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The infinite summation formulas of the Multiple Lauricella's hypergeometric functions

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Abstract. *In this article, inspired by the work of Wang on some infinite summation formulas of Appell functions in two variables and by the help of some famous summation theorems, is introduced some new inverse pair symbolic operators with the multidimensional analogues. The properties of symbolic operators are studied and the infinite summation formulas for the four multiple Lauricella's functions are obtained.*

Keywords. Appell functions, multiple Lauricella's hypergeometric functions, the inverse pair symbolic operators, Poole's formula, infinity summation formulas.

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1 Introduction and definitions

Hypergeometric functions in one and more variables occur naturally in a wide variety of problems in applied mathematics, statistics, operations research, theoretical physics, and engineering sciences. For instance, Srivastava and Kashyap [24] presented a number of interesting applications of hypergeometric functions in queuing theory and related stochastic processes. The work of Niukkanen [21] on the multiple hypergeometric functions is motivated by various physical and quantum chemical applications of such functions. Especially, many problems in gas dynamics lead to degenerate second-order partial differential equations, which are solvable in terms of multiple hypergeometric functions. Among examples,

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we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [14].

The success of the theory of hypergeometric functions in one variable has stimulated the development of a corresponding theory in two and more variables. Appell [1] has defined in 1880 four functions $F_i\,(i = 1, 2, 3, 4)$, all of which are analogous of Gaussian hypergeometric functions $F(a, b; c; z)$. A great merit in the further development of the theory of the hypergeometric series in two variables belongs to Horn [19], who gave a general definition and classification order of double hypergeometric series. He has investigated the convergence of hypergeometric series of two variables and established the systems of partial differential equations which they satisfy. Horn investigated in particular hypergeometric series of second order and found some series, which are either expressible in terms of one variable or are products of two hypergeometric series of one variable. According his investigations, there are essentially 34 (14 complete and 20 confluent) convergent series of second order.

Lauricella further generalized the four Applell series F_1, F_2, F_3, F_4 to the case of n variables and defined multiple hypergeometric series denoted by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$. These functions have important applications (see, [20, p. 114]). For instance, explicit fundamental solutions of a multidimensional singular elliptic equation are expressed in terms of the Lauricella's hypergeometric function $F_A^{(n)}$ [10], [11]. For a given multiple hypergeometric function, it is useful to fund a decomposition formula, which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions, involving fewer variables. The familiar operator method of Burcnall and Chaundy (see, [7], [8]) has been used by them rather extensively for finding decomposition formulas for hypergeometric functions of two variables in terms of the classical Gaussian hypergeometric function of one variable. In the papers [3], [4], [5], [6] interesting results were obtained on the study of the double hypergeometric functions. Recently, in the work [18], are obtained the formulas of analytic continuation for the Lauricella's hypergeometric functions in three variables.

Following the works [7], [8], Hasanov and Srivastava [16], [17] introduced operators generalizing the Burcnall–Chaundy operators and found expansion formulas for many triple hypergeometric functions, and they proved recurrent formulas when the dimension of hypergeometric function exceeds three. However, due to the recurrence, additional difficulties may arise in the applications of those decomposition formulas. Recently, in the works [12], [13], [15], some new non-recurrence decomposition formulas for n-variable Lauricella functions $F_A^{(n)}$ and $F_B^{(n)}$ are obtained and directly applied to the solving boundary value problems for multidimensional singular elliptic equation.

In the present paper, inspired by the work [25] of Wang, we establish the infinite summation formulas for Lauricella functions in n variables $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$.

The plan of our paper is as follows. In Section 2 we briefly give some preliminary information, which will be used later. In Section 3, we define inverse pair symbolic operators $H_x(a, b)$, $\bar{H}_x(a, b)$ and their multidimensional analogues. In Sections 4 and 5, we write the infinity summation formulas associated with one-dimensional and multi-dimensional inverse pair operators, respectively.

2 Preliminaries

Throughout this work it is convenient to employ the Pochhammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n = \lambda(\lambda + 1)...(\lambda + n - 1),\ n = 1, 2, ...; \quad (\lambda)_0 \equiv 1$$

for which the following equalities are true:

$$(1)_n = n!,\ \ (\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda),\ \ (\lambda)_{n+k} = (\lambda)_n(\lambda+n)_k,$$

where $\Gamma(z)$ is a famous Euler's gamma-function.

