\frac{\gamma}{q}+\alpha-\gamma} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}. \tag{4.16}$$

Then from (4.16), also from (4.2) and (4.15), we obtain

$$M_G^\alpha f(chx) \lesssim \left(sh\frac{r}{2}\right)^\alpha M_G f(chx) + \sup_{s>r} \left(sh\frac{s}{2}\right)^{\frac{\gamma}{q}+\alpha-\gamma} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}$$

$$= \left(sh\frac{r}{2}\right)^\alpha M_G^\alpha f(chx) + \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \omega(x,s)^{\frac{1}{p}} \frac{\left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}}{\omega(x,s)^{\frac{1}{p}}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^\alpha M_G f(chx) + \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)} \sup_{s>r} \left(sh\frac{s}{2}\right)^\alpha \omega(x,s)^{\frac{1}{p}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^\alpha M_G f(chx) + \left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}.$$

Shoosing $sh\frac{r}{2} = \left(\frac{\|f\|_{M_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}}{M_G f(chx)} \right)^{\frac{q-p}{\alpha q}}$, we get

$$|M_G^\alpha f(chx)| \lesssim (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}^{1-\frac{p}{q}}.$$

for any $x \in \mathbb{R}_+$.

If we take $\omega_1 = \omega_2 = \omega^{\frac{1}{p}}$ into account in condition (3.2) then conditions (4.14) and (3.2) will be equivalent and the statement of this theorem will follow from the boundedness of maximal operator M_G on the space $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$, which was proved in Corollary 3.2.

Indeed,

$$\|M_G^\alpha f\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)} = \sup_{x,s\in\mathbb{R}_+} \omega(x,s)^{-\frac{1}{q}} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G^\alpha f \right\|_{L_{q,\lambda}(H_s)}$$

$$\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x,s\in\mathbb{R}_+} \omega(x,s)^{-\frac{1}{q}} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G f \right\|_{L_{p,\lambda}(H_s)}^{\frac{p}{q}}$$

$$\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}} \left(\sup_{x,s\in\mathbb{R}_+} \omega(x,s)^{-\frac{1}{p}} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{p}} \left\| A_{chx}^\lambda M_G f \right\|_{L_{p,\lambda}(H_s)} \right)^{\frac{p}{q}}$$

$$\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|M_G f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}^{\frac{p}{q}} \lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}},$$

if $1<p<q<\infty$ and

$$\|M_G^\alpha f\|_{W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}} = \sup_{x,s\in\mathbb{R}_+} \omega(x,s)^{-\frac{1}{q}} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left\|A^\lambda_{chx} M_G^\alpha f\right\|_{WL_{q,\lambda}(H_s)}$$

$$\lesssim \|f\|^{1-\frac{1}{q}}_{\mathcal{M}_{1,\gamma,\omega}} \sup_{x,s\in\mathbb{R}_+} \omega(x,s)^{-\frac{1}{q}} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left\|A^\lambda_{chx} M_G f\right\|^{\frac{1}{q}}_{WL_{1,\lambda}(H_s)}$$

$$\lesssim \|f\|^{1-\frac{1}{q}}_{\mathcal{M}_{1,\gamma,\omega}} \left(\sup_{x,s\in\mathbb{R}_+} \omega(x,s)^{-1} \left(sh\frac{s}{2}\right)^{-\gamma} \left\|A^\lambda_{chx} M_G f\right\|_{WL_{1,\lambda}(H_s)}\right)^{\frac{1}{q}}$$

$$\lesssim \|f\|^{1-\frac{1}{q}}_{\mathcal{M}_{1,\gamma,\omega}} \|M_G f\|^{\frac{1}{q}}_{\mathcal{M}_{1,\gamma,\omega}} \lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega}},$$

if $1<q<\infty$.

In case $\omega(x,r)=\left(sh\frac{r}{2}\right)^{\nu-\gamma}$, $0<\nu<\gamma$ from Theorem 4.2 for fractional maximal operator M_G^α, we obtain the following Adams type result [1] .

Corollary 4.3. *Let $0<\alpha<\gamma, 1\le p<\frac{\gamma}{\alpha}$, $0<\nu<\gamma-\alpha p$ and conditions (3.4) holds. Then when $p>1$ operator M_G^α is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$ and when $p=1$operator M_G^α is bounded from $L_{1,\lambda,\nu}(\mathbb{R}_+)$ to $WL_{q,\lambda,\nu}(\mathbb{R}_+)$.*

From this in particular when $\gamma=2\lambda+1$ we get Theorem G.

Theorem 4.4. *Let $0<\alpha<\gamma$, $1\le p<q<\infty$ and $\omega\in\Omega_p^\gamma$.*

(i) *If $\omega(x,r)$ satisfies the condition (3.2), then (4.15) is a sufficient condition for the boundedness of M_G^α from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$.*

If $1<p<q<\infty$, then the condition (4.15) is a sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

(ii) *If $\omega\in\Phi_p^\gamma$, then the condition*

$$\left(sh\frac{r}{2}\right)^{\alpha}\omega(r)\lesssim\left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}}, r>0 \tag{4.17}$$

in necessary for the boundedness M_G^α from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$, and from $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ to $M_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$ if $1<p<q<\infty$.

(iii) *If $\omega\in\Phi_p^\gamma$, then the condition (4.17) in necessary and sufficient for the boundedness M_G^α from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$.*

Moreover, if $1<p<q<\infty$, then the condition (4.17) is necessary and sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. The first part of the theorem follows from Theorem 4.2 when $\omega(x,r)=\omega^p(r)$. We will now prove the second part of this theorem. Let $H_0=(0,r_0)$ and $x\in H_0$. By Lemma 3.4 $\left(sh\frac{r_0}{2}\right)^\alpha\lesssim M_G^\alpha\chi_{H_0}(chx)$. Moreover, by Lemma 2.4 and Lemma 3.4, we obtain

$$\left(sh\frac{r_0}{2}\right)^\alpha \lesssim |H_0|_\lambda^{-\frac{1}{q}} \|M_G^\alpha\chi_{H_0}\|_{L_{q,\lambda}(H_0)} \lesssim \omega(r_0)^{\frac{p}{q}} \|M_G^\alpha\chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)}$$

$$\lesssim \omega(r_0)^{\frac{p}{q}} \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} \lesssim \omega(r_0)^{\frac{p}{q}-1}$$

or

$$\left(sh\frac{r_0}{2}\right)^\alpha \omega(r_0)^{1-\frac{p}{q}} \lesssim 1 \Leftrightarrow \omega(r_0)\lesssim\left(sh\frac{r_0}{2}\right)^{-\frac{\alpha p}{q-p}}$$

$$\Leftrightarrow \left(sh\frac{r_0}{2}\right)^{\alpha}\omega(r_0) \lesssim \left(sh\frac{r_0}{2}\right)^{-\frac{\alpha p}{q-p}}.$$

The third part of this theorem follows from the first and second parts of theorem.

