Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **44** (1), 117-127 (2024).
https://doi.org/10.30546/2617-7900.44.1.2024.117

About uniform analogues of strongly paracompact and Lindelöf spaces

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Received: 08.12.2023 / Revised: 15.03.2023 / Accepted: 12.05.2023

Abstract. *This paper introduces uniform analogues of strongly paracompact and Lindelöf spaces and investigates their connection with other uniform properties of compactness type, and also establishes the characterization of these classes by using finitely additive open covers, compactification and ω-mappings.*

Keywords. ω-mapping, strong uniform R-paracompactness, strong uniform B-paracompactness, strong uniform P-paracompactness, uniform R-Lindelöfness.

Mathematics Subject Classification (2010): 54D20, 54E15.

1 Introduction

As it is known the strong paracompactness and Lindelöfness play an important role in General Topology. Therefore, the search of uniform analogues of strong paracompactness and Lindelöfness is an important and interesting problem in the theory of uniform spaces. The uniformization problem of the topological spaces theory, i.e. finding and studying of uniform analogues of classes of topological spaces and continuous mappings is relevant. Important properties of the compactness type of topological spaces include paracompact, strongly paracompact, and Lindelöf spaces.There are various approaches to the definition of uniform paracompactness, strongly uniform paracompactness, and uniform Lindelöf property of uniform spaces. For example, uniformly R-paracompact [1], uniformly B-paracompact [2], uniformly F-paracompact [3], uniformly P-paracompact [4], strongly uniformly A-paracompact [5], uniformly B-Lindelöf [2], and uniformly I-Lindelöf spaces [6].

In the works [11]-[22] the most important properties of uniformly paracompact topological groups and metric spaces, uniformly para-Lindelöf, uniformly Hypocompact, countably

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uniformly paracompact, uniformly superparacompact, uniformly completely paracompact and strongly uniformly paracompact spaces were obtained.

In the works [23]-[30] paracompact-types of uniform spaces and uniformly continuous mappings were studied.

In this paper we study other variants of strongly uniformly paracompact and uniformly Lindelöf spaces, namely, strongly uniformly R-paracompact, strongly uniformly B-paracompact, strongly uniformly P-paracompact and uniformly R-Lindelöf spaces.

2 Preliminaries and Denotations

Throughout this paper all uniform spaces are assumed to be Hausdorff, mappings are uniformly continuous. For covers α and β of the set X , the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, for covers α and β of a set X , we have: $\alpha \wedge \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. The cover α finitely additive if $\alpha^{\angle} = \alpha$, $\alpha^{\angle} = \{\cup\alpha_0 : \alpha_0 \subset \alpha \text{ is finite}\}$, $\alpha(x) = \cup St(\alpha, x)$, $x \in X$, $\alpha(H) = \cup St(\alpha, H)$, $H \subset X$. A cover α of the uniform space (X, U) is called *uniformly star finite*, if there exists a uniform cover $\beta \in U$ such that every $\alpha(B)$ meets α only for a finite number of elements of α [2]; a uniform space (X, U) is called *uniformly R-paracompact*, if every open cover of (X, U) has a uniformly locally open refinement [1]; a uniform space (X, U) is called *uniformly B-paracompact*, if for each finitely additive open cover γ of (X, U) there exists sequence uniform cover $\{\alpha_i : i \in N\} \subset U$, such that following condition is fulfilled: for each point $x \in X$ there exist a number $i \in N$ and $\Gamma \in \gamma$ such that $\alpha_i(x) \subset \Gamma$ (BP) [2]; a uniform space (X, U) is called *uniformly P-paracompact*, if for each open cover γ of (X, U) there exists a sequence of uniform covers $\{\alpha_i : i \in N\} \subset U$ such that the condition (BP) is fulfilled [4]; a uniform space (X, U) is called *uniformly B-Lindelöf*, if it is both uniformly B-paracompact and $\aleph_0$-bounded [2]; a uniform space (X, U) is called *$\aleph_0$-bounded*, if the uniformity U has a base consisting of countable covers [2]; a uniform space (X, U) is called *uniformly locally Lindelöf*, if the uniformity U contains a uniform cover, the closure of each its element is Lindelöf [4]; a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called *precompact*, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and finite uniform cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [2]; a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called *uniformly perfect*, if it is both precompact and perfect [2]; a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called *uniformly open*, if f maps each open uniform cover $\alpha \in U$ to an open uniform cover $f\alpha \in V$ [2]. A uniformly continuous mapping $f : (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called *strongly uniformly open*, if for each $\alpha \in U$ there exists $\beta \in V$ such that $f(\alpha(x)) \supset \beta(f(x))$ for any $x \in X$ [2]. τ_U is the topology generated by the uniformity U and U_X denotes the universal uniformity.