It is known that the Euler gamma-function $\Gamma(a)$ has property [9, p. 17, Eq. (2)]

$$\Gamma(a+m) = \Gamma(a)(a)_m.$$

Here $(a)_m$ is a Pochhammer symbol, for which the equality $(a)_{m+n} = (a)_m(a+m)_n$ and its particular case $(a)_{2m} = (a)_m(a+m)_m$ are true [9, p. 67, Eq. (5)].

A function

$$F(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k,\ c \neq 0,-1,-2,... \tag{2.1}$$

is known as the Gaussian hypergeometric function.

If $\mathrm{Re}c > \mathrm{Re}b > 0$, we have Euler's formula [9, p. 114, Eq.(1)]

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt. \tag{2.2}$$

Here the right-hand side of (2.2) is a one-valued analytic function of the argument z within the domain $|arg(1-z)| < \pi$. Therefore, the function (2.2) gives also analytic continuation of $F(a,b;c;z)$.

The integral representation (2.2) allows to derive the Boltz formula [9, p. 105, Eq. (3)]

$$F(a,b;c;z) = (1-z)^{-b} F\left(c-a,b;c;\frac{z}{z-1}\right) \tag{2.3}$$

and get the value of the Gaussian function in unity (the summation formula) [9, p. 73, Eq. (73)]

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \ \mathrm{Re}(c-a-b) > 0,\ \ c \neq 0,-1,-2,... \tag{2.4}$$

When we consider the properties of a hypergeometric function, it is very important to study the infinite sums of this function. For example, the following formula shows that the infinite sum of a hypergeometric function can give an elementary function [23, p. 413, Eq. 6.7.1(8)]

$$\sum_{k=0}^{\infty} \frac{(a)_k(c-b)_k}{k!(c)_k} z^k F(a+k,b;c+k;z) = (1-z)^{-a}. \tag{2.5}$$

The double Appell hypergeometric functions are defined as follows [1]:

$$F_1\big(a,b,b';c;x,y\big) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\,(b')_n}{(c)_{m+n} m!n!} x^m y^n, \tag{2.6}$$

$$F_2\big(a,b,b';c,c';x,y\big) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\,(b')_n}{(c)_m\,(c')_n\, m!n!} x^m y^n, \tag{2.7}$$

$$F_3\big(a,a',b,b';c;x,y\big) = \sum_{m,n=0}^{\infty} \frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{(c)_{m+n} m!n!} x^m y^n, \tag{2.8}$$

$$F_4\big(a,b;c,c';x,y\big)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m\,(c')_n\,m!n!}x^m y^n. \tag{2.9}$$

We introduce the following notations:

$$\mathbf{a}:=\big(a_1,...,a_n\big),\ \ \mathbf{b}:=\big(b_1,...,b_n\big),\ \ \mathbf{c}:=\big(c_1,...,c_n\big);\ \ \mathbf{x}:=\big(x_1,...,x_n\big),$$

$$(\mathbf{a})_{\mathbf{k}}:=\prod_{j=1}^{n}\left(a_j\right)_{k_j};\ \ \mathbf{x}^{\mathbf{k}}:=\prod_{j=1}^{n}x_j^{k_j},$$

$$\mathbf{k}:=\left(k_1,...,k_n\right),\ \ |\mathbf{k}|:=k_1+...+k_n,\ k_1\ge 0,...,k_n\ge 0;\ K!:=k_1!k_2!...k_n!,$$

$$\mathbf{l}:=\left(l_1,...,l_n\right),\ \ |\mathbf{l}|:=l_1+...+l_n,\ l_1\ge 0,...,l_n\ge 0;\ L!:=l_1!l_2!...l_n!.$$

The Lauricella hypergeometric functions in three (real or complex) variables are defined as following [2, p. 114]

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!\left(\mathbf{c}\right)_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}\ \ \left(|x_1|+...+|x_n|<1\right), \tag{2.10}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(\mathbf{a})_{\mathbf{k}}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}},\ \ \left(|x_1|<1,\ ...,\ |x_n|<1\right), \tag{2.11}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}(b)_{|\mathbf{k}|}}{K!\left(\mathbf{c}\right)_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}\ \ \left(\sqrt{|x_1|}+...+\sqrt{|x_n|}<1\right), \tag{2.12}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}},\ \ \left(|x_1|<1,\ ...,\ |x_n|<1\right). \tag{2.13}$$

In all definitions (2.10)–(2.13), as usual, the denominator parameters c, c_1, ..., c_n are neither zero nor a negative integer.