The following theorem is an analogue of Adams -Gunawan type result, (see [25, Theorem 4.8]).

Theorem 4.5. *Let* $0<\alpha<\gamma,\ 1\le p<q<\infty,\ \omega\in\Omega_p^{\gamma}$ *and satisfy (3.2), and*

$$\left(sh\frac{r}{2}\right)^{\alpha}\omega(x,r)+\sup_{s>r}\left(sh\frac{s}{2}\right)^{\alpha}\omega(x,s)\lesssim\omega(x,r)^{\frac{p}{q}},r>0. \tag{4.18}$$

Then M_G^{α} *is bounded from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\lambda,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$ *when* $p>1$ *and from* $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ *when* $p=1$.

Proof. Let $1\le p<\infty$ and $f\in\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$. By Theorem 4.1, we have (4.1). Then from (4.18), we get

$$\begin{aligned}M_G^{\alpha}f(chx)&\lesssim\left(sh\frac{r}{2}\right)^{\alpha}M_Gf(chx)+\sup_{s>r}\left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}}\left\|A_{chx}^{\lambda}f\right\|_{L_{p,\lambda}(H_s)}\\&\lesssim\left(sh\frac{r}{2}\right)^{\alpha}M_Gf(chx)+\sup_{s>r}\left(sh\frac{s}{2}\right)^{\alpha}\omega(x,s)\,\|f\|_{\mathcal{M}_{p,\lambda,\omega}}\\&\lesssim\min\left\{\omega(x,r)^{\frac{p}{q}-1}M_Gf(chx),\omega(x,r)^{\frac{p}{q}}\|f\|_{M_{p,\gamma,\omega}}\right\}.\end{aligned}$$

If $\omega(x,r)^{\frac{p}{q}-1}M_Gf(chx)\ \le\ \omega(x,r)^{\frac{p}{q}}\|f\|_{\mathcal{M}_{p,\gamma,\omega}}$, then $\omega(x,r)\ \ge\ M_Gf(chx)\|f\|^{-1}_{\mathcal{M}_{p,\gamma,\omega}}$ and, we have

$$\begin{aligned}M_G^{\alpha}f(chx)&\lesssim\omega(x,r)^{\frac{p}{q}-1}M_Gf(chx)\\&\lesssim M_Gf(chx)\left(\frac{\|f\|_{\mathcal{M}_{p,\lambda,\omega}}}{M_Gf(chx)}\right)^{1-\frac{p}{q}}\lesssim(M_Gf(chx))^{\frac{p}{q}}\,\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}}.\end{aligned}$$

If $\omega(x,r)^{\frac{p}{q}-1}M_G^{\alpha}f(chx)\ge\omega(x,r)^{\frac{p}{q}}\,\|f\|_{\mathcal{M}_{p,\gamma,\omega}}$ then $\omega(x,r)\le M_Gf(chx)\|f\|^{-1}_{\mathcal{M}_{p,\gamma,\omega}}$ and we have

$$M_G^{\alpha}f(chx)\lesssim\omega(x,r)^{\frac{p}{q}}\,\|f\|_{\mathcal{M}_{p,\gamma,\omega}}\lesssim\left(\frac{M_Gf(chx)}{\|f\|_{\mathcal{M}_{p,\gamma,\omega}}}\right)^{\frac{p}{q}}\|f\|_{\mathcal{M}_{p,\gamma,\omega}}\lesssim(M_Gf(chx))^{\frac{p}{q}}\,\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}}.$$

Thus, we obtain

$$M_G^{\alpha}f(chx)\lesssim(M_Gf(chx))^{\frac{p}{q}}\,\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}}. \tag{4.19}$$

From Corollary 3.2 and (4.19), we have

$$\begin{aligned}\|M_G^{\alpha}f\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}}&\lesssim\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}}\left\|(M_Gf)^{\frac{p}{q}}\right\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{p}{q}}}}\\&\lesssim\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}}\,\|M_Gf\|^{\frac{p}{q}}_{\mathcal{M}_{q,\gamma,\omega}}\lesssim\|f\|_{\mathcal{M}_{p,\gamma,\omega}},\end{aligned}$$

if $1<p<q<\infty$ and

$$\|M_G^{\alpha}f\|_{W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}}\lesssim\|f\|^{1-\frac{1}{q}}_{\mathcal{M}_{1,\gamma,\omega}}\,\|M_Gf\|^{\frac{1}{q}}_{\mathcal{M}_{1,\gamma,\omega}}\lesssim\|f\|_{\mathcal{M}_{1,\gamma,\omega}}$$

if $p=1$.

The following result is an analogue of Theorem 4.9 from [25].

Theorem 4.6. *Let* $0 < \alpha < \gamma,\ \ 1 \le p < q < \infty$, *and* $\omega \in \Omega_p^\gamma$.

(i) *If* $\omega(x,r)$ *satisfies the condition (4.2), then the condition (4.18) is sufficient for the boundedness* M_G^α *from* $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$.

Moreover, if $p > 1$, *then the condition (4.18) is sufficient for the boundedness* M_G^α *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

(ii) *If* $\omega \in \Phi_p^\gamma$, *then the condition*

$$\left(sh\frac{r}{2}\right)^\alpha \omega(r) \lesssim \omega(r)^{\frac{p}{q}},\ r > 0 \tag{4.20}$$

is necessary for the boundedness M_G^α *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$ *if* $p > 1$, *also from* $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ *if* $p = 1$.

(iii) *If* $\omega \in \Phi_p^\gamma$ *then the condition (4.20) is necessary and sufficient for the boundedness* M_G^α *from* $\mathcal{M}_{1,\lambda,\omega}(\mathbb{R}_+)$ *to* $W\mathcal{M}_{q,\lambda,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$.

Moreover, if $p > 1$ *the condition (4.20) is necessary and sufficient for the boundedness* M_G^α *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. The first part of this theorem follows from Theorem 4.5.

We will prove the second part. Let $H_0 = (0, r_0)$ and $x \in H_0$. By Lemma 2.4 and Lemma 3.3, we get

$$\left(sh\frac{r_0}{2}\right)^\alpha \lesssim |H_0|_\lambda^{-\frac{1}{q}} \|M_G^\alpha \chi_{H_0}\|_{L_{q,\lambda}(H_0)}$$

$$\lesssim \omega(r_0)^{\frac{p}{q}} \|M_G^\alpha \chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)} \lesssim \omega(r_0)^{\frac{p}{q}} \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} \lesssim \omega(r_0)^{\frac{p}{q}-1}$$

or

$$\left(sh\frac{r_0}{2}\right)^\alpha \omega(r_0)^{1-\frac{p}{q}} \lesssim 1 \Leftrightarrow \left(sh\frac{r_0}{2}\right)^\alpha \omega(r_0) \lesssim \omega(r_0)^{\frac{p}{q}}, r_0 > 0.$$

Since, this true for any $x \in \mathbb{R}_+$ and $r_0 > 0$, then our statement is proved.

The third part follows from the first and second parts of the theorem.