3 Strongly uniformly paracompact and uniformly Lindelöf spaces

A uniform space (X, U) is called *strongly uniformly R-paracompact*, if every open cover of (X, U) has a uniformly star finite open refinement.

If a uniform space (X, U) is strongly uniformly R-paracompact, then the topological space (X, τ_U) is strong paracompact. Conversely, if a Tychonoff space (X, τ) is strongly paracompact, then the uniform space (X, U_X), where U_X is the universal uniformity, is strongly uniformly R-paracompact. Indeed, let α be an arbitrary open cover of the space (X, τ_U). Then there exists a uniformly star finite open cover β refines α. Since every uniformly star finite open cover is a star finite open cover, then the cover β is a star finite one.

Thus, the space (X, τ_U) is strongly paracompact; conversely, let space (X, τ) be strongly paracompact. Then the set of all open covers form the base of the universal uniformity U_X of the space (X, τ). It is evident that the uniform space (X, U_X) is strongly uniformly R-paracompact.

Theorem 3.1 *Let (X, U) be a uniform space, bX be its an arbitrary compactification. A uniform space (X, U) is strongly uniformly R-paracompact if and only if for any compactum $K \subset bX \backslash X$ there exists a uniformly star finite open cover α such that $[A]_{bX} \cap K = \varnothing$ for all $A \in \alpha$.*

Proof. *Necessity*. Let (X, U) be a strongly uniformly R-paracompact space and $K \subset bX \backslash X$ be an arbitrary compactum. Then for each point $x \in X$ there exists an open neighborhood O_x in bX such that $[O_x]_{bX} \cap K = \varnothing$. Assume $\beta = \{O_x \cap X : x \in X\}$. It is evident that β is the open cover of a space (X, U). A uniformly star finite open cover γ is a refinement of β. Then $[\Gamma]_{bX} \cap K = \varnothing$ for any $\Gamma \in \gamma$.

Sufficiency. Let α be an arbitrary open cover of a space (X, U). Then there exists open family β in bX such that $\beta \wedge \{X\} = \alpha$. We put $K = bX \backslash \cup \beta$. Then for compactum K there exists uniform star finite open cover γ such that $[\Gamma]_{bX} \cap K = \varnothing$ for any $\Gamma \in \gamma$. Since the set $[\Gamma]_{bX}$ is compactum there are $B_1, B_2, ..., B_n \in \beta$ such that $[\Gamma]_{bX} \subset \overset{n}{\underset{i=1}{\cup}} B_i$. Therefore $\Gamma \subset \overset{n}{\underset{i=1}{\cup}} A_i$, where $\overset{n}{\underset{i=1}{\cup}} A_i \in \alpha$. Hence, (X, U) is a strongly uniformly R-paracompact space.

The following theorem is an inner characterization for strongly uniformly R-paracompact spaces.

Theorem 3.2 *For uniform space (X, U) the following are equivalent:*

1) *(X, U) is strongly uniformly R-paracompact;*
2) *(X, U) is uniformly R-paracompact and topological space (X, τ_U) is strongly paracompact.*

Proof. $1) \Rightarrow 2)$. It is evident.

$2) \Rightarrow 1)$. Let α be an arbitrary open cover of a uniform space (X, U). Since the topological space (X, τ_U) is strongly paracompact, then there exists a star finite open cover β is a refinement of α.Moreover, (X, U) is uniformly R-paracompact there is a uniformly local finite open cover γ, that refines β. We show that β is a uniformly star finite cover. Since a cover γ is uniformly locally finite, then there exists a uniform cover $\lambda \in U$ such that $St(L, \gamma)$ is finite for any $L \in \lambda$. Let $L \in \lambda$ be an arbitrary element. Since $\gamma \succ \beta$, then for any $G \in St(L, \gamma)$ there exists $B \in \beta$ such that $G \subset B$. By the star finite of β we have that $St(B, \beta)$ is finite, and moreover $St(G, \beta)$ is finite for any $G \in St(L, \gamma)$. Hence, $St(L, \beta)$ is finite for any $L \in \lambda$. $\beta(L)$ meets only finite number of elements of the cover β, in that β is a star finite cover. Thus, a uniform space (X, U) is uniformly R-paracompact.