Clearly, we have

$$F_A^{(2)}=F_2,\ \ F_B^{(2)}=F_3,\ \ F_C^{(2)}=F_4,\ \ F_D^{(2)}=F_1,\ \ F_A^{(1)}=F_B^{(1)}=F_C^{(1)}=F_D^{(1)}\equiv F,$$

where F_1, ..., F_4 are the Appell series defined by (2.6)–(2.9) and F is Gaussian hypergeometric function defined in (2.1).

Indeed, in all definitions (2.10)–(2.13) a number of variables is natural: $n=1,2,...$. However, in our further studies, if $n=0$, then we accept that

$$F_A^{(0)}=F_B^{(0)}=F_C^{(0)}=F_D^{(0)}\equiv 1.$$

The following summation formula [2, p. 117]

$$F_D^{(n)}\left(a,\mathbf{b};c;1,...,1\right)=\frac{\Gamma(c)\Gamma\left(c-a-B\right)}{\Gamma(c-a)\Gamma\left(c-B\right)},\ \ \mathrm{Re}(c-a-B)>0 \tag{2.14}$$

is valid, where $B:=b_1+...+b_n$.

It is easy to see that the formula (2.14) generalizes the famous Gaussian summation formula (2.4).

3 The inverse pair operators $H_x(a,b)$ and $\bar{H}_x(a,b)$

Burchnall and Chaundy [7], [8] systematically presented a number of expansions for some double hypergeometric functions in series of simpler hypergeometric functions. However, the Burchnall–Chaundy method is limited to functions of two variables.

Developing the idea of Burchnall and Chaundy, we introduce the following mutually inverse symbolic operators for n variables:

$$H_{\mathbf{x}}(\alpha,\beta) := \frac{\Gamma\big(\alpha+\delta_1+...+\delta_n\big)\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma\big(\beta+\delta_1+...+\delta_n\big)} \tag{3.1}$$

and

$$\bar{H}_{\mathbf{x}}(\alpha,\beta) := \frac{\Gamma(\alpha)\,\Gamma\big(\beta+\delta_1+...+\delta_n\big)}{\Gamma(\beta)\,\Gamma\big(\alpha+\delta_1+...+\delta_n\big)}, \tag{3.2}$$

where

$$\mathbf{x} := \big(x_1,...,x_n\big);\;\; \delta_j = x_j\frac{\partial}{\partial x_j},\;\; j=1,...,n.$$

In the one-dimensional case, these operators look like

$$H_{x_j}(\alpha,\beta) := \frac{\Gamma(\alpha+\delta_j)\,\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta+\delta_j)},\;\; j=1,...,n \tag{3.3}$$

and

$$\bar{H}_{x_j}(\alpha,\beta) := \frac{\Gamma(\alpha)\,\Gamma(\beta+\delta_j)}{\Gamma(\beta)\,\Gamma(\alpha+\delta_j)},\;\; j=1,...,n. \tag{3.4}$$

Using the Gaussian formula (2.4) for $F(a,b;c;1)$, we get

$$H_{x_j}(\alpha,\beta) = \sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r\,(-\delta_j)_r}{(\beta)_r\,r!},\;\; j=1,...,n \tag{3.5}$$

and

$$\bar{H}_{x_j}(\alpha,\beta) = \sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r\,(-\delta_j)_r}{(1-\alpha-\delta_j)_r\,r!},\;\; j=1,...,n. \tag{3.6}$$