5 Fractional maximal commutator on generalized *G*- Morrey spaces $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$

In this section we characterise the boundedness of fractional maximal commutator on generalized G-Morrey spaces $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$.

The result obtained are analoques of relevant Spanne-Guliyev results, which were obtained in [24,25] for fractional maximal operator

$$M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha}{Q}-1} \int_{B(x,r)} |f(y)|dy, \quad 0 \le \alpha < Q,$$

and k-th $(k = 1, 2, ...)$ order fractional maximal commutator

$$M_{b,\alpha,k} f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha}{Q}-1} \int_{B(x,r)} |b(x) - b(y)|^k |f(y)|dy,$$

where $B(x,r)$ is an open ball centered at the point x with radius $r > 0$, and $|B(x,r)|$ is its Lebesgue measure.

51 Spanne-Guliyev type result

Firstly we will consider the BMO_G Gegenbauer space, which was introduced in [31].

Definition 5.1. *Denote by $BMO_G(\mathbb{R}_+)$ the Gegenbauer- BMO space (G-BMO space) as the set of locally integrable functions on $\mathbb{R}_+ = (0,\infty)$ such that*

$$\|f\|_{BMO_G(\mathbb{R}_+)} = \sup_{x,r\in\mathbb{R}_+} \frac{1}{|H_r|} \int\limits_{H_r} \left| A^\lambda_{chy} f(chx) - f_{H_r}(chx) \right| sh^{2\lambda} y dy < \infty,$$

where

$$f_{H_r}(chx) = \frac{1}{|H_r|} \int\limits_{H_r} A^\lambda_{chy} f(chx) sh^{2\lambda} y dy$$

and $H_r = (0,r)$.

According to the definition

$$BMO_G(\mathbb{R}_+) := \left\{ f \in L^{loc}_{1,\lambda}(\mathbb{R}_+) : \|f\|_{BMO_G(R_+)} < \infty \right\}.$$

In [31] for the operator M^α_G commutator k-th ($k = 1, 2, \ldots$) order is defined as follows:

$$M^{b,\alpha,k}_G f(chx) = \sup_{r>0} |H_r|^{\frac{\alpha}{\gamma}-1} \int\limits_{H_r} \left| A^\lambda_{chy} f(chx) - b_{H_r}(chx) \right|^k A^\lambda_{chy} |f(chx)| sh^{2\lambda} y dy,$$

where $b \in L^{loc}_{1,\lambda}(\mathbb{R}_+)$ and $0 < \alpha < \gamma$.

We will later need the following preliminary statement.

Lemma 5.2. *Let $1 < p < \infty$, $0 < \alpha < \frac{\gamma}{p}$, (2.19) holds and $b \in BMO_G(\mathbb{R}_+)$. Then for any $f \in L^{loc}_{1,\lambda}(\mathbb{R}_+)$ and any interval $H_r = (0,r)$, $r > 0$ the following inequality*

$$\left\| A^\lambda_{chx} M^{b,\alpha,k}_G \right\|_{L_{q,\lambda}(H_r)} \lesssim \left(sh\frac{r}{2} \right)^{\frac{\gamma}{q}} \|b\|^k_{BMO_G(R_+)} \sup_{s>r} \left(sh\frac{s}{2} \right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}} \right)^{k\gamma} \right) \left\| A^\lambda_{chx} f \right\|_{L_{p,\lambda}(H_s)}$$

is true.

Proof. Write $f = f_1 + f_2$ where $f_1 = f\chi_{H_r}$ and $f_2 = f\chi_{(H_r)^c}$ where χ_{H_s} -is a characteristic function of the set H_s. Then we obtain

$$\left\| A^\lambda_{chx} M^{b,\alpha,k}_G f \right\|_{L_{q,\lambda}(H_r)} \le \left\| A^\lambda_{chx} M^{b,\alpha,k}_G f_1 \right\|_{L_{q,\lambda}(H_r)} + \left\| A^\lambda_{chx} M^{b,\alpha,k}_G f_2 \right\|_{L_{q,\lambda}(H_r)}. \quad (5.1)$$

By the boundedness of $M^{b,\alpha,k}_G$ from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$ (see [31, Theorem 4.1]), we have (see prof of Lemma 2.6)

$$\left\| A^\lambda_{chx} M^{b,\alpha,k}_G f_1 \right\|_{L_{q,\lambda}(H_r)} \lesssim \|b\|^k_{BMO_G} \left\| A^\lambda_{chx} f_1 \right\|_{L_{p,\lambda}(H_r)} \lesssim \|b\|^k_{BMO_G} \left\| A^\lambda_{chx} f \right\|_{L_{p,\lambda}(H_r)}.$$

Let $x \in \mathbb{R}_+$ and $H_s \cap (H_r)^c \neq \emptyset$ then $s > r$ and we have (see the proof of Lemma 2.6)

$$M^{b,\alpha,k}_G f(chx) \lesssim \sup_{s>r} |H_r|^{\frac{\alpha}{\gamma}-1} \int\limits_{H_s} \left| A^\lambda_{chy} b(chx) - b_{H_s}(chx) \right|^k A^\lambda_{chy} |f(chx)| sh^{2\lambda} y dy. \quad (5.2)$$

From (5.2), Lemma 2.5, we obtain

$$\left\|A_{chx}^{\lambda} M_G^{b,\alpha,k} f\right\|_{L_{q,\lambda}(H_r)} \leq \left\|M_G^{b,\alpha,k} f_2\right\|_{L_{q,\lambda}(H_r)}$$

$$\lesssim \left\{\int\limits_{H_r} \left[\sup_{s>r} |H_s|^{\frac{\alpha}{\gamma}-1} \int\limits_{H_s} \left|A_{chy}^{\lambda} b(chx) - b_{H_s}(chx)\right|^k A_{chy}^{\lambda} |f(chx)| \, sh^{2\lambda} y dy\right]^q sh^{2\lambda} x dx\right\}^{\frac{1}{q}}$$

$$\lesssim \left\{\int\limits_{H_r} \left[\sup_{s>r} |H_s|^{\frac{\alpha}{\gamma}-1} \int\limits_{H_s} \left|A_{chy}^{\lambda} b(chx) - b_{H_r}(chx)\right|^k A_{chy}^{\lambda} |f(chx)| \, sh^{2\lambda} y dy\right]^q sh^{2\lambda} x dx\right\}^{\frac{1}{q}}$$

$$+ \left\{\int\limits_{H_r} \left[\sup_{s>r} |H_s|^{\frac{\alpha}{\gamma}-1} \int\limits_{H_s} |b_{H_r}(chx) - b_{H_s}(chx)|^k A_{chy}^{\lambda} |f(chx)| \, sh^{2\lambda} y dy\right]^q sh^{2\lambda} x dx\right\}^{\frac{1}{q}}$$

$$= A_1 + A_2. \tag{5.3}$$

We will estimate A_1. By applying Holder's inequality and considering the following relations (see [31, Corollary 2.8])

$$\sup_{x,r\in\mathbb{R}_+} \left(\frac{1}{|H_s|_\lambda} \int\limits_{H_s} \left|A_{chy}^{\lambda} f(chx) - b_{H_s}(chx)\right|^p sh^{2\lambda} y dy\right)^{\frac{1}{p}} \approx \|f\|_{BMO_G}, \tag{5.4}$$

we have

$$\int\limits_{H_s} \left|A_{chy}^{\lambda} b(chx) - b_{H_s}(chx)\right|^k A_{chy}^{\lambda} |f(chx)| \, sh^{2\lambda} y dy$$

$$\lesssim \left(\int\limits_{H_s} \left|A_{chy}^{\lambda} b(chx) - b_{H_s}(chx)\right|^{kp'} sh^{2\lambda} y dy\right)^{\frac{1}{p'}} \left(\int\limits_{H_s} A_{chy}^{\lambda} |f(chx)|^p \, sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\lesssim |H_s|^{\frac{1}{p'}} \|b\|_{BMO_G}^k \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_s)}.$$

