Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **44**(4), 5-12 (2024).
https://doi.org/10.30546/2617-7900.44.4.2024.05

Anisotropic maximal commutator and commutator of anisotropic maximal operator on Lorentz spaces

Ali Akbulut***, Fatai A. Isayev, Ayhan Serbetci**

Received: 02.02.2024 / Revised: 23.08.2024 / Accepted: 04.09.2024

Abstract. *In this paper we consider the anisotropic maximal commutator M_b^d and the commutator of the anisotropic maximal operator $[b, M^d]$ on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$. We obtain necessary and sufficient conditions for the boundedness of the operators M_b^d and $[b, M^d]$ on $L^{p,q}(\mathbb{R}^n)$ when b belongs to the bounded mean oscillation space $BMO(\mathbb{R}^n)$, whereby some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$ are obtained.*

Keywords. Lorentz spaces, anisotropic maximal operator, commutator, BMO spaces.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35.

1 Introduction

The aim of this paper is to study anisotropic maximal commutator operator M_b^d and commutator of anisotropic maximal operator $[b, M^d]$ in the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$. We give necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator operator M_b^d and commutator of anisotropic maximal operator $[b, M^d]$ on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of the bounded mean oscillation space $BMO(\mathbb{R}^n)$.

Let $\mathbb{R}^n$ be the n-dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$ and S^{n-1} denote the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $\mathcal{E}(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}\mathcal{E}(x,r)$ denote the set $\mathbb{R}^n \backslash \mathcal{E}(x,r)$. Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \big(t^{d_1} x_1, \dots, t^{d_n} x_n\big)$. By [5,9], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous

* Corresponding author

A. Akbulut
Department of Mathematics, Kirsehir Ahi Evran University, Kirsehir, Turkey
E-mail: akbulut72@gmail.com

F.A. Isayev
Institute of Mathematics and Mechanics, Baku, Azerbaijan
E-mail: isayevfatai@yahoo.com

A. Serbetci
Department of Mathematics, Ankara University, Ankara, Turkey
E-mail: Ayhan.Serbetci@ankara.edu.tr

metric space (see [5,7,9]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let also $\Pi_d(x,r) = \{y \in \mathbb{R}^n : \max_{1\le i \le n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelopiped, ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \setminus \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1,\dots,1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = B(x,r)$. Note that in the standard parabolic case $d = (1,\dots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

For $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the anisotropic maximal operator M is defined by

$$M^d f(x) = \sup_{r>0} |\mathcal{E}(x,r)|^{-1} \int_{\mathcal{E}(x,r)} |f(y)| dy,$$

where $|\mathcal{E}(x,r)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x,r)$. If $d = \mathbf{1}$, then $M \equiv M^d$ is the classical Hardy-Littlewood maximal operator.

The study of anisotropic maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance [11]).

The anisotropic maximal commutator generated by the operator M^d and $b \in L^1_{\rm loc}(\mathbb{R}^n)$ is defined by

$$M^d_b f(x) = \sup_{r>0} |\mathcal{E}(x,r)|^{-1} \int_{\mathcal{E}(x,r)} |b(x) - b(y)|\, |f(y)|\, dy.$$

The commutator generated by the operator M^d and a suitable function b is defined by

$$[b, M^d] f(x) = b(x) M^d f(x) - M^d(bf)(x).$$

Obviously, the operators M^d_b and $[b, M^d]$ essentially differ from each other since M^d_b is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear.

The operators M, M^d_b and $[b, M^d]$ play an important role in real and harmonic analysis and applications (see, for instance [1–3, 10, 14–16, 18]). The nonlinear commutator of Hardy-Littlewood maximal function $[b, M]$ can be used in studying the product of a function in H^1 and a function in BMO [6]. The boundedness of the anisotropic maximal operator M^d on $L^p(\mathbb{R}^n)$ is one of the most fundamental results in harmonic analysis. It has been extended to a range of other function spaces, and to many variations of the standard maximal operator. The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance [8, 11]).

This paper is organized as follows. In Section 2 we give some definitions and auxiliary results. In Section 3 we obtain necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator M^d_b on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ on $L^{p,q}(\mathbb{R}^n)$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Definition and some basic properties

We start with the definition of Lorentz spaces. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar L^p spaces, also they are appear to be useful in the general interpolation theory.