Similarly, using the summation formula (2.14) for $F_D^{(n)}\big(a,\mathbf{b};c;1,...,1\big)$, we get

$$H_{\mathbf{x}}(\alpha,\beta) = \sum_{|\mathbf{k}|=0}^{\infty}\frac{(\beta-\alpha)_{|\mathbf{k}|}}{K!(\beta)_{|\mathbf{k}|}}\prod_{j=1}^{n}(-\delta_j)_{k_j} \tag{3.7}$$

and

$$\bar{H}_{\mathbf{x}}(\alpha,\beta) = \sum_{|\mathbf{k}|=0}^{\infty}\frac{(\beta-\alpha)_{|\mathbf{k}|}}{K!(1-\alpha-\delta)_{|\mathbf{k}|}}\prod_{j=1}^{n}(-\delta_j)_{k_j}\,. \tag{3.8}$$

Note, that for every analytic function $f(z)$ the following Poole's formula [22, p. 26, Eq. (33)]

$$\left(-z\frac{\partial}{\partial z}\right)_r\{f(z)\} = (-1)^r z^r\frac{d^r}{dz^r}\{f(z)\} \tag{3.9}$$

is valid.

The companion to the Poole's formula (3.9), the following operator identity [22, p. 93, Eq. (7)]

$$\left(a + z\frac{\partial}{\partial z}\right)_r \{f(z)\} = z^{1-a}\frac{d^r}{dz^r}\left\{z^{a+r-1}f(z)\right\} \tag{3.10}$$

is valid for every analytic function $f(z)$.

The operator identities (3.9) and (3.10) can be proved by the mathematical induction method.

4 The infinity summation formulas associated with one-dimensional inverse pair operators

Consider the inverse pair operators $H_{x_j}(\alpha,\beta)$ and $\bar{H}_{x_j}(\alpha,\beta)$, defined in (3.3) and (3.4), respectively ($1 \le j \le n$). A composition of these operators r times will be denoted by

$$H^r_{\mathbf{x}_r}(\mathbf{a}_r,\, \mathbf{b}_r) := H_{x_1}(a_1,b_1)\, H_{x_2}(a_2,b_2) \ldots H_{x_r}(a_r,b_r),$$

$$\bar{H}^r_{\mathbf{x}_r}(\mathbf{a}_r,\, \mathbf{b}_r) := \bar{H}_{x_1}(a_1,b_1)\, \bar{H}_{x_2}(a_2,b_2) \ldots \bar{H}_{x_r}(a_r,b_r),$$

where $\mathbf{a}_r := \left(a_1, ..., a_r\right)$, $\mathbf{b}_r := \left(b_1, ..., b_r\right)$ and $\mathbf{x}_r := \left(x_1, ..., x_r\right)$ are vectors with r components ($1 \le r \le n$).

Theorem 4.1 *Let n be a number of the variables of Lauricella's functions defined in (2.10)–(2.13). If r is a natural number and $1 \le r \le n$, then the following symbolic forms hold*

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = H^r_{\mathbf{x}_r}(\mathbf{b}_r,\mathbf{d}_r)\, F_A^{(n)}(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};\mathbf{c};\mathbf{x}), \tag{4.1}$$

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \bar{H}^r_{\mathbf{x}_r}(\mathbf{d}_r,\mathbf{b}_r)\, F_A^{(n)}(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};\mathbf{c};\mathbf{x}), \tag{4.2}$$

$$\begin{gathered} F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = H^r_{\mathbf{x}_r}(\mathbf{b}_r,\mathbf{c}_r)\,(1-X_r)^{-a} \times \\ \times F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\, \mathbf{c}_{r+1,n};\, \frac{x_{r+1}}{1-X_r}, ..., \frac{x_n}{1-X_r}\right); \end{gathered} \tag{4.3}$$