From this, and (2.1), we obtain

$$A_1 \leq \|b\|_{BMO_G}^k |H_r|^{\frac{1}{q}} \sup_{s>r} |H_s|^{\frac{\alpha}{\gamma}-\frac{1}{p}} \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_s)}$$

$$\approx \|b\|_{BMO}^k \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_s)}. \tag{5.5}$$

For estimate A_2, again using Holder's inequality and the inequality (see [31])

$$|f_{H_1}(chx) - b_{H_2}(chx)| \leq \frac{|H_2|_\lambda}{|H_1|_\lambda} \|f\|_{BMO_G}, H_1 \subset H_2,$$

and (2.1), we get

$$A_2 \lesssim \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} |b_{H_s}(chx) - b_{H_r}(chx)|^k \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_s)}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}} \|b\|^k_{BMO_G} \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left(\frac{|H_s|_\lambda}{|H_r|_\lambda}\right)^k \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_s)}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}} \|b\|^k_{BMO_G} \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma} \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_s)}. \tag{5.6}$$

Summing (5.4) and (5.5), we obtain

$$\left\|A^\lambda_{chx} M^{b,\alpha,k}_G f_2\right\|_{L_{q,\lambda}(H_r)}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}} \|b\|^k_{BMO_G} \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_s)}$$

$$= \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}} \|b\|^k_{BMO_G} \sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_s)}.$$

Finally from (5.1), we have

$$\left\|A^\lambda_{chx} M^{b,\alpha,k}_G f\right\|_{L_{q,\lambda}(H_r)} \lesssim \|b\|^k_{BMO_G} \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_r)}$$

$$+ \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}} \|b\|^k_{BMO_G} \sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_s)}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{\gamma}{q}} \|b\|^k_{BMO_G} \sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_s)}.$$

Lemma is proved.

The following theorem is Spanne-Guliyev type result (see [24, Theorem 4.1]).

Theorem 5.3. *Let $1 < p < \infty, 0 \le \alpha < \frac{\gamma}{p}$, (2.19) holds, $b \in BMO_G(\mathbb{R}_+)$ and the pair (ω_1, ω_2) satisfy the condition*

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \omega_1(x,s) \lesssim \omega_2(x,r). \tag{5.7}$$

Then the operator $M^{b,\alpha,k}_G$ is bounded from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

Moreover,

$$\left\|M^{b,\alpha,k}_G f\right\|_{\mathcal{M}_{q,\gamma,\omega_2}} \lesssim \|b\|^k_{BMO_G} \|f\|_{\mathcal{M}_{p,\gamma,\omega_1}}.$$

is true.

Proof. By Lemma 5.2 we can write

$$\left\|M^{b,\alpha,k}_G f\right\|_{\mathcal{M}_{q,\gamma,\omega_2}} = \sup_{x,r\in\mathbb{R}_+} \omega_2(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}} \left\|A^\lambda_{chx} M^{b,\alpha,k}_G\right\|_{L_{q,\lambda}(H_r)}$$

$$\lesssim \|b\|^k_{BMO_G} \sup_{x,r\in\mathbb{R}_+} \omega_2(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}} \sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_s)}$$

$$\lesssim \|b\|^k_{BMO_G} \sup_{x,r\in\mathbb{R}_+} \omega_1(x,r)^{-1} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}} \left\|A^\lambda_{chx} f\right\|_{L_{p,\lambda}(H_r)} \lesssim \|f\|^k_{BMO_G} \|f\|_{\mathcal{M}_{p,\gamma,\omega_1}}.$$

Theorem is proved.

In the case when $\alpha = 0$ and $p = q$ we get the following result.

Corollary 5.4. *Let* $1 < p < \infty$, $b \in BMO_G(\mathbb{R}_+)$ *and the pair* (ω_1, ω_2) *satisfy the condition*

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \omega_1(x,s) \lesssim \omega_2(x,r). \tag{5.8}$$

Then the operator $M_G^{b,k}$ *is bounded from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

Moreover, we have

$$\left\|M_G^{b,k} f\right\|_{\mathcal{M}_{q,\gamma,\omega_2}} \lesssim \|b\|_{BMO_G}^k \|f\|_{\mathcal{M}_{p,\gamma,\omega_1}}.$$

52 Adams-Guliyev type result

The following theorem is Adams-Guliyev type result (see [24, Theorem 4.2]).

Theorem 5.5. *Let* $1 < p < q < \infty, 0 < \alpha < \frac{\gamma}{p},\ \ b \in BMO_G(\mathbb{R}_+)$ *and* $\omega(x,r)$ *satisfy the conditions:*

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{-\gamma} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right)^p \omega(x,s) \lesssim \omega(x,r), \tag{5.9}$$

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \omega(x,s)^{\frac{1}{p}} \lesssim \left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}}. \tag{5.10}$$

Then the operator $M_G^{b,\alpha,k}$ *is bounded from* $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$.

Proof. Let $1 < p < q < \infty$,$0 < \alpha < \frac{\gamma}{p}$ and $f \in \mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$. Let $f = f_1 + f_2$ where $f_1 = f\chi_{H_r}$ and $f_2 = f\chi_{(H_r)^c}$.