Suppose that f is a measurable function on $\mathbb{R}^n$, then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\},$$

where

$$d_f(s) := |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad \forall s > 0.$$

Definition 2.1 *[4] The Lorentz space $L^{p,q} \equiv L^{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty$ is the collection of all measurable functions f on $\mathbb{R}^n$ such the quantity*

$$\|f\|_{L^{p,q}} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)} \tag{2.1}$$

is finite. Clearly $L^{p,p} \equiv L^p$ and $L^{p,1} \equiv WL^p$. The functional $\|\cdot\|_{L^{p,q}}$ is a norm if and only if either $1 \leq q \leq p$ or $p = q = \infty$.

Lemma 2.1 *[4] Let $1 < p, p', q, q' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Suppose that $f \in L^{p,q}(\mathbb{R}^n)$ and $f \in L^{p',q'}(\mathbb{R}^n)$. Then*

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq 2\|f\|_{L^{p,q}(\mathbb{R}^n)} \|g\|_{L^{p',q'}(\mathbb{R}^n)}.$$

The following result completely characterizes the boundedness of M^d on Lorentz spaces.

Lemma 2.2 *[4] Let $1 \leq p, q \leq \infty$.*
(i) *If $1 < p \leq \infty$, then the operator M^d is bounded on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.*
(ii) *If $p = 1$, then the operator M^d is bounded from $L^{1,q}(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$.*

3 The boundedness of the anisotropic maximal commutator operator M_b^d on $L^{p,q}(\mathbb{R}^n)$ Lorentz spaces

In this section we find necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator M_b^d on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.

Definition 3.1 *We define the bounded mean oscillation space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norm*

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y) - f_{\mathcal{E}(x,t)}| dy < \infty,$$

where $f_{\mathcal{E}(x,t)} = |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} f(y) dy$.

Lemma 3.1 *([2, Corollary 1.11]) If $b \in BMO(\mathbb{R}^n)$, then there exists a positive constant C such that*

$$M_b^d f(x) \le C\|b\|_* M^d\big(M^d f\big)(x) \tag{3.1}$$

for almost every $x \in \mathbb{R}^n$ and for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

The following theorem is the first of our main results.

Theorem 3.1 *Let $p, q \in (1, \infty)$. The following assertions are equivalent:*

(i) *$b \in BMO(\mathbb{R}^n)$.*

(ii) *The operator M_b^d is bounded on $L^{p,q}(\mathbb{R}^n)$.*

(iii) *There exist a constant $C > 0$ such that*

$$\sup_{\mathcal{E}} \frac{\big\|\big(b(\cdot) - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\big\|_{L^{p,q}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}(\mathbb{R}^n)}} \le C. \tag{3.2}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_{\mathcal{E}} \frac{\big\|\big(b(\cdot) - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\big\|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \le C. \tag{3.3}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Lemmas 2.2 and 3.1, we get

$$\begin{aligned}\|M_b^d f\|_{L^{p,q}} &\lesssim \|b\|_* \|M^d\big(M^d f\big)\|_{L^{p,q}} \\ &\lesssim \|b\|_* \|M^d f\|_{L^{p,q}} \\ &\lesssim \|b\|_* \|f\|_{L^{p,q}}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that M_b^d is bounded on $L^{p,q}(\mathbb{R}^n)$. Let $\mathcal{E} = \mathcal{E}(x, r)$ be a fixed ellipsoid. We consider $f = \chi_{\mathcal{E}}$. It is easy to compute that

$$\|\chi_{\mathcal{E}}\|_{L^{p,q}} \approx r^{\frac{|d|}{p}}. \tag{3.4}$$

On the other hand, for all $x \in B$ we have

$$\begin{aligned}\big|b(x) - b_{\mathcal{E}}\big| &\le \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b(x) - b(y)|dy \\ &= \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b(x) - b(y)|\,\chi_{\mathcal{E}}(y)dy \\ &\le M_b^d(\chi_{\mathcal{E}})(x).\end{aligned}$$

Since M_b^d is bounded on $L^{p,q}(\mathbb{R}^n)$, then by (3.4) we obtain

$$\frac{\|\big(b - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\|_{L^{p,q}}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}}} \le \frac{\|M_b^d(\chi_{\mathcal{E}})\|_{L^{p,q}}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}}} \lesssim \frac{\|\chi_{\mathcal{E}}\|_{L^{p,q}}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}}} = 1, \tag{3.5}$$

which implies that (3.2) holds since the ball $B \subset \mathbb{R}^n$ is arbitrary.