$$\begin{gathered} F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r}, ..., \frac{x_n}{1-X_r}\right) = \\ = (1-X_r)^a\, \bar{H}^r_{\mathbf{x}_r}(\mathbf{b}_r,\mathbf{c}_r)\, F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}); \end{gathered} \tag{4.4}$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x}) = H^r_{\mathbf{x}_r}(\mathbf{a}_r,\mathbf{d}_r)\, F_B^{(n)}(\mathbf{d}_r,\mathbf{a}_{r+1,n};c;\mathbf{x}), \tag{4.5}$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x}) = \bar{H}^r_{\mathbf{x}_r}(\mathbf{d}_r,\mathbf{a}_r)\, F_B^{(n)}(\mathbf{d}_r,\mathbf{a}_{r+1,n};c;\mathbf{x}); \tag{4.6}$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x}) = H^r_{\mathbf{x}_r}(\mathbf{d}_r,\mathbf{c}_r)\, F_C^{(n)}(a,b;\mathbf{d}_r,\mathbf{c}_{r+1,n};\mathbf{x}); \tag{4.7}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) = H^r_{\mathbf{x}_r}(\mathbf{b}_r,\mathbf{d}_r)\, F_D^{(n)}(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};c;\mathbf{x}), \tag{4.8}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) = \bar{H}^r_{\mathbf{x}_r}(\mathbf{d}_r,\mathbf{b}_r)\, F_D^{(n)}(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};c;\mathbf{x}), \tag{4.9}$$

where

$$\mathbf{b}_{r+1,n} := \left(b_{r+1}, ..., b_n\right),\;\; \mathbf{b}_{n+1,n} = \emptyset,\;\; \mathbf{c}_{r+1,n} := \left(c_{r+1}, ..., c_n\right),\;\; \mathbf{c}_{n+1,n} = \emptyset,$$

$$\mathbf{d}_r := \left(d_1, ..., d_r\right),\;\; X_r := x_1 + ... + x_r,\;\; 1 \le r \le n.$$

The symbolic forms (4.1)–(4.9) are used to obtain a large number of infinite summation formulas of the multiple Lauricella's hypergeometric functions. Namely, using the formulas (3.5), (3.6) and applying them many times, by virtue of Poole's formulas (3.9) and (3.10), we get

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(d_j-b_j)_{k_j}}{k_j!\,(c_j)_{k_j}}(-x_j)^{k_j}\right]\times$$

$$\times F_A^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{d}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right),\tag{4.10}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(b_j-d_j)_{k_j}}{k_j!\,(c_j)_{k_j}}x_j^{k_j}\right]\times$$

$$\times F_A^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{d}_r,\mathbf{b}_{r+1,n};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right),\tag{4.11}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=(1-X_r)^{-a}\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(c_j-b_j)_{k_j}}{k_j!\,(c_j)_{k_j}}\left(\frac{-x_j}{1-X_r}\right)^{k_j}\right]\times$$

$$\times F_A^{(n-r)}\left(a+|\mathbf{k}_r|,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right);\tag{4.12}$$

$$F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right)=(1-X_r)^{a}\times$$

$$\times\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(c_j-b_j)_{k_j}}{k_j!\,(c_j)_{k_j}}x_j^{k_j}\right]\cdot F_A^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{b};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right);\tag{4.13}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}\frac{1}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(d_j-a_j)_{k_j}(b_j)_{k_j}}{k_j!}(-x_j)^{k_j}\right]\times$$

$$\times F_B^{(n)}\left(\mathbf{d}_r+\mathbf{k}_r,\mathbf{a}_{r+1,n},\mathbf{b}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}\right),\tag{4.14}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}\frac{1}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(a_j-d_j)_{k_j}(b_j)_{k_j}}{k_j!}x_j^{k_j}\right]\times$$

$$\times F_B^{(n)}\left(\mathbf{d}_r,\mathbf{a}_{r+1,n},\mathbf{b}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}\right),\tag{4.15}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}(b)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(c_j-d_j)_{k_j}}{k_j!\,(c_j)_{k_j}}(-x_j)^{k_j}\right]\times$$

$$\times F_C^{(n)}\left(a+|\mathbf{k}_r|,b+|\mathbf{k}_r|;\mathbf{d}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}\right);\tag{4.16}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}\frac{(a)_{|\mathbf{k}_r|}}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(d_j-b_j)_{k_j}}{k_j!}(-x_j)^{k_j}\right]\times$$

$$\times F_D^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{d}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}\right),\tag{4.17}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}_r|=0}^{\infty}\frac{(a)_{|\mathbf{k}_r|}}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(b_j-d_j)_{k_j}}{k_j!}x_j^{k_j}\right]\times$$

$$\times F_D^{(n)}\left(a+|\mathbf{k}_r|,\mathbf{d}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}\right). \tag{4.18}$$

In all formulas (4.10)–(4.18), n and r are natural numbers and $1 \le r \le n$.