By doing the same as in Section 5.1, for $p \in (1, \infty)$, we can write

$$M_G^{b,\alpha,k} f_2(chx) \lesssim \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\gamma} \int_{H_s} \left|A_{chy}^{\lambda} b(chx) - b_{H_s}(chx)\right|^k A_{chy}^{\lambda} |f_2(chx)|\, sh^{2\lambda} y dy$$

$$\lesssim \|b\|_{BMO_G}^k \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_s)}. \tag{5.11}$$

From (4.2) follows that

$$M_G^{b,\alpha,k} f_1(chx) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G^{b,k} f(chx). \tag{5.12}$$

From (5.9) - (5.11) we obtain

$$M_G^{b,\alpha,k} f(chx) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G^{b,k} f(chx)$$

$$+ \|b\|_{BMO_G}^k \sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \left\|A_{chx}^{\lambda} f\right\|_{L_{p,\lambda}(H_s)} \tag{5.13}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G^{b,k} f(chx) + \|b\|_{BMO_G}^k \sup_{s>r} \omega(x,s)^{\frac{1}{p}} \left(sh\frac{s}{2}\right)^{\alpha} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{k\gamma}\right) \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G^{b,k} f(chx) + \|b\|_{BMO_G}^{k} \left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}. \tag{5.14}$$

Soosing

$$sh\frac{r}{2} = \left(\frac{\|b\|_{BMO_G}^{k} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}}{M_G^{b,k} f(chx)} \right)^{\frac{q-p}{\alpha q}},$$

for any $x \in \mathbb{R}_+$, we have

$$M_G^{b,\alpha,k} f(chx) \lesssim \|b\|_{BMO_G}^{k\left(1-\frac{p}{q}\right)} \left(M_G^{b,k} f(chx)\right) \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}^{1-\frac{p}{q}}.$$

Now the statement of the theorem follows from the the boundedness of commutator $M_G^{b,k}$ on $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ which was proved in Corollary 5.4 by condition (5.8). Indeed,

$$\left\|M_G^{b,\alpha,k} f\right\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}} = \sup_{x,r\in\mathbb{R}_+} \omega(x,r)^{-\frac{1}{q}} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}} \left\|A_{chx}^{\lambda} M_G^{b,\alpha,k} f\right\|_{L_{q,\lambda}(H_r)}$$

$$\lesssim \|b\|_{BMO_G}^{k\left(1-\frac{p}{q}\right)} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x,r\in\mathbb{R}_+} \omega(x,r)^{-\frac{1}{q}} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}} \left\|A_{chx}^{\lambda} M_G^{b,k} f\right\|_{L_{p,\lambda}(H_r)}$$

$$\lesssim \|b\|_{BMO_G}^{k\left(1-\frac{p}{q}\right)} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left(\sup_{x,r\in\mathbb{R}_+} \omega(x,r)^{-\frac{1}{p}} \left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}} \left\|A_{chx}^{\lambda} M_G^{b,k} f\right\|_{L_{p,\lambda}(H_r)} \right)^{\frac{p}{q}}$$

$$\lesssim \|b\|_{BMO_G}^{k\left(1-\frac{p}{q}\right)} \left\|M_G^{b,k} f\right\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}^{1-\frac{p}{q}} \lesssim \|b\|_{BMO_G}^{k\left(1-\frac{p}{q}\right)} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}}.$$

In case when $\omega(x,r) = \left(sh\frac{r}{2}\right)^{\nu-\gamma}, 0 < \nu < \gamma$, from the Theorem 5.5 follows Adams-Guliyev type result for commutator of Gegenbauer fractional maximal operator M_G^{α}.

Corollary 5.6. *Let* $0 < \alpha < \gamma,\ 1 < p < \frac{\gamma}{\alpha},\ 0 < \nu < \gamma - \alpha p$, *condition (3.4) holds and* $b \in BMO_G(\mathbb{R}_+)$. *Then the operator* $M_G^{b,\alpha,k}$ *is bounded from* $L_{p,\lambda}(\mathbb{R}_+)$ *to* $L_{q,\lambda}(\mathbb{R}_+)$.

We will use the following estimates to prove the main results.

Lemma 5.7. Let $b \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$ and $H_0 = (0, r_0) \subset H_r = (0, r)$. Then the estimate

$$\|b\|_{BMO_G} \left(sh\frac{r_0}{2}\right)^{\alpha} \lesssim M_G^{b,\alpha} \chi_{H_0}(chx)$$

is true for any $x \in H_0$.

Proof. We choose c_0 large enough to satisfy the inequality $r \le c_0 r_0$. Then by Lemma 3.3 when $0 \le t < c_0$ the inequality $t \le sht \le e^{c_0} t$ is true and by (2.1), we get (see proof of Lemma 3.4)

$$|H_r|_{\lambda}^{1-\frac{\alpha}{\gamma}} \lesssim |H_0|_{\lambda}^{1-\frac{\alpha}{\gamma}}. \tag{5.15}$$

From the definition and the estimate (5.14) and (2.1), we get

$$M_G^{b,\alpha} \chi_{H_0}(chx) = \sup_{r>0} |H_r|_{\lambda}^{\frac{\alpha}{\gamma}-1} \int_{H_r} \left|A_{chy}^{\lambda} b(chx) - b_{H_r}(chx)\right| A_{chy}^{\lambda} \chi_{H_0}(chx) sh^{2\lambda} y dy$$

$$\geq |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int\limits_{H_r\cap H_0} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| sh^{2\lambda} ydy$$

$$\gtrsim |H_0|_\lambda^{\frac{\alpha}{\gamma}-1} \int\limits_{H_0} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| sh^{2\lambda} ydy \approx \left(sh\frac{r_0}{2}\right)^\alpha \|b\|_{BMO}.$$

The following theorem is an analogue of Theorem 5.5 from [25] and one of the main results of this paper.

Theorem 5.8. *Let* $b \in BMO_G(\mathbb{R}_+) \setminus \{\text{const}\}, p, q \in [1,\infty), 0 \leq \alpha < \gamma, \omega_1 \in \Omega_p^\gamma$ *and* $\omega_2 \in \Omega_q^\gamma$.

(i) *Let* $1 < p < \frac{\gamma}{\alpha}$, *and (2.19) holds, then the condition*

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^\gamma\right) \omega_1(x,s) \lesssim \omega_2(x,r), r > 0$$

is sufficient for the boundedness $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

(ii) *If* $\omega_1 \in \Phi_p^\gamma$, *then the condition*

$$\left(sh\frac{r}{2}\right)^\alpha \omega_1(r) \lesssim \omega_2(r), \quad s > 0 \tag{5.16}$$

is necessary for the boundedness $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$

(iii) *Let* $1 < p < \frac{\gamma}{\alpha}$ *and (2.19) holds. If* $\omega_1 \in \Phi_p^\gamma$ *safisfies the condition*

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^\gamma\right) \omega_1(s) \lesssim \left(sh\frac{r}{2}\right)^\alpha \omega_1(r),$$

for any $r > 0$, *then the condition (5.15) is necessary and sufficient for the boundedness* $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

Proof. The first part of our theorem follows from Theorem 5.3 when $k = 1$.

We will prove the second part of this theorem. Note that $\|b\|_{BMO_G}$ is a seminorm and $\|b\|_{BMO_G} = 0$ is and only if b is a constant almost everywhere (see [31, §2]). Therefore, if $b \in BMO_G(\mathbb{R}_+) \setminus \{const\}$ then $\|b\|_{BMO_G} > 0$. For any $r > 0$ there exists a point $x_0 \in H_0 \subset H_r$ shuch that $\left\| A_{chy}^\lambda b - b_{H_0} \right\|_{L_{1,\lambda}(H_0)} > 0$ but in other cases it is a constant .

