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On an example of Wiman-Valiron type application of general estimation in mathematical physics

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Abstract. In the paper we give some example on applications of Wiman-Valiron type general estimations for parabolic equations in Hilbert space to specific problems of mathematical physics. These examples illustrate the behavior of the solution of problems in the neighborhood of the singular point (0 or ∞) depending on the growth of Fourier coefficients of the initial function.

Keywords. evalution equation, Hilbert space, Wiman-Valiron type estimation, parabolic equation, spectrum spectral asymptotic formula.

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1 Introduction

In the Hilbert space H we consider an evolution equations of the form

$$u'(t) + A(t)u(t) = 0, \tag{1.1}$$

where $A(t)$ for each t is a positive-definite self-adjoint operator with a discrete spectrum. Let $\{\lambda_k(t)\}$ be a sequence of eigen-values, $\{\varphi_k(t)\}$ be an orthonormal system of eigen-functions of the operator $A(t)$.

Let $N(\lambda)$ be the amount of the eigen-values λ_k of the operator $A(t)$ not exceeding the given number $\lambda > 0$.

We introduce the following denotations for the solution $u(t)$ of the equation (1.1):

$$M(t) = \|u(t)\|, \mu(t) = \max_k |(u(t), \varphi_k(t))|. \tag{1.2}$$

The main problem stated and solved in theory of Wiman-Valiron type estimates is in obtaining the upper estimates of the function $M(t)$ through the function $\mu(t)$ in the neighborhood of the singular point (0 or ∞) .

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Let $\Psi(y) > 0,\ y > 0$ and the condition of the form

$$\int_1^\infty \left(\int_1^y \psi(t)dt\right)^{-\alpha} dy < \infty, \quad \alpha > 0 \tag{1.3}$$

be fulfilled (the number $\alpha > 0$ is determined by the operator $A(t)$). Under the Wiman-Valiron type estimations we understand the fulfillment of the asymptotic inequality of type

$$M(t) \le \psi(\mu(t)). \tag{1.4}$$

Such kind estimates were established in N.M. Suleymanov's doctoral dissertation defended in M.V. Lomonosov MSU in 1982 on the basis of which the monograph "Probability, entire functions and Wiman-Varilon type estimates for evolution equations" (Moscow, MSU Publ. 2012) was written. It should be noted that when establishing Wiman-Valirion type estimates in the above mentioned initial works of N.M. Suleymanov, spectral asymptotics of operators known to 1980 year have played an important role. But the main, sometimes the most important and final results on spectral asymptotics were obtained only in the eighties and nineties years in the papers of Ivriy, Safarov, Vassiliev, Kostyuchenko and others. In the present paper we widedy use the last achievements of spectral asymptotics exact asymptotic formulas obtained in the paper of Yu. Safarov and D. Vassiliev (1997), and also the formulas of Yu. Safarov and Yu. Netrusov (2005).

The results of Wiman-Valiron and Rosenbloom obtained at the beginning of the last century for entire functions served as a model or such estimates. Let us briefly describe these results.

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function,

$$M(r) = \max_{|z|=r} |f(z)|,\ \mu(r) = \max_n |a_n| r^n,\ |z| \le r.$$

The following inequality is valid:

$$M(r) \le \mu(r)\left(log\mu(r)\right)^{1/2+\varepsilon},\ \varepsilon > 0. \tag{$W-V$}$$

And this inequality can be violated only on the set $E \subset (0,\infty)$ for which $\int\limits_E dr/r < \infty$ (E is an exceptional set).

In [4] Rosenbloom proved a more general and strong elegant result: For some class of functions $\psi(y)$ he established an estimate of the form:

$$M(r) \le \mu(r)\sqrt{\psi(\mu(r))}, \tag{R}$$

outside the set $E \subset (0,\infty)$ of finite logarithmic measure.

2 General Wiman-Valiron type estimation and its implementation

In this paper established (R) type estimates where the role of the functions $M(r)$ and $\mu(r)$ in (R) are played by the functions $M(t)$ and $\mu(t)$ from (1.2). We give the Wiman-Valiron type main estimate obtained in the papers of N.M.Suleymanov [1-4] for parabolic equations in Hilbert space.

Imposing asymptotic character conditions on the class of operators $A(t)$ in (1.1) and on function $N(\lambda)$ in [1] for equations (1.1) the following general estimations was established

$$\frac{\|u(t)\|}{\sqrt{\psi(t^{-\beta}\log\|u(t)\|)}} \le t^{-\frac{\alpha}{2}}\,\mu(t), \alpha > 0, 0 < \beta < 1. \tag{S}$$

This estimate is valid outside, possibly, some set of finite logarithmic measure.

In this paper we give two examples of application of the estimate (S') to specific problems of mathematical physics illustrating the character of behaviour of the norm of the solution of the equation (1.1) as $t \to 0$ depending on the behaviour of Fouvier coefficients of the initial function.