If $n = r$ in the equality (4.12), then we obtain a famous formula for the Lauricella's function $F_A^{(n)}$ [2, p. 116, Eq. (9)]:

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right) = \left(1-X_n\right)^{-a} F_A^{(n)}\left(a,\mathbf{c}-\mathbf{b};\mathbf{c};\frac{x_1}{X_n-1},...,\frac{x_n}{X_n-1}\right). \tag{4.19}$$

We note that the formula (4.19) is a natural generalization of the Boltz formula (2.3).

If $n = r$ in the equality (4.13), then the infinity summation of the Lauricella's function $F_A^{(n)}$ is an elementary function:

$$\sum_{|\mathbf{k}|=0}^{\infty}(a)_{|\mathbf{k}|}\prod_{j=1}^{n}\left[\frac{(c_j-b_j)_{k_j}}{k_j!\,(c_j)_{k_j}}x_j^{k_j}\right]\cdot F_A^{(n)}\left(a+|\mathbf{k}|,\mathbf{b};\mathbf{c}+\mathbf{k};\mathbf{x}\right) = \left(1-X_n\right)^{-a}. \tag{4.20}$$

It is easy to see, that the equality (4.20) generalizes a famous infinity summation formula for the Gaussian hypergeometric function (2.5).

5 The infinity summation formulas associated with multi-dimensional inverse pair operators

In this section, we consider the multi-dimensional inverse pair operators defined in (3.1) and (3.2).

Theorem 5.1 *The following symbolic forms are valid:*

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right) = H_{\mathbf{x}}\left(a,d\right)F_A^{(n)}\left(d,\mathbf{b};\mathbf{c};\mathbf{x}\right), \tag{5.1}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right) = \bar{H}_{\mathbf{x}}\left(d,a\right)F_A^{(n)}\left(d,\mathbf{b};\mathbf{c};\mathbf{x}\right); \tag{5.2}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right) = H_{\mathbf{x}}\left(d,c\right)F_B^{(n)}\left(\mathbf{a},\mathbf{b};d;\mathbf{x}\right), \tag{5.3}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right) = \bar{H}_{\mathbf{x}}\left(c,d\right)F_B^{(n)}\left(\mathbf{a},\mathbf{b};d;\mathbf{x}\right); \tag{5.4}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right) = H_{\mathbf{x}}\left(a,d\right)F_C^{(n)}\left(d,b;\mathbf{c};\mathbf{x}\right), \tag{5.5}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right) = \bar{H}_{\mathbf{x}}\left(d,a\right)F_C^{(n)}\left(d,b;\mathbf{c};\mathbf{x}\right), \tag{5.6}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right) = H_{\mathbf{x}}\left(a,d_1\right)H_{\mathbf{x}}\left(b,d_2\right)F_C^{(n)}\left(d_1,d_2;\mathbf{c};\mathbf{x}\right), \tag{5.7}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right) = \bar{H}_{\mathbf{x}}\left(d_1,a\right)\bar{H}_{\mathbf{x}}\left(d_2,b\right)F_C^{(n)}\left(d_1,d_2;\mathbf{c};\mathbf{x}\right); \tag{5.8}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right) = H_{\mathbf{x}}\left(a,d\right)F_D^{(n)}\left(d,\mathbf{b};c;\mathbf{x}\right), \tag{5.9}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right) = \bar{H}_{\mathbf{x}}\left(d,a\right)F_D^{(n)}\left(d,\mathbf{b};c;\mathbf{x}\right), \tag{5.10}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=H_{\mathbf{x}}\left(d,c\right)F_D^{(n)}\left(a,\mathbf{b};d;\mathbf{x}\right), \tag{5.11}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=H_{\mathbf{x}}\left(a,c\right)\prod_{j=1}^{n}\left(1-x_j\right)^{-b_j}; \tag{5.12}$$

$$\prod_{j=1}^{n}\left(1-x_j\right)^{-b_j}=\bar{H}_{\mathbf{x}}(a,c)F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right). \tag{5.13}$$