Let $b \in BMO_G(\mathbb{R}_+) \setminus \{const\}$ then by Hölder's inequality, we get

$$\left\| A_{chy}^\lambda b - b_{H_0} \right\|_{L_{1,\lambda}(H_0)} = \int\limits_0^{r_0} A_{chy}^\lambda \left| b_{H_r}(chx) - b_{H_0}(chx) \right| sh^{2\lambda} tdt$$

$$\leq \left\| A_{chy}^\lambda b - b_{H_0} \right\|_{L_{q,\lambda}(H_0)} \left(\int\limits_0^{r_0} sh^{2\lambda} tdt \right)^{\frac{1}{q}} = |H_0|_\lambda^{\frac{1}{q}} \left\| A_{chy}^\lambda b - b_{H_0} \right\|_{L_{q,\lambda}(H_0)}.$$

Thus $\left\| A_{chy}^\lambda b - b_{H_0} \right\|_{L_{q,\lambda}(H_0)} \geq |H_0|_\lambda^{\frac{1}{q}} \|b\|_{BMO_G}$. Using inequality $A_{chy}^\lambda 1 = 1$ and Lemma 2.4, we can write

$$\left(sh\frac{r_0}{2}\right)^\alpha \lesssim \frac{\left\| A_{chy}^\lambda M_G^{b,\alpha} \chi_{H_0} \right\|_{L_{q,\lambda}(H_0)}}{\left\| A_{chy}^\lambda b \right\|_{BMO_G} |H_0|_\lambda^{\frac{1}{q}}} \lesssim \left\| M_G^{b,\alpha} \chi_{H_0} \right\|_{L_{q,\lambda}(H_0)} |H_0|_\lambda^{-\frac{1}{q}}$$

$$\lesssim \omega_2(r_0) \left\| M_G^{b,\alpha} \chi_{H_0} \right\|_{\mathcal{M}_{q,\gamma,\omega_2}} \lesssim \omega_2(r_0) \left\| \chi_{H_0} \right\|_{\mathcal{M}_{p,\gamma,\omega_1}} \lesssim \frac{\omega_2(r_0)}{\omega_1(r_0)}.$$

The third statement of the theorem follows from the first and the second parts.

53 Adams-Guliyev and Adams-Gunavan type results

In this section we characterize the boundedness of the operator $M_G^{b,\alpha}$ on generalized G-Morrey spaces.

The following result is an analogue of Theorem 5.7 from [25].

Theorem 5.9. (Adams-Guliyev type results) *Let* $1 < p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $b \in BMO_G(\mathbb{R}_+)$ *and let* $\varphi \in \Omega_p^\gamma$ *satisfy the conditions:*

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{-\frac{\gamma}{p}} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^\gamma\right) \varphi(x,s) \lesssim \varphi(x,r), \tag{5.17}$$

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^\gamma\right) \varphi(x,s) \lesssim \left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}}. \tag{5.18}$$

Then the operator $M_G^{b,\alpha}$ *is bounded from* $\mathcal{M}_{p,\gamma,\varphi}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\varphi^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. We get the statement of this theorem if we take into account $\omega(x,r)^{\frac{1}{p}} = \varphi(x,r)$ in Theorem 5.5.

The following theorem is an analogue of the Theorem 5.8 from [25] and this is one the most important results of this paper.

Theorem 5.10. *Let* $1 < p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $b \in BMO_G(\mathbb{R}_+)$ *and* $\omega \in \Omega_p^\gamma$

(i) *If* $\omega(x,r)$ *satisfies the condition (5.16), then condition (5.17) is sufficient for the boundedness* $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

(ii) *If* $\omega \in \Phi_p^\gamma$, *then the condition*

$$\left(sh\frac{s}{2}\right)^{\alpha} \omega(r) \lesssim \left(sh\frac{r}{2}\right)^{-\frac{\alpha p}{q-p}} \tag{5.19}$$

is necessary for th boundedness $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

(iii) *If* $\omega \in \Phi_p^\gamma$ *satisfies the condition*

$$\sup_{s>r} \left(sh\frac{s}{2}\right)^{\alpha} \left(1 + \left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^\gamma\right) \omega(s) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} \omega(r), \tag{5.20}$$

for all $r > 0$, *then the conditions (5.18) necessary and sufficient for the on of* $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. The first part of the theorem follows from Theorem 5.6. We will prove the second part of this theorem. Let $b \in BMO_G(\mathbb{R}_+) \setminus \{const\}$, $H_0 = (0,r_0)$, $x \in H_0 \subset H_r = (0,r)$, then $\left\| A_{chy}^\lambda b - b_{H_0} \right\|_{L_{1,\lambda}(H_0)} > 0$.

By Lemma 5.7

$$\|b\|_{BMO_G} \left(sh\frac{r_0}{2}\right)^{\alpha} \lesssim M_G^{b,\alpha} \chi_{H_0}(chx).$$

Therefore, by Lemma 2.4 and relation (5.3) we have

$$\left(sh\frac{r_0}{2}\right)^{\alpha} \lesssim \left\|A^{\lambda}_{chy}M^{b,\alpha}_{G}\chi_{H_0}\right\|_{L_{q,\lambda}(H_0)} |H_0|_{\lambda}^{-\frac{1}{q}}$$

$$\lesssim \omega_r(r_0)^{\frac{p}{q}} \left\|M^{b,\alpha}_{G}\chi_{H_0}\right\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}} \lesssim \omega(r_0)^{\frac{p}{q}} \left\|\chi_{H_0}\right\|_{\mathcal{M}_{p,\gamma,\omega}} \lesssim \omega_r(r_0)^{\frac{p}{q}-1}$$

or

$$\left(sh\frac{r_0}{2}\right)^{\alpha}\omega(r_0)^{1-\frac{p}{q}} \lesssim 1 \Leftrightarrow \left(sh\frac{r_0}{2}\right)^{\alpha}\omega(r_0) \lesssim \left(sh\frac{r_0}{2}\right)^{-\frac{\alpha p}{q-p}},$$

for any $r_0 > 0$.

Since the last relation is true for every $r \in \mathbb{R}_+$, we are done.

The third statement of the theorem follows from first a second parts of the theorem.

The following result is an analogue of Theorem 5.9 in [25].