1^0. Let $\Omega \subset R^d$ $(d \geq 2)$ be a domain with a boundary Γ and Q be a cylinder

$$Q = \Omega \times [0,T], T < \infty \text{ and } \sigma = \Gamma \times [0,T], H = L^2(\Omega).$$

Let us consider a second-order differential operator of the form

$$A(t)u(x,t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x,t)\frac{\partial u}{\partial x_j}\right) + cu, \tag{2.1}$$

where

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \alpha\left|\xi\right|^2, \ \ \alpha > 0, t \in [0,T], \ \ c(x,t) \geq 0, \ \xi \in R^n. \tag{2.2}$$

Let us consider a Cauchy -Dirichlet (or Cauchy-Neumann) type boundary value problem

$$\left.\begin{array}{r} \frac{\partial u}{\partial t} + A(t)u = 0 \ in \ \ Q \\ u = 0 \qquad \ on \ \ \sigma \\ u(x,0) = u_0(x) \ in \ \ \Omega \end{array}\right\}. \tag{2.3}$$

Note that if $A(t)$ is an elliptic differential operator, then the operator $\frac{\partial}{\partial t} + A$ will be a parabolic operator. In the case $A(t) = -\Delta_D$, where Δ_D is a Laplace Dirichlet operator, then the solution of the equation in (2.3) is represented in the form of the sum of Fourier-series:

$$u(x,t) = \sum_{k=0}^{\infty} c_k(0)e^{-t\lambda_k}\varphi_k(x), \ \lambda_k > 0,$$

where $c_k(0)$ are Fourier coefficients of the function $u_0(x)$ by the system of eigen functions $\{\varphi_k(x)\}$ of the operator Δ_D. We have:

$$\|u(\cdot,t)\|^2 = \sum_{k=0}^{\infty} c_k^2(0)e^{-2t\lambda_k}.$$

According to our method we obtain:

$$\mu^2(t) = \max_k c_k^2(0)\, e^{-2t\lambda_k}.$$

Let us consider a very special case: Suppose:

$$\lambda_n = \frac{1}{2}n^p, \ p \geq 1, \ c_n^2(0) = n^{p\beta}, \ 0 < \beta \leq 1.$$

We obtain:

$$\|u(\cdot,t)\|^2 = \sum_{n=0}^{\infty} n^p e^{-tn} = (-1)^p \frac{d^p}{dt^p}\left(\frac{1}{1-e^{-t}}\right) \sim \frac{c}{t^{p+1}}, \ \ t \to 0, \ c = const.$$

On the other hand,

$$\mu^2(t) = \max_n n^p e^{-tn} = \frac{c}{t^p}.$$

Consequently, we obtain an estimate of the form:

$$M(t) \le \psi(\mu(t)), \;\; \psi(y) = \frac{y}{\sqrt{t}}, \; t \to 0.$$

We now consider a more general case of the example.

For that we should know asymptotic formulas for the functions of distribution of eigenvalues of the operator $A(t)$ in (7). For the operators Δ_D and Δ_N such formulas are known. Recently, such formulas with more exact estimate of the residue for the operators Δ_D and Δ_N $b \to R^n$. In were obtained in the papers of Yu. Jafarov and Netrusov [9], Safarov and Vasilev [10], Kostyuchenko [1] and other authors particular, In the paper of Safarov and Netrusov (2005) the following exact unimravable asymptotic formulas were obtained for $N(\lambda)$:

1^0 $N_D(\lambda) = \lambda^d + O(\lambda^{d-\alpha}), 0 < \alpha < 1$ for the operator $\Delta_D, d \ge 2$.

2^0 $N_D(\lambda) = \lambda^d + O(\lambda^{\frac{d-1}{\alpha}}), 0 < \alpha < 1$ for the operator Δ_N.

Here $N_D(\lambda)$ is the function of distribution of eigen values of the operator $A(t) = -\Delta_D$ in $\Omega \subset R^d$ when establishing the $W-V$ type main estimates the main conditions improved on the function $N(\lambda)$ of the operator $A(t)$ is (1.1) as follows :

For $\lambda > \delta > 0, \; \lambda \to \infty$ the following inequality is fulfilled :

$$\Delta N(\lambda, \gamma) \le c\delta\lambda^s(1 + \lambda^\nu), s \ge 0, 0 < \nu < 1. \tag{2.4}$$

For the operator $A(t) = -\Delta_D$ this condition is fulfilled for $s = d-\alpha-1, \nu = \alpha, d \ge 2$. Consequently, we have the inequality of the from:

$$\Delta N_D(\lambda, \delta) = c\delta\lambda^{d-1-\alpha}(1 + \lambda^\alpha).$$

Then in accordance with general theory of N.M. Suleymanov papers (see [1-4]) we have an inequality of the form:

$$\Delta N_D(g', g'') = c\sqrt{g''}|g'|^s(1 + |g'|^\alpha), \;\; s = d - 1 - \alpha, \tag{2.5}$$

where $\Delta N_D(\lambda, \delta) = N_D(\lambda + \delta) - N_D(\lambda - \delta)$.