Theorem 3.3 *For locally compact space (X, U) the following are equivalent:*

1) *(X, U) is uniformly locally compact;*
2) *(X, U) is strongly uniformly R-paracompact.*

Proof. $1) \Rightarrow 2)$ follows from Theorem 3.2.

$2) \Rightarrow 1)$. From local compactness of a space (X, U) it follows that for each point $x \in X$ there exists an open neighborhood O_x such that $[O_x]$ is compact. A family $\alpha = \{O_x : x \in X\}$ forms an open cover of (X, U). A uniformly star finite open cover β refines α. Each $B \in \beta$ in some set $\overset{n}{\underset{i=1}{\cup}} O_{x_i}$, hence by monotonity of the closure operator we have

$[B] \subset [\bigcup_{i=1}^{n} O_{x_i}]$, it means that $[B]$ is compact. So, $[\beta] = \{[B] : B \in \beta\}$ is a uniform cover consisting of compact subsets. Therefore, (X, U) is uniformly locally compact.

Theorem 3.4 *The uniform space (X, U) is strongly uniformly R-paracompact if and only if every finitely open cover of (X, U) has a uniformly star finite open refinement.*

Proof. *Necessity*. Let (X, U) be a strongly uniformly R-paracompact space and α be an arbitrary finitely additive open cover. Then there exists a uniformly star finite open cover β of a space (X, U) such that $\beta \succ \alpha$. Hence, a space (X, U) is strongly uniformly R-paracompact.

Sufficiency. Suppose that uniformly star finite open cover can be refined in any finitely additive open cover of a space (X, U). Let us show that (X, U) is strongly uniformly R-paracompact space. Let α be an arbitrary open cover. By strongly uniformly R-paracompactness of (X, U) a uniformly star finite open cover β of (X, U) is a refinement of a finitely additive open cover $\alpha^{\angle}$. For each $B \in \beta$ we choose $A_B \in \alpha^{\angle}$ such that $B \subset A_B$, where $A_B = \bigcup_{i=1}^{n} A_i$, $A_i \in \alpha$, $i = 1, 2, ..., n$. Assume $\alpha_0 = \bigcup\{\alpha_B : B \in \beta\}$, $\alpha_B = \{B \cap A_i : i = 1, 2, ..., n\}$. Then α_0 is uniformly star finite open cover of the space (X, U) refines an open cover α. Hence, a space (X, U) is strongly uniformly R-paracompact.

Proposition 3.1 *Real line R with the natural uniformity is a strongly uniformly R-paracompact space. Space R^n with the natural uniformity is strongly uniformly R-paracompact.*

Proof. Let α be an arbitrary finitely additive open cover of R. Assume $\beta = \{(n-1, n+1) : n = 0, \pm 1, \pm 2, ...\}$. Then β is uniform star finite open cover of R, wherein $[n-1, n+1]$ is compactum. It means that there exists a finite family $\{A_1, A_2, ..., A_n\}$, $A_i \in \alpha$, $i = 1, 2, ..., n$ and $(n-1, n+1) \subset [n-1, n+1] \subset \bigcup_{i=1}^{n} A_i$. So, $\beta \succ \alpha^{\angle} = \alpha$. Hence, R with the natural uniformity is strongly uniformly R-paracompact space. The second part of Proposition is proved by analogue.

Infinite discrete uniform space (X, U_D), where U_D is a discrete uniformity of a cardinality $\tau \geq \aleph_0$, is also strongly uniformly R-paracompact, but it is not a compact uniform space.

Proposition 3.2 *Every strongly uniformly R-paracompact space is complete.*

Proof. It easy follows from that fact: Every uniformly R-paracompact space is complete.

Uniform space (X, U) is called *strongly uniformly P-paracompact*, if it is uniformly P-paracompact and its topological space (X, τ_U) is strongly paracompact space.

Uniform space (X, U) is called *strongly uniformly B-paracompact*, if it is uniformly B-paracompact and its topological space (X, τ_U) is a strongly paracompact space.

By the definitions of strong uniform R-paracompactness, strong uniform B-paracompactness and strong uniform P-paracompactness it follows that any strongly uniformly R-paracompact space is strongly uniformly B-paracompact. Any strongly uniformly P-paracompact space is strongly uniformly B-paracompact.