$(iii) \Rightarrow (iv)$. Assume that (3.2) holds, we will prove (3.3). For any fixed ellipsoid $\mathcal{E}$, by Lemma 2.1 and (3.2), (3.4), it is easy to see

$$\begin{aligned}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b(x) - b(y)|dy &\lesssim \frac{1}{|\mathcal{E}|}\,\|\big(b - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\|_{L^{p,q}}\,\|\chi_{\mathcal{E}}\|_{L^{p',r'}} \\ &\lesssim \frac{\|\big(b - b_{\mathcal{E}}\big)\chi_{\mathcal{E}}\|_{L^{p,q}}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}}} \\ &\lesssim 1.\end{aligned}$$

$(iv) \Rightarrow (i)$. For any fixed ellipsoid $\mathcal{E}$, we have

$$\begin{aligned}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|dy &= \frac{\|(b-b_{\mathcal{E}})\chi_{\mathcal{E}}\|_{L^1}}{|\mathcal{E}|}\\ &\le \sup_{\mathcal{E}}\frac{\|(b-b_{\mathcal{E}})\chi_{\mathcal{E}}\|_{L^1}}{|\mathcal{E}|}\\ &\lesssim 1,\end{aligned}$$

which implies that $b \in BMO(\mathbb{R}^n)$. Thus the proof of Theorem 3.1 is completed.

4 The boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$

In this section we find necessary and sufficient conditions for the boundedness of the commutator of the anisotropic maximal operator $[b, M^d]$ on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.

Let b be a function defined on $\mathbb{R}^n$ and denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0\\ |b(x)|, & \text{if } b(x) < 0\end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M^d]$ and M^d_b are valid:

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned}\big|[b, M^d]f(x)\big| &= \big|b(x)M^d f(x) - M^d(bf)(x)\big|\\ &= \big|M^d(b(x)f)(x) - M^d(bf)(x)\big|\\ &\le M^d(|b(x)-b|f)(x)\\ &= M^d_b f(x).\end{aligned}$$

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M^d]f(x)| \le M^d_b f(x) + 2b^-(x)\, M^d f(x), \qquad x \in \mathbb{R}^n \tag{4.1}$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example [12,18]).

Obviously, the M^d_b and $[b, M^d]$ operators are essentially different from each other because M^d_b is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear.

Let $\mathcal{E} = \mathcal{E}(x, r)$ be a fixed ellipsoid. Denote by $M^d_{\mathcal{E}} f$ the local maximal function of f:

$$M^d_{\mathcal{E}} f(x) := \sup_{\mathcal{E}'\ni x:\, \mathcal{E}'\subset\mathcal{E}} \frac{1}{|\mathcal{E}'|}\int_{\mathcal{E}'}|f(y)|\,dy,\ x \in \mathbb{R}^n.$$

Applying Theorem 3.1, we obtain the following result which is the second of our main results.

Theorem 4.1 *Let $p, q \in (1, \infty)$. The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(ii) *The operator* $[b, M^d]$ *is bounded on* $L^{p,q}(\mathbb{R}^n)$.