As can be seen from the theorem, each of $F_A^{(n)}$ and $F_B^{(n)}$ has two symbolic forms, and each of $F_C^{(n)}$ and $F_D^{(n)}$ has four symbolic forms. The symbolic forms (5.1)–(5.13) are used to obtain a large number of the infinity summation formulas of multiple Lauricella's functions. For this purpose, in addition to the formulas (3.7) and (3.8), we will also use the following equalities:

$$H_{\mathbf{x}}\left(a,d_1\right)H_{\mathbf{x}}\left(b,d_2\right)=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{\left(d_1-a\right)_{|\mathbf{k}|}\left(d_2-b\right)_{|\mathbf{l}|}\left(b\right)_{|\mathbf{k}|}}{K!L!\left(d_1\right)_{|\mathbf{k}|}\left(d_2\right)_{|\mathbf{k}|+|\mathbf{l}|}}\prod_{j=1}^{n}\left(-\delta_j\right)_{k_j+l_j}, \tag{5.14}$$

$$\bar{H}_{\mathbf{x}}\left(d_1,a\right)\bar{H}_{\mathbf{x}}\left(d_2,b\right)=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\left(a-d_1\right)_{|\mathbf{k}|}}{K!L!\left(b-d_2\right)_{|\mathbf{k}|}\left(1-d_1-\delta_1-...-\delta_n\right)_{|\mathbf{k}|}}\times$$

$$\times\frac{\left(b-d_2\right)_{|\mathbf{k}|+|\mathbf{l}|}\left(b\right)_{|\mathbf{k}|}}{\left(1-d_2-\delta_1-...-\delta_n\right)_{|\mathbf{k}|+|\mathbf{l}|}}\prod_{j=1}^{n}\left(-\delta_j\right)_{k_j+l_j}. \tag{5.15}$$

Applying the formulas (3.7) and (3.8) twice, one can easily obtain the equalities (5.14) and (5.15), respectively.

Therefore, we have the following infinity summation formulas:

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\left(d-a\right)_{|\mathbf{k}|}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}$$

$$\times F_A^{(n)}\left(d+|\mathbf{k}|,\mathbf{b}+\mathbf{k};\mathbf{c}+\mathbf{k};\mathbf{x}\right), \tag{5.16}$$

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{\left(a-d\right)_{|\mathbf{k}|}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_A^{(n)}\left(d,\mathbf{b}+\mathbf{k};\mathbf{c}+\mathbf{k};\mathbf{x}\right); \tag{5.17}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\left(c-d\right)_{|\mathbf{k}|}\left(\mathbf{a}\right)_{\mathbf{k}}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!\left(d\right)_{|\mathbf{k}|}\left(c\right)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}$$

$$\times F_B^{(n)}\left(\mathbf{a}+\mathbf{k},\mathbf{b}+\mathbf{k};d+|\mathbf{k}|;\mathbf{x}\right), \tag{5.18}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{\left(d-c\right)_{|\mathbf{k}|}\left(\mathbf{a}\right)_{\mathbf{k}}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!\left(d\right)_{|\mathbf{k}|}\left(c\right)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_B^{(n)}\left(\mathbf{a}+\mathbf{k},\mathbf{b}+\mathbf{k};d;\mathbf{x}\right); \tag{5.19}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\left(d-a\right)_{|\mathbf{k}|}\left(b\right)_{|\mathbf{k}|}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}$$

$$\times F_C^{(n)}\left(d+|\mathbf{k}|, b+|\mathbf{k}|; \mathbf{c}+\mathbf{k}; \mathbf{x}\right), \tag{5.20}$$

$$F_C^{(n)}\left(a, b; \mathbf{c}; \mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty} \frac{(a-d)_{|\mathbf{k}|}(b)_{|\mathbf{k}|}}{K!(\mathbf{c})_{\mathbf{k}}} \mathbf{x}^{\mathbf{k}} F_C^{(n)}\left(d, b+|\mathbf{k}|; \mathbf{c}+\mathbf{k}; \mathbf{x}\right), \tag{5.21}$$