Proposition 3.3 *Every separably metrizable uniform space (X, U) is strongly uniformly P-paracompact (strongly uniformly B-paracompact).*

Proof. Let (X, U) be separably metrizable uniform space. Let α be an arbitrary (finitely additive) open cover of the space (X, U). Since the space (X, U) is metrizable there exists a countable base $\{\alpha_n\} \subset U$. Let $x \in X$ be an arbitrary point. Then there is $A \in \alpha$, such that $A \ni x$. Since A is open, then there exists a number $n \in N$ such that $\alpha_n(x) \subset A$.

Hence it follows that (X, U) is uniformly $P(B)$-paracompact. Since (X, U) is separable metrizable, then (X, τ_U) is Lindelöf, i.e. it is strongly paracompact. So, the uniform space (X, U) is strongly uniformly $P(B)$-paracompact.

In uniform topology the special interest is the question of identifying and studying those uniform properties such that for any finitely additive open cover have a uniformly continuous ω-mapping onto some metrizable space. A.A. Borubaev has studied this problem and in one of the seminars he posed the following analogous problem: "To find and investigate those uniform spaces such in which for any (finitely additive) open cover has a uniformly continuous ω-mapping onto some strongly paracompact metrizable space".

Theorem 3.5 *The uniform space (X, U) is strongly uniformly P-paracompact (strongly uniformly B-paracompact) if and only if for each (finitely additive) open cover ω of space (X, U) there exists a uniformly continuous ω-mapping f of the uniform space (X, U) onto some strongly paracompact metrizable uniform space (Y, V).*

Proof. *Necessity*. Let (X, U) be a strongly uniformly P-paracompact (strongly uniformly B-paracompact) space and ω is any (finitely additive) open cover of the space (X, U). Then for ω there exists a normal sequence of uniform covers $\{\alpha_n\}$, that satisfies (BP)-condition. For a normal sequence of uniform covers $\{\alpha_n\}$ there exists a pseudometrics ρ on X such that $\alpha_{n+1}(x) \subset \{y : \rho(x, y) < \frac{1}{2^{n+1}}\} \subset \alpha_n(x)$ for any $x \in X$ and $n \in N$. For any $x, y \in X$ $x \sim y$ if and only if $\rho(x, y) = 0$. Let Y be the quotient of the set X, $f : X \to Y$ be mapping of a set X into quotient set Y_ω. On the quotient set we define metric by the rule: for any two $y_1, y_2 \in Y$ assume $\rho(y_1, y_2) = d(f^{-1}y_1, f^{-1}y_2)$. It is clear, that the metric ρ induces uniformity V_ω on Y_ω and the mapping $f : (X, U) \to (Y_\omega, V_\omega)$ is uniformly continuous. Let $y \in Y$ be an arbitrary chosen point and x be any point of $f^{-1}y$. Then there exist $n \in N$ and $L \in \omega$ such that the star $\alpha_n(x)$ in L. We suppose $U_y = \{y \in Y : \sigma(y, y) < \frac{1}{2^{n+2}}\}$. Then $f^{-1}U_y \subset \{x \in X : \sigma(x, x) \leq \frac{1}{2^{n+1}}\} \subset \alpha_n(x) \subset L$. So, f is an ω-mapping.

Sufficiency. Let for every (finitely additive) open cover ω of the space (X, U) there exists uniformly continuous ω-mapping f of uniform space (X, U) onto some strongly paracompact metrizable uniform space (Y, V). We show that the uniform space (X, U) is strongly uniformly P-paracompact (strongly uniformly B-paracompact). Let ω be an arbitrary open (finitely additive) cover of the uniform space (X, U) and f be a uniformly continuous mapping of uniform space (X, U) to strongly paracompact metrizable uniform space (Y, V). Then there exists a sequence of uniform covers $\{\beta_n\}$ of (Y, V). Assume $\{\alpha_n\}$, where $\alpha_n = f^{-1}(\beta_n)$. Clearly that $\{\alpha_n\}$ is the sequence of uniform covers of (X, U). We show that for each point $x \in X$ there exists a number $n \in N$ and $W \in \omega$ such that $\alpha_n(x) \subset W$. Let $x \in X$ be an arbitrary point. Then there exists $U_x \in \omega$ such that $U_x \ni x$. Since the set U_x is open, then there exists a number $n \in N$ such that $\alpha_n(x) \subset U_x$.