(iii) *There exist a constant $C > 0$ such that*

$$\sup_{\mathcal{E}} \frac{\left\| \big(b(\cdot) - M^d_{\mathcal{E}}(b)(\cdot)\big)\chi_{\mathcal{E}} \right\|_{L^{p,q}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}(\mathbb{R}^n)}} \le C. \tag{4.2}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_{\mathcal{E}} \frac{\left\| \big(b(\cdot) - M^d_{\mathcal{E}}(b)(\cdot)\big)\chi_{\mathcal{E}} \right\|_{L^{1}(\mathbb{R}^n)}}{|\mathcal{E}|} \le C. \tag{4.3}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$. Combining Lemma 2.2 and Theorem 3.1, and inequality (4.1), we get

$$\begin{aligned} \|[b, M^d]f\|_{L^{p,q}} &\le \|M^d_{\mathcal{E}} f + 2b^- M^d f\|_{L^{p,q}} \\ &\le \|M^d_{\mathcal{E}} f\|_{L^{p,q}} + \|b^-\|_{L^\infty} \|M^d f\|_{L^{p,q}} \\ &\lesssim \big(\|b\|_* + \|b^-\|_{L^\infty}\big) \|f\|_{L^{p,q}}. \end{aligned}$$

Thus, we obtain that $[b, M^d]$ is bounded on $L^{p,q}(\mathbb{R}^n)$.

$(ii) \Rightarrow (iii)$. Assume that $[b, M^d]$ is bounded on $L^{p,q}(\mathbb{R}^n)$. Since

$$M^d(b\chi_{\mathcal{E}})\chi_{\mathcal{E}} = M^d_{\mathcal{E}}(b) \quad \text{and} \quad M^d(\chi_{\mathcal{E}})\chi_{\mathcal{E}} = \chi_{\mathcal{E}},$$

we have

$$\begin{aligned} |M^d_{\mathcal{E}}(b) - b\chi_{\mathcal{E}}| &= |M^d(b\chi_{\mathcal{E}})\chi_{\mathcal{E}} - bM^d(\chi_{\mathcal{E}})\chi_{\mathcal{E}}| \\ &\le |M^d(b\chi_{\mathcal{E}}) - bM^d(\chi_{\mathcal{E}})| = |[b, M^d]\chi_{\mathcal{E}}|. \end{aligned}$$

Hence

$$\|M^d_{\mathcal{E}}(b) - b\chi_{\mathcal{E}}\|_{L^{p,q}(\mathbb{R}^n)} \le \|[b, M^d]\chi_{\mathcal{E}}\|_{L^{p,q}(\mathbb{R}^n)}.$$

Thus we get

$$\frac{\|\big(b - M^d_{\mathcal{E}}(b)\big)\chi_{\mathcal{E}}\|_{L^{p,q}}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}}} \le \frac{\|[b, M^d](\chi_{\mathcal{E}})\|_{L^{p,q}}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}}} \lesssim \frac{\|\chi_{\mathcal{E}}\|_{L^{p,q}}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}}} = 1,$$

which deduces that (iii).

$(iii) \Rightarrow (iv)$. Assume that (4.2) holds, then for any fixed ellipsoid $\mathcal{E}$, by Lemma 2.1, we conclude that

$$\begin{aligned} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(x) - M^d_{\mathcal{E}}(b)(x)| dx &\lesssim \frac{1}{|\mathcal{E}|} \|\big(b - M^d_{\mathcal{E}}(b)\big)\chi_{\mathcal{E}}\|_{L^{p,q}} \|\chi_{\mathcal{E}}\|_{L^{p',r'}} \\ &\lesssim \frac{\|\big(b - M^d_{\mathcal{E}}(b)\big)\chi_{\mathcal{E}}\|_{L^{p,q}}}{\|\chi_{\mathcal{E}}\|_{L^{p,q}}} \\ &\lesssim 1. \end{aligned}$$

$(iv) \Rightarrow (i)$. Assume that (4.3) holds, we will prove $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.

Denote by

$$E := \{x \in \mathcal{E} : \, b(x) \le b_{\mathcal{E}}\}, \quad F := \{x \in \mathcal{E} : \, b(x) > b_{\mathcal{E}}\}.$$

Since

$$\int_E |b(t) - b_{\mathcal{E}}| \, dt = \int_F |b(t) - b_{\mathcal{E}}| \, dt,$$

in view of the inequality $b(x) \le b_{\mathcal{E}} \le M^d_{\mathcal{E}}(b)$, $x \in E$, we get

$$\begin{aligned} \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b - b_{\mathcal{E}}| &= \frac{2}{|\mathcal{E}|}\int_{E} |b - b_{\mathcal{E}}| \\ &\le \frac{2}{|\mathcal{E}|}\int_{E} |b - M^d_{\mathcal{E}}(b)| \\ &\le \frac{2}{|\mathcal{E}|}\int_{\mathcal{E}} |b - M^d_{\mathcal{E}}(b)| \lesssim c. \end{aligned}$$

Consequently, $b \in BMO(\mathbb{R}^n)$.