Theorem 5.11. (Adams-Gunavan type result). *Let* $1 < p < q < \infty,\ \ 0 < \alpha < \frac{\gamma}{p}$, $b \in BMO_G(\mathbb{R}_+)$ *and* $\omega(x,r)$ *satisfy condition (5.7) and*

$$\left(sh\frac{r}{2}\right)^{\alpha}\omega(x,r) + \sup_{s>r}\left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{\gamma}\right)\left(sh\frac{r}{2}\right)^{\alpha}\omega(x,s) \lesssim \omega(x,r)^{\frac{p}{q}}. \tag{5.21}$$

Then $M^{b,\alpha}_G$ *is bounded from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. Let $1 < p < q < \infty,\ \ 0 < \alpha < \frac{\gamma}{p}$ and $f \in \mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$. We write $f = f_1 + f_2$, where $f_1 = f\chi_{H_r}$ and $f_2 = f\chi_{(H_r)^c}$. From (5.12) when $k = 1$, we have

$$M^{b,\alpha}_G f(chx) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M^b_G f(chx)$$

$$+\|b\|_{BMO_G}\sup_{s>r}\left(sh\frac{s}{2}\right)^{\alpha-\frac{\gamma}{p}}\left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{\gamma}\right)\|f\|_{L_{p,\lambda}(H_s)}. \tag{5.22}$$

Then from (5.20) and (5.21) we get

$$M^{b,\alpha}_G f(chx) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M^b_G f(chx)$$

$$+\|b\|_{BMO_G}\|f\|_{\mathcal{M}_{p,\gamma,\omega}}\sup_{s>r}\left(sh\frac{s}{2}\right)^{\alpha}\left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{\gamma}\right)\omega(x,s)$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M^b_G f(chx) + \|b\|_{BMO_G}\|f\|_{\mathcal{M}_{p,\gamma,\omega}}\omega(x,r)^{\frac{p}{q}}$$

(see proof of Theorem 4.5)

$$\lesssim \min\left\{\omega(x,r)^{\frac{p}{q}-1}M^b_G f(chx), \omega(x,r)^{\frac{p}{q}}\|f\|_{\mathcal{M}_{p,\gamma,\omega}}\right\}$$

$$\lesssim \left(M^b_G f(chx)\right)^{\frac{p}{q}}\|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}}. \tag{5.23}$$

Taking in (5.7) $\omega_1(x,s) = \omega(x,s)\left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}}$, $\omega_2(x,r) = \omega(x,r)\left(sh\frac{r}{2}\right)^{-\frac{\gamma}{q}}$ and taking into account (5.22), obtain

$$\left\|M^{b,\alpha}_G f\right\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}} \lesssim \|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}}\left\|\left(M^b_G f\right)^{\frac{p}{q}}\right\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}}$$

$$\lesssim \|f\|^{1-\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}}\left\|M^b_G f\right\|^{\frac{p}{q}}_{\mathcal{M}_{p,\gamma,\omega}} \lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega}}.$$

The following theorem is an alalogue of the Theorem 5.10 in [25].

Theorem 5.12. *Let* $1 < p < q < \infty,\ \ 0 < \alpha < \gamma$, $b \in BMO_G(\mathbb{R}_+)\backslash \{const\}$ *and* $\omega \in \Omega_p^\gamma$.

(i) *If* $1 < p < \infty$ *and* $\omega(s)$ *satisfies the condition*

$$\sup_{s>r}\left(sh\frac{s}{2}\right)^{-\frac{\gamma}{p}}\left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{\gamma}\right)\omega_1(x,s)\lesssim\omega_2(x,r),\ r>0,$$

then the condition

$$\left(sh\frac{r}{2}\right)^{\alpha}\omega(r)+\sup_{s>r}\left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{\gamma}\right)\left(sh\frac{s}{2}\right)^{\alpha}\omega(s)\lesssim\omega(r)^{\frac{p}{q}},r>0$$

is sufficient for the boundedness $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

(ii) *If* $\omega \in \Phi_p^\gamma$, *then the condition*

$$\left(sh\frac{r}{2}\right)^{\alpha}\lesssim\omega(r)^{\frac{p}{q}-1} \tag{5.24}$$

is necessary for the boundedness $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

(iii) *Let* $1 < p < \infty$ *If* $\omega \in \Phi_p^\gamma$ *and satisfies the condition*

$$\sup_{s>r}\left(sh\frac{s}{2}\right)^{\alpha}\left(1+\left(\frac{sh\frac{s}{2}}{sh\frac{r}{2}}\right)^{\gamma}\right)\omega(s)\lesssim\left(sh\frac{r}{2}\right)^{\alpha}\omega(r)$$

for any $r > 0$, *then the condition (5.23) is necessary and sufficient for the boundedness* $M_G^{b,\alpha}$ *from* $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ *to* $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. The first part of the theorem follows from the Theorem 5.11.

We shall prove the second part. Let $b \in BMO_G(\mathbb{R}_+)\backslash\{const\}$, $H_0 = (0, r_0)$ and $x \in H_0 \subset H_r = (0, r)$, then $\left\|A^\lambda_{chy}b - b_{H_0}\right\|_{L_{1,\lambda}(H_0)} > 0$. By Lemma 5.7

$$\left\|A^\lambda_{chy}b-b_{H_0}\right\|_{L_{1,\lambda}(H_0)}\left(sh\frac{r_0}{2}\right)^{\alpha}\|b\|_{BMO_G}\lesssim M_G^{b,\alpha}\chi_{H_0}(chx).$$

Therefore, by Lemma 2.4 and (5.3), we have

$$\left(sh\frac{r_0}{2}\right)^{\alpha}\lesssim\left\|M_G^{b,\alpha}\chi_{H_0}\right\|_{L_{q,\lambda}(H_0)}|H_0|_\lambda^{-\frac{1}{q}}$$

$$\lesssim\omega(r_0)^{\frac{p}{q}}\left\|M_G^{b,\alpha}\chi_{H_0}\right\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}}\lesssim\omega(r_0)^{\frac{p}{q}}\left\|\chi_{H_0}\right\|_{\mathcal{M}_{p,\gamma,\omega}}\lesssim\omega(r_0)^{\frac{p}{q}-1}.$$

Since this is true for every $r_0 > 0$, we are done. The third statement of the theorem follows from the first and second parts of the theorem.

In the case, when $\omega(x,r) = \left(sh\frac{r}{2}\right)^{\frac{\nu-\gamma}{p}}$, $0 < \nu < \gamma$ from Theorem 5.12, we get the following Adams type result for the G -fractional maximal commutator $M_G^{b,\alpha}$.

Corollary 5.13. *Let* $b \in BMO_G(\mathbb{R}_+)\backslash\{const\}$, $0 < \alpha < \gamma$, $1 < p < q < \infty$ $0 < \nu < \gamma - \alpha p$. *Then* $M_G^{b,\alpha}$ *is bounded from* $L_{p,\lambda}(\mathbb{R}_+)$ *to* $L_{q,\lambda}(\mathbb{R}_+)$ *if and only if condition (3.4) holds.*