And now we prove, that (X, τ_U) is a strongly paracompact space. It is sufficient to show that is a refinement of any finitely additive open cover. Since the mapping f is ω-mapping, then every point $y \in Y$ has a neighborhood U_y, whose preimage $f^{-1}U_y$ is contained in one element at least of the cover ω. Let $\{U_y : y \in Y\}$. Clearly, it is an open cover of the space (Y, τ_V), in which a star finite open cover β is a refinement. It is evident, that star finite open cover $f^{-1}\beta$ refines the cover ω, i.e. the space (X, τ_U) is strongly paracompact. Thus, the uniform space (X, U) is strongly uniformly P-paracompact (strongly uniformly B-paracompact).

This theorem is a uniform analogue of Dowker-Ponomarev-Fedorchuk-Shediva (Trnkova) Theorem [8].

A uniform space $I^\tau \times D_\tau$, $\tau > \aleph_0$, as the product of Tychonoff cube I^τ by a discrete uniform space D_τ, is strongly uniformly B-paracompact, but it is not strongly uniformly P-paracompact. Uniform space (Q, U_Q) of rational number with the natural uniformity U_Q is incomplete. So, it is not strongly uniformly R-paracompact, although its topological space (Q, τ_{U_Q}) is strongly paracompact.

Strong uniform R-paracompactness, strong uniform P-paracompactness and strong uniform B-paracompactness are inherented by taking closed subspaces and by disjoint sums of uniform spaces.

Proposition 3.4 *Any uniformly locally compact space is strongly uniformly R-paracompact (strongly uniformly B-paracompact).*

Proof. Let a uniform space (X, U) be uniformly locally compact. Then the topological space (X, τ_U) is locally compact and paracompact, i.e. strongly paracompact. By Theorem 3.2 the uniform space (X, U) is strongly uniformly R-paracompact (strongly uniformly B-paracompact) space.

Proposition 3.4 is a uniform analogue of the Arhangelskii Theorem (see [7]): Any locally compact group is strongly paracompact.

Proposition 3.5 *Any uniformly R-paracompact (uniformly P-paracompact, uniformly B-paracompact, respectively) space (X, U), its topological space is locally compact, is strongly uniformly R-paracompact (strongly uniformly P-paracompact, strongly uniformly B-paracompact, respectively).*

Proof. Let (X, U) be a uniformly R-paracompact (uniformly P-paracompact, uniformly B-paracompact, respectively) space and its topological space (X, τ_U) be locally compact. Then the space (X, τ_U) is paracompact. Hence, the topological space (X, τ_U) is strongly paracompact. Then by Theorem 3.2 the uniform space (X, U) is strongly uniformly R-paracompact (strongly uniformly P-paracompact, strongly uniformly B-paracompact, respectively).

Lemma 3.1 *Every (uniformly) perfect mapping $f : (X, U) \to (Y, V)$ between uniform spaces (X, U) and (Y, V) is a ω-mapping for any finitely additive open cover ω of the space (X, U).*

Proof. Let ω be an arbitrary open cover of the space (X, U). It is easy to see that a cover $\alpha = \{f^{-1}y : y \in Y\}$ refines the cover ω. For each $f^{-1}y \in \alpha$ we choose $W_y \in \omega$ such that $f^{-1}y \subset W_y$. Then since the mapping f is closed there exists a neighborhood $O_y \ni y$ such that $f^{-1}O_y \subset W_y$.

Lemma 3.2 *Let $f : (X, U) \to (Y, V)$ be uniformly continuous mapping between uniform spaces (X, U) and (Y, V). If β is uniformly star finite open cover of the space (Y, V), then $f^{-1}\beta$ is uniformly star finite open cover of the space (X, U).*

Proof. Let β be uniformly star finite open cover of the space (Y, V). Then by uniform continuity of a mapping f a cover $f^{-1}\beta$ is open in (X, U). Let $\gamma \in V$ be a uniform cover such that $\beta(\Gamma)$ meets only finite number of elements of β, i.e. $\beta(\Gamma) \subset \bigcup_{i=1}^{n} B_i$. Hence, $f^{-1}\beta(f^{-1}\Gamma) \subset \bigcup_{i=1}^{n} f^{-1}B_i$, where $f^{-1}B_i \in f^{-1}\beta$, $f^{-1}\Gamma \in f^{-1}\gamma$, $f^{-1}\gamma \in U$. It means, that $f^{-1}\beta$ is uniformly star finite open cover of the space (X, U).