In order to show that $b^- \in L^\infty(\mathbb{R}^n)$, note that $M^d_{\mathcal{E}}(b) \ge |b|$. Hence

$$0 \le b^- = |b| - b^+ \le M^d_{\mathcal{E}}(b) - b^+ + b^- = M^d_{\mathcal{E}}(b) - b.$$

Thus

$$(b^-)_{\mathcal{E}} \le c,$$

and by the Lebesgue Differentiation theorem we get that

$$0 \le b^-(x) = \lim_{|\mathcal{E}|\to 0} \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} b^-(y)dy \le c \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Remark 4.1 Note that in the case of $d = \mathbf{1} \equiv (1, \dots, 1)$ from Theorem 3.1 we get [13, Theorem 3.1] and Theorem 4.1 we get [13, Theorem 4.2].

Acknowledgements
The authors thank the referee(s) for careful reading the paper and useful comments.

References

1. Abasova, G.A., Omarova, M.N.: *Commutator of anisotropic maximal function with BMO functions on total anisotropic Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **43**(1), 3-15 (2022).
2. Agcayazi, M., Gogatishvili, A., Koca, K., Mustafayev, R.: *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan, **67** (2), 581-593 (2015).
3. Bastero, J., Milman, M., Ruiz, F.J.: *Commutators for the maximal and sharp functions*, Proc. Amer. Math. Soc. **128** (11), 3329-3334 (2000)(electronic).
4. Bennett, C., Sharpley, R.: *Interpolation of Operators,* Academic Press, Boston, 1988.
5. Besov, O.V., Il'in, V.P., Lizorkin, P.I.: *The L_p-estimates of a certain class of non-isotropically singular integrals,* (Russian) Dokl. Akad. Nauk SSSR **169**, 1250-1253 (1966).
6. Bonami, A., Iwaniec, T., Jones, P., Zinsmeister, M.: *On the product of functions in BMO and H_1*, Ann. Inst. Fourier Grenoble **57**(5), 1405-1439 (2007).
7. Bramanti, M., Cerutti, M.C.: *Commutators of singular integrals on homogeneous spaces*, Boll. Un. Mat. Ital. B **10**(7), 843-883 (1996).
8. Coifman, R.R., Rochberg, R., Weiss, G.: *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (3), 611-635 (1976).
9. Fabes, E.B., Rivère, N.: *Singular integrals with mixed homogeneity*, Studia Math. **27**, 19-38 (1966).
10. Garcia-Cuerva, J., Harboure, E., Segovia, C., Torrea, J.L.: *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J. **40**, 1397-1420 (1991).

11. Grafakos, L.: Modern Fourier analysis, *2nd ed., Graduate Texts in Mathematics*, **250**, *Springer, New York*, (2009).
12. Guliyev, V.S.: *Some characterizations of BMO spaces via commutators in Orlicz spaces on stratified Lie groups*, Results Math. **77**, article number 42, (2022).
13. Guliyev, V.S.: *Maximal commutator and commutator of maximal function on Lorentz spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **44**(4), 1-10 (2024).
14. Hu, G., Yang, D.: *Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type*, J. Math. Anal. Appl. **354**, 249-262 (2009).
15. Janson, S.: *Mean oscillation and commutators of singular integral operators*, Ark. Mat. **16**, 263-270 (1978).
16. Segovia, C., Torrea, J.L.: *Weighted inequalities for commutators of fractional and singular integrals*, Publ. Mat. **35**, 209-235 (1991).
17. Zhang, P., Wu, J.: *Commutators of the fractional maximal functions*, Acta Math. Sinica (Chin. Ser.) 52, 1235-1238 (2009).
18. Zhang, P., Wu, J., Sun, J.: *Commutators of some maximal functions with Lipschitz function on Orlicz spaces*, Mediterr. J. Math. **15** (6), Paper No. 216, 13 pp. (2018).