By the key lemma in N.M.Suleymanov's monograph [1], p. 84 we have the following of the form:

$$e^{2g(t)} \le \mu^2(t)\Delta N_D(g', g''), \;\; g(t) = \frac{1}{2}\log(u(t), u(t)).$$

We choose the function ψ in such a way that an inequality of the form to be fulfilled

$$\Delta N_D(g', g'') \le t^{-\alpha}\, \psi(t^{-\beta}\, g(t)), \;\; \alpha > 0, 0 < \beta < 1. \tag{2.6}$$

Then we obtain

$$e^{2g(t)} \le \mu^2(t)\, t^{-\alpha}\, \psi(t^{-\beta} g(t)).$$

Hence we have:

$$\frac{\|u(t)\|}{\sqrt{\psi(t^{-\beta}\log\|u(t)\|)}} \le \mu(t)t^{-\frac{\alpha}{2}}.$$

In particular, for $\psi(y) = y^R$ we obtain:

$$\frac{\|u(t)\|}{(\log\|u(t)\|)^{k/2}} \le \mu(t)\, t^{-\gamma}, \;\; \gamma = \frac{\alpha + \beta}{2}.$$

This is equivalent to the inequality:

$$\|u(t)\|_{L^2(\Omega)} \le \widetilde{\mu}(t) \left(\log \widetilde{\mu}(t)\right)^{\frac{1}{2}+\varepsilon},$$

where $\widetilde{\mu}(t) = t^{-\gamma}\mu(t),\ \mu(t) = \max\limits_k |C_k(0)| e^{-t\lambda_k},\ \ C_k(0) = (u_0(x), \varphi_k(x))$.

Thus, the (S') type estimates for specific examples of the Cauchy-Dirichlet and Cauchy-Neumann problem for $A(t) = -\Delta_D$ as $A(t) = -\Delta_D$ characterizes the behavior of the solution as $t \to 0$ depending on the behavior of the Fourier estimates of initial dates solution of problem (2.3) when similar estimations are obtained for the solutions of the problem (2.3), $A(t) = (-\Delta)^m, m \ge 1$.

Let us consider the spectral problem

$$Av = \lambda^m v,\ \ \beta^{(j)} v\Big|_{\partial\Omega} = 0,\ \lambda > 0. \tag{2.7}$$

For some discrete values $\lambda = \lambda_k, 0 < \lambda_1 < \lambda_2, ..., k \to \infty$ there exist non-trivial solution of this problem, where $\Omega \subset R^n (n \ge 2)$ is a domain.

For the function $N(\lambda)$ for the problem (2.7) in Yu. Safarov and D. Vasilev's paper it was established exact asymptotic formula of the form:

$$N(\lambda) = c_0, \lambda^n\ + c_1\lambda^{n-1} + 0(\lambda^{n-1}),\ \lambda \to +\infty$$

Let $O(\lambda^{n-1}) = \lambda^{n-1-\tau}, 0 < \tau < 1$. For $\lambda > \delta > 0, \lambda \to \infty$ we obtain:

$$\Delta N(\lambda, \delta) = c_0\delta\lambda^{n-1} + c_1\delta\lambda^{n-2} + c_3\delta\lambda^{n-2-\tau} = c\delta\lambda^{n-2-\tau}[1 + \lambda^\tau + \lambda^{1+\tau}].$$

We have

$$\Delta N(g', g'') = c\sqrt{g''}\,(g')^{n-2-\tau}\ \left[1 + (g')^\tau| + (g')^{1+\tau}\right].$$

We find the function $\psi(y)$ for which the following inequality is fulfilled:

$$\Delta N(g', g'') \le t^{-\alpha}\sqrt{\psi(t^{-\beta} g)},\ \ \alpha > 0, 0 < \beta < 1.$$

By virtue of the key lemma we obtain

$$e^{2g(t)} \le \mu^2(t)\Delta N(g', g''),$$

which is equivalent to the estimate

$$\frac{\|u(t)\|}{\sqrt{\psi(t^{-\beta}\log\|u(t)\|)}} \le \mu(t) t^{-\frac{\alpha}{2}}.$$

In the case $\psi(y) = y^k\ \ (k > 2)$ we obtain

$$\|u(t)\| \le \widetilde{\mu}(t) \left(\log\widetilde{\mu}(t)\right)^{\frac{k}{4}},\ \widetilde{\mu} = t^{-\gamma}\mu(t),\ \ \gamma = \frac{2\alpha + k\beta}{4}.$$

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