Theorem 3.6 *Let $f : (X, U) \to (Y, V)$ be an ω-mapping between uniform spaces (X, U) and (Y, V). If the uniform space (Y, V) is strongly uniformly R-paracompact, then the space (X, U) is also strongly uniformly R-paracompact.*

Proof. Let ω be an arbitrary finitely additive open cover of the space (X, U). Then for each point $y \in Y$ there exists a neighborhood $O_y \ni y$ such that $O_y \subset W$ for some $W \in \omega$. Let $\beta = \{O_y : y \in Y\}$. Then there exists uniformly star finite open cover γ of the space (Y, V) such that $\gamma \succ \beta$. By Lemma 3.1 it follows that $f^{-1}\gamma$ is uniformly star finite open cover of the space (X, U). Evidently that $f^{-1}\gamma \succ \omega$. Hence, (X, U) is strongly uniformly R-paracompact.

Corollary 3.1 *Strong uniform R-paracompactness is preserved by preimages of perfect (uniformly perfect) mappings.*

Strong uniform B-paracompactness is not a uniform invariant on uniformly perfect mappings. But the following uniform analogue of Hanai Theorem [10] takes place.

Theorem 3.7 *The preimage of a strongly uniformly B-paracompact space under (uniformly) perfect mappings is strongly uniformly B-paracompact.*

Proof. Let $f : (X, U) \to (Y, V)$ be a perfect mapping of a uniform space (X, U) onto a strongly uniformly B-paracompact uniform space (Y, V) and ω be an arbitrary finitely additive open cover of uniform space (X, U). Clearly that cover $\{f^{-1}y : y \in Y\}$ is refined in cover ω. Since the mapping f is closed, it implies that $\beta = \{f^{\#}W : W \in \omega\}$ is finitely additive open cover of uniform space (Y, V), where a small image $f^{\#}W = Y \backslash f(X \backslash W)$. Since the space (Y, V) is strongly uniformly B-paracompact, then there exists a sequence of uniform covers $\{\beta_n\}$ satisfying the next condition: for each point $y \in Y$ there exists a number $n \in N$ and $f^{\#}W$ of β such that $\beta_n(y) \subset f^{\#}W$. It is easy to see that $f^{-1}\beta$ refines the cover ω. Let $x \in X$ be an arbitrary point. Then for point $y \in Y$ there exists a number $n \in N$ and $f^{\#}W$ of β such that $\beta_n(y) \subset f^{\#}W$. Hence, there exists a uniform cover α_n such that $f\alpha_n$ refines β_n It is easy to see that the star $\alpha_n(x)$ is contained in $f^{-1}(f\alpha_n(y))$, and the last is contained in $f^{-1}\beta_n(y)$ and $f^{-1}\beta_n(y) \subset W$. Taking into account that f is a perfect mapping, (X, τ_U) is a strong paracompact space. So uniform space (X, U) is strongly uniformly B-paracompact.

Corollary 3.2 *The preimage of strongly uniformly P-paracompact space under (uniformly) perfect mappings is strongly uniformly P-paracompact.*

The following theorem is a uniform analogue of V.I. Ponomarev Theorem [9] on the preservation of strong paracompactness by the image of open perfect mappings.

Theorem 3.8 *The image of a strongly uniformly B-paracompact space under a strongly uniformly open uniformly perfect mappings is strongly uniformly B-paracompact.*

Proof. Let a uniform space (X, U) be strongly uniformly B-paracompact. Let λ be any finitely additive open cover of uniform space (Y, V). For finitely additive open cover $\alpha = f^{-1}\lambda$ there exists a sequence of uniform covers $\{\alpha_i\}$ satisfying the condition: for any point $x \in X$ there exist index $i \in N$ and $A \in \alpha$ such that $\alpha_i(x) \subset A$. Since the mapping $f : (X, U) \to (Y, V)$ is strongly uniformly open, then for each uniform cover α_i there exists a uniform cover λ_i satisfying the condition: $f(\alpha_i(x)) \supset \lambda_i(f(x))$ for each point $x \in X$. Hence, it follows that for any point $y \in Y$ there exist $i \in N$ and $L \in \lambda$ such that $\lambda_i(x) \subset L$. So, the uniform space (X, U) is uniformly B-paracompact. As it is known ([9]) strong paracompactness is preserved by the image of open perfect mappings. Based on this fact, we conclude that topological space (Y, τ_V) is strongly paracompact. Thus, the image of (Y, V) is strongly uniformly B-paracompact.