References

1. Adams, O.R.: *A note on Riesz potentials*, Duke Math. J. **42**,765-778 (1975).
2. Akbulut, A., Guliyev, V.S., Mustafayev, R.: *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces,* Math. Bohem. **137**(1), 27-43 (2012).
3. Burenkov, V.I., Gogatashvili, A., Guliyev, V.S., Mustafayev, R.Ch.: *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **35**(1), 67-87 (2011).
4. Burenkov, V.I., Guliyev, H.V.: *Necessary and sufficient condition for baundedness of the maximal operator in local Morrey-type spaces*, Studia Math. **163**(2), 157-176 (2004).
5. Burenkov, V.I., Guliyev, V.S.: *Necessary and sufficient condition of the Riesz potential in local Morrey-type spaces,* Potential Anal. **30**(3), 211-249 (2009).
6. Burenkov, V.I., Guliyev, V.S., Guliyev, H.V.: *Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces*, Dokl. Math. **75**(1), 103-107 (2007).
7. Burenkov, V.I., Guliyev, H.V., Guliyev, V.S.: *Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces*, J. Comput. Appl. Math. **208**(1), 280-301 (2007).
8. Burenkov, V.I., Gogatishvili, A., Guliyev, V.S.: *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **35**(1), 67-87 (2011).
9. Chiarenza, F., Frasca, M.: *Morrey spaces and Hardy-Littlewood maximal function,* Rend. Mat. Appl. **7**(3-4), 273-279 (1987).
10. Durand, L., Fishbane, P.M., Simmons, L.M.: *Expansion formulas and addition theorems for Gegenbauer functions,* J. Math. Phys. **17**(11), 1993-1948 (1976).
11. Eroglu, A., Guliyev, V.S., Azizov, C.V.: *Characterizations for the fractional integral operators in generalized Morrey spaces on Garnot groups*, Math. Notes, **102**(5), 127-139 (2017).
12. Eridani, H.G.: *On the boundedness of a generalized fractional integral on generalized Morrey spaces*, Tamkang J. Math. **33**(4), 335-340 (2002).
13. Eridani, Utoyo, M.I., Gunawan, H.: *A characterization for fractional integrals on generalized Morrey spaces*, Anal. Theory Appl. **28**(3), 263-268 (2012).
14. Di Fazio, V., Ragusa, M.A.: *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients,* J. Funct. Anal. **112**(2), 241-256 (1993).
15. Guliyev, V.S., Ibragimov, E.Dzh.: *Conditions for $L_{p,\lambda}$- Boundedness of the Reisz Potential Generated by the Gegenbauer Differential operator*, Mat. Zametki, **105**(5), 685-695 (2019).
16. Guliyev, V.S., Ibrahimov, E.J: *Necessary and sufficient conditions for the boundedness of the Gegenbauer-Riesz potential on Morrey spaces,* Georgian Math. J. **25**(2), 1-14 (2018).
17. Guliyev, V.S., Ibrahimov, E.J.: *Necessary and sufficient condition for the boundedness of the Gegenbauer-Riesz potential in modified Morrey Spaces,* Trans. A. Razmadze Math. Inst. **173**, 37-52 (2019).
18. Guliyev, V.S., Ibrahimov, E.J., Ekincioglu, E., Jafarova, S.Ar.: *Inequality of o Neil-type for convolutions associated with Geganbauer differential operator and applications*, J. Math. Study. **53**(1), 90-124 (2020).
19. Guliyev, V.S., Shukurov, P.S.: *On the boundedness of the fractional maximal operator, Riesz potential and their commutators in generalized Morrey spaces.* Advances in harmonic analysis and operator theory, Oper. Theory Adv. Appl. **229**, 175-199 (2013).
20. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948, 20 pp. (2009).

21. Guliyev, V.S.: *Sobolev's theorem for the anisotropic Riesz-Bessel potential in Morrey-Bessel spaces*, Dokl. Akad. Nauk. **367**(2), 155-156 (1999).
22. Guliyev, V.S., Hasanov, J.J.: *Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces*, J. Math. Anal. Appl. **347**(1), 113-122 (2008).
23. Guliyev, V.S., Omarova, M.N., Ragusa, M.A., Scapelatto, A.: *Commtautors and generalized local Morrey spaces*. J. Math. Anal. Appl. **457**(2), 1388-1402 (2018).
24. Guliyev, V.S., Akbulut, A., Mammadov, J.: *Boundedness of fractional maximal operator and their higner order commutator in generalized Morrey spaces on Carnot groups,* Acta Math. Sci. **33**(5), 1329-1346 (2013).
25. Guliyev, V., Ekincioglu, J., Kaya, I., Safarov, Z.: *Characterizations for the fraxtional maximal commutators operator in generalized Morrey spaces on Carnot groups,* Integral Transforms Spec. Funct. **30**(6), 453-470 (2019).
26. Gunawan, H.A.: *A note on the generalized fractional integral operators,* J. Indows. Math. Soc. **9**, 39-43 (2003).
27. Ibrahimov, E.J., Akbulut, A.: *The Hardy-Littlewood-Sobolev theorem for Riesz potential generated by Gegenbauer differential operator,* Trans. A. Razmadze Math. Inst. **170** (2), 166-199 (2016).
28. Ibrahimov, E.J., Guliyev, V.S., Jafarova, S.Ar.: *On weighted boundedness of fractional maximal operator and the Riesz-Gegenbauer potential generated by Gegenbauer differential operator,* Trans. A. Razmadze Math. Inst. **173**(3), 45-74 (2019).
29. Ibrahimov, E.J., Jafarova, S.Ar., Ekincioglu, S.E.: *Maximal and potential operators associated with Gegenbauer differential operator on generalized Morrey spaces,* Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **46**(1), 129-143 (2020).
30. Ibrahimov, E.J., Dadashova, G.A., Ekincioglu, S.E.: *On the boundedness of G- maximal operator and G-Riesz potential in the generalized Morrey spaces.* Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, **40**(1), 1-15 (2020).
31. Ibrahimov, E.J., Dadashova, G.A., Jafarova, S.Ar.: *Boundedness of higner order commutators of G-fractional integral and G- fractional maximal operator with G-BMO functions.* Trans. A. Razmadze Math. Inst. **474**(3), 325-341 (2020).
32. Ibrahimov, E.J.: *On Gegenbauer transformation on the half-line*, Georgion Math. J. **18**(3), 497-515 (2011).
33. Komori, Y., Mizuhara, T.: *Notes on commutators and Morrey spaces*, Hokkaido Math. J. **32**, 345-353 (2003).
34. Morrey, C.B.: *On the solution of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**(1), 126-166 (1938).
35. Meskhi, A., H. Rafeiza, Zaighum, M.A.: *Interpolation on variable Morrey spaces defined on quasi-metric measure spaces*, J. Funct. Anal. **270**(10), 3946-3961 (2016).
36. Nakai, E.: *Hardy-Littlewood maximal operator, singular integral operator and the Riesz potential on generalized Morrey spaces*, Math. Nachr. **166**, 95-105 (1994).
37. Peetre, J.: *On the theory $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal. **4**, 71-87 (1969).
38. Sawano, Y., Sugano, S., Tanako, H.: *Generalized fractional integral operators and fractional maximal operators in the frame work of Morrey spaces* , Trans. Amer. Math. Soc. **363**(12), 6481-6503 (2011).
39. Shirai, S.: *Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces*, Hokkaido Math. J. **35**, 683-696 (2006).
40. Softova, L.: *Singular integral and commutators in generalized Morrey space,* Acta Math. Sin. **22**(3), 757-766 (2006).