$$F_C^{(n)}\left(a, b; \mathbf{c}; \mathbf{x}\right)=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty} \frac{(-1)^{|\mathbf{k}|+|\mathbf{l}|}\left(d_1-a\right)_{|\mathbf{k}|}\left(d_2-b\right)_{|\mathbf{l}|}(b)_{|\mathbf{k}|}\left(d_1\right)_{|\mathbf{k}|+|\mathbf{l}|}}{K!L!\left(d_1\right)_{|\mathbf{k}|}(\mathbf{c})_{\mathbf{k}+\mathbf{l}}} \mathbf{x}^{\mathbf{k}+\mathbf{l}} \times$$

$$\times F_C^{(n)}\left(d_1+|\mathbf{k}|+|\mathbf{l}|, d_2+|\mathbf{k}|+|\mathbf{l}|; \mathbf{c}+\mathbf{k}+\mathbf{l}; \mathbf{x}\right), \tag{5.22}$$

$$F_C^{(n)}\left(a, b; \mathbf{c}; \mathbf{x}\right)=\sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty} \frac{(-1)^{|\mathbf{l}|}\left(a-d_1\right)_{|\mathbf{k}|}\left(b-d_2\right)_{|\mathbf{k}|+|\mathbf{l}|}(b)_{|\mathbf{k}|}}{K!L!\left(b-d_2\right)_{|\mathbf{k}|}(\mathbf{c})_{\mathbf{k}+\mathbf{l}}} \mathbf{x}^{\mathbf{k}+\mathbf{l}}$$

$$\times F_C^{(n)}\left(d_1+|\mathbf{l}|, d_2+|\mathbf{l}|; \mathbf{c}+\mathbf{k}+\mathbf{l}; \mathbf{x}\right); \tag{5.23}$$

$$F_D^{(n)}\left(a, \mathbf{b}; c; \mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty} \frac{(-1)^{|\mathbf{k}|}(d-a)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}} \mathbf{x}^{\mathbf{k}}$$

$$\times F_D^{(n)}\left(d+|\mathbf{k}|, \mathbf{b}+\mathbf{k}; c+|\mathbf{k}|; \mathbf{x}\right), \tag{5.24}$$

$$F_D^{(n)}\left(a, \mathbf{b}; c; \mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty} \frac{(a-d)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}} \mathbf{x}^{\mathbf{k}} F_D^{(n)}\left(d, \mathbf{b}+\mathbf{k}; c+|\mathbf{k}|; \mathbf{x}\right), \tag{5.25}$$

$$F_D^{(n)}\left(a, \mathbf{b}; c; \mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty} \frac{(-1)^{|\mathbf{k}|}(a)_{|\mathbf{k}|}(c-d)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(d)_{|\mathbf{k}|}(c)_{|\mathbf{k}|}} \mathbf{x}^{\mathbf{k}}$$

$$\times F_D^{(n)}\left(a+|\mathbf{k}|, \mathbf{b}+\mathbf{k}; d+|\mathbf{k}|; \mathbf{x}\right), \tag{5.26}$$

$$F_D^{(n)}\left(a, \mathbf{b}; c; \mathbf{x}\right)=\prod_{j=1}^{n}\left[\left(1-x_j\right)^{-b_j}\right] \cdot F_D^{(n)}\left(c-a, \mathbf{b}; c; \frac{x_1}{x_1-1}, ..., \frac{x_n}{x_n-1}\right); \tag{5.27}$$

$$\prod_{j=1}^{n}\left[\left(1-x_j\right)^{-b_j}\right]=\sum_{|\mathbf{k}|=0}^{\infty} \frac{(c-a)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}} \mathbf{x}^{\mathbf{k}} F_D^{(n)}\left(a, \mathbf{b}+\mathbf{k}; c+|\mathbf{k}|; \mathbf{x}\right). \tag{5.28}$$

The infinity summation formulas (4.10)–(4.18) and (5.16)–(5.28) can be proved without symbolic methods by comparing coefficients of equal powers of $x_1, x_2, ..., x_n$ on both sides.

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