Proposition 3.6 *Product $(X, U) \times (Y, V)$ of a strongly uniformly B-paracompact space (X, U) by a compact space (Y, V) is strongly uniformly B-paracompact.*

Proof. It is known, that projection $f : (X, U) \times (Y, V) \to (X, U)$ is uniformly perfect mapping of the product $(X, U) \times (Y, V)$ on the strong uniformly B-paracompact space (X, U). Then, according to Theorem 3.7, the product $(X, U) \times (Y, V)$ is strongly uniformly B-paracompact.

Corollary 3.3 *Product of any discrete uniform space by a compact uniform space is strongly uniformly B-paracompact.*

The following theorem shows equivalence of various kinds of strong uniform paracompactness in the class of $\aleph_0$-bounded uniform spaces.

Theorem 3.9 *For $\aleph_0$-bounded uniform space (X, U) the following are equivalent:*

1) (X, U) *is strongly uniformly P-paracompact;*
2) (X, U) *is strongly uniformly B-paracompact;*
3) *Topological space (X, τ_U) is Lindelöf.*

Lemma 3.3 *If for a uniform space (X, U) its topological space (X, τ_U) is Lindelöf, then uniform space (X, U) is strongly uniformly P-paracompact.*

Proof. Since the space (X, τ_U) is Lindelöf, then it is strongly paracompact. By Lemma 2.10 (see [4]) a uniform space (X, U) is uniformly P-paracompact. Hence, uniform space (X, U) is strongly uniformly P-paracompact.

Corollary 3.4 *For a Tychonoff space (X, τ) the following are equivalent:*

1) *Topological space (X, τ) is Lindelöf;*
2) *For each uniformity U such that $\tau_U = \tau$, uniform space (X, U) is strongly uniformly P-paracompact;*
3) *For each uniformity U such that $\tau_U = \tau$, uniform space (X, U) is strongly uniformly B-paracompact.*

A uniform space (X, U) is called *uniformly R-Lindelöf*, if it is uniformly R-paracompact and $\aleph_0$-bounded.

Proposition 3.7 *If (X, U) is a uniformly R-Lindelöf space, then the topological space (X, τ_U) is Lindelöf. Conversely, if (X, τ) is Lindelöf then the uniform space (X, U_X) is uniformly R-Lindelöf.*

Proof. Let α be an arbitrary open cover of (X, τ_U). Then for cover α exists a uniformly locally finite open cover β such that $\beta \succ \alpha$. Since the space (X, U) is $\aleph_0$-bounded the cover β contains a countable uniform cover β_0 . Then for any $i \in N$ we have that $B_i \subset \bigcup\limits_{j=1}^{k} A_j$. The system $\{B_i \bigcap A_j\}$, $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ forms a countable open cover which is refinement of α. Consequently, the space (X, τ_U) is Lindelöf. Conversely, if (X, τ) is Lindelöf, then the system of all open covers forms a base of universal uniformity U_X of the space (X, τ). It follows from this that (X, U_X) is uniformly R-Lindelöf.

Proposition 3.8 *Every compact uniform space (X, U) is uniformly R-Lindelöf.*

Proof. Since any compact uniform space is precompact, i.e. $\aleph_0$-bounded. It is clear that every compact space is uniform R-paracompact. Consequently, the uniform space (X, U) is uniformly R-Lindelöf.

Proposition 3.9 *Every uniformly R-Lindelöf space (X, U) is uniformly locally Lindelöf.*

Proof. From the facts that each closed subspace of a Lindelöf space is Lindelöf, one concludes that (X, U) is uniformly locally Lindelöf.

Proposition 3.10 *Every uniformly R-Lindelöf space (X, U) is uniformly B-Lindelöf.*

Proof. Let (X, U) be a uniformly R-Lindelöf space. It is known, every uniformly R-paracompact space is uniformly B-paracompact [2]. Hence, the uniform space is uniformly B-Lindelöf.

Proposition 3.11 *Every uniformly R-Lindelöf space (X, U) is strongly uniformly R-paracompact.*

Proof. Let (X, U) be a uniformly R-Lindelöf space. According to Theorem 3.2 the space (X, U) is strongly uniformly R-paracompact.

Proposition 3.12 *Every uniformly R-Lindelöf space (X, U) is complete.*

Proof. The completeness of (X, U) follows from Propositions 3.2 and 3.11.

Incomplete separably metrizable spaces are not uniformly R-Lindelöf.

Lemma 3.4 *Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to uniform space (Y, V). If a space (X, U) is $\aleph_0$-bounded, then (Y, V) is also $\aleph_0$-bounded.*

Proof. Let $\beta \in V$ be an arbitrary uniform cover. Then $f^{-1}\beta \in U$. According to Proposition 1.1.6. [2, p. 40] the cover $f^{-1}\beta$ has a countable subcover $f^{-1}\beta_0$. Then β_0 is a countable subcover of β. Therefore the space (Y, V) is $\aleph_0$-bounded.

Lemma 3.5 *Let $f : (X, U) \to (Y, V)$ be a precompact mapping of a uniform space (X, U) to a uniform space (Y, V). If a space (Y, V) is $\aleph_0$-bounded, then (X, U) is also $\aleph_0$-bounded.*

Proof. Let $f : (X, U) \to (Y, V)$ be a precompact mapping between uniform spaces (X, U) and (Y, V) and $\alpha \in U$ be an arbitrary uniform cover of (X, U). Then by virtue of the precompactness of f there exist a countable cover $\beta \in V$ and a finite cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ f^{-1}\alpha$. But the cover $f^{-1}\beta \wedge \gamma$ is countable. Therefore the space (X, U) is $\aleph_0$-bounded.

Lemma 3.6 *Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) and (Y, V). If β is uniformly locally finite cover of the space (Y, V), then $f^{-1}\beta$ is a uniformly locally finite cover of the space (X, U).*

Proof. By the conditions of the Lemma there exists a uniform cover $\alpha \in V$ such that $|St(A, \beta)|$ is finite for each $B \in \beta$, i.e. for each $A \in \alpha$ there exist elements $B_i \in \beta$, $i = 1, 2, ..., n$ such that $A \subset \bigcup\limits_{i=1}^{n} B_i$. Since the mapping f is uniformly continuous, the cover $f^{-1}\alpha$ is uniform, i.e. $f^{-1}\alpha \in U$. Hence, $f^{-1}A \subset \bigcup\limits_{i=1}^{n} f^{-1}B_i$, $f^{-1}A \in f^{-1}\alpha$, $f^{-1}B_i \in f^{-1}\beta$. So, $f^{-1}\beta$ is uniformly locally finite cover of (X, U).

Theorem 3.10 *Let $f : (X, U) \to (Y, V)$ be a uniformly perfect mapping of a uniform space (X, U) onto a uniform space (Y, V). Then uniform R-Lindelöfness converse both to direction of image and to one of preimage.*

Proof. The uniform R-Lindelöfness of the space (Y, V) follows from the Theorem 2.3.9 [2, p. 99] and Lemma 3.4, and the uniform R-Lindelöfness of the space (X, U) follows from the Theorem 2.3.9 [2, p. 100] and Lemmas 3.5, 3.6.

Theorem 3.11 *Let $f : (X, U) \to (Y, V)$ be a uniformly open mapping of a uniform space (X, U) onto a uniform space (Y, V). If (X, U) is uniformly R-Lindelöf space, then (Y, V) is also uniformly R-Lindelöf.*

Proof. Let f be a uniformly open mapping of a uniform space (X, U) onto a uniform space (Y, V) and α be an arbitrary finitely additive open cover of (Y, V). Then $f^{-1}\alpha$ is a finitely additive open cover of the space (X, U) and due to the uniform R-paracompactness we have$f^{-1}\alpha \in U$. Since f is uniformly open, $\alpha \in V$. Then (Y, V) is uniformly R-paracompact. According to Lemma 3.4 the space (Y, V) is $\aleph_0$-bounded. So, the space (Y, V) is uniformly R-Lindelöf.

4 Conclusions

Thus, paper introduces new concepts such as strongly uniformly R-paracompact, strongly uniformly B-paracompact, strongly uniformly P-paracompact, and uniformly R-Lindelöf spaces and investigates their connection with other uniform properties of compactness type, and also establishes the characterization of these classes by using finitely additive open covers, compactification and ω-mappings. In particular, the problem stated by A.A. Borubaev about ω-mappings is solved.

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