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Maximal commutator and commutator of maximal operator on Lorentz spaces

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Abstract. *In this paper, we study the boundedness of the maximal commutator M_b and the commutators of the maximal operator $[b, M]$ in the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$. We give necessary and sufficient conditions for the boundedness of the operators M_b and $[b, M]$ on Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ when b belongs to $BMO(\mathbb{R}^n)$ spaces, whereby some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$ spaces are obtained.*

Keywords. Maximal operator, commutator, Lorentz space, BMO space

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1 Introduction

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance [7]). For $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the maximal operator M is defined by

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the ball of radius r centered at $x \in \mathbb{R}^n$, ${}^{\complement}B(x,r)$ is its complement and $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$.

The maximal commutator generated by the operator M and $b \in L^1_{\rm loc}(\mathbb{R}^n)$ is defined by

$$M_b f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy.$$

The commutators generated by the operator M and a suitable function b is defined by

$$[b, M]f(x) = b(x)Mf(x) - M(bf)(x).$$

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Obviously, the operators M_b and $[b, M]$ essentially differ from each other since M_b is positive and sublinear and $[b, M]$ is neither positive nor sublinear. The operators M, $[b, M]$ and M_b play an important role in real and harmonic analysis and applications (see, for instance [1,2,6,10–12,14]).

The commutator estimates have many important applications, for example, in studying the regularity and boundedness of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and in characterizing certain function spaces (see, for instance [5,7]). The boundedness of the Hardy-Littlewood maximal operator M on $L^p(\mathbb{R}^n)$ is one of the most fundamental results in harmonic analysis. It has been extended to a range of other function spaces, and to many variations of the standard maximal operator. In particular, one can study commutators of M with BMO functions b. These turn out to be L^p bounded for $1 < p < \infty$ if and only if $b \in BMO$ and $b^- \equiv -\min\{b, 0\} \in L^\infty(\mathbb{R}^n)$ [2]. This is useful, for instance, when studying the product of an H^1 function with a BMO function [4]. Note that, the boundedness of the operator M_b on L^p spaces was proved by Garcia-Cuerva et al. [6].

In this paper we obtain necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and commutators of maximal operator $[b, M]$ on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$. We give some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

The structure of the paper is as follows. In Section 2 we give some definitions and auxiliary results. In Section 3 we obtain necessary and sufficient conditions for the boundedness of the maximal commutator M_b on $L^{p,q}(\mathbb{R}^n)$ Lorentz spaces. In Section 4 we give necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b, M]$ on $L^{p,q}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Definition and some basic properties

We start with the definition of Lorentz spaces. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar L^p spaces, also they are appear to be useful in the general interpolation theory.

Suppose that f is a measurable function on $\mathbb{R}^n$, then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\},$$

where

$$d_f(s) := |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad \forall s > 0.$$

Definition 2.1 *[3] The Lorentz space $L^{p,q} \equiv L^{p,q}(\mathbb{R}^n)$, $0 < p, q \le \infty$ is the collection of all measurable functions f on $\mathbb{R}^n$ such the quantity*

$$\|f\|_{L^{p,q}} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)} \tag{2.1}$$

is finite. Clearly $L^{p,p} \equiv L^p$ and $L^{p,1} \equiv WL^p$. The functional $\|\cdot\|_{L^{p,q}}$ is a norm if and only if either $1 \le q \le p$ or $p = q = \infty$.

Lemma 2.1 *[3] Let $1 < p, p', q, q' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Suppose that $f \in L^{p,q}(\mathbb{R}^n)$ and $f \in L^{p',q'}(\mathbb{R}^n)$. Then*

$$\|fg\|_{L^1(\mathbb{R}^n)} \le 2\|f\|_{L^{p,q}(\mathbb{R}^n)} \|g\|_{L^{p',q'}(\mathbb{R}^n)}.$$

The following result completely characterizes the boundedness of M on Lorentz spaces.

Lemma 2.2 *[3] Let $1 \le p, q \le \infty$.*
(i) *If $1 < p \le \infty$, then the operator M is bounded on the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.*
(ii) *If $p = 1$, then the operator M is bounded from $L^{1,q}(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$.*

3 $L^{p,q}$-boundedness of the maximal commutator operator M_b

In this section we find necessary and sufficient conditions for the boundedness of the maximal commutator M_b on $L^{p,q}(\mathbb{R}^n)$ Lorentz spaces.

Definition 3.1 *We define the space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norm*

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| dy < \infty,$$

where $f_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} f(y)dy$.

Lemma 3.1 *([1, Corollary 1.11]) If $b \in BMO(\mathbb{R}^n)$, then there exists a positive constant C such that*

$$M_b f(x) \le C\|b\|_* M^2 f(x) \tag{3.1}$$

for almost every $x \in \mathbb{R}^n$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 3.1 *Let $p, q \in (1, \infty)$. The following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$.*
(ii) *The operator M_b is bounded on $L^{p,q}(\mathbb{R}^n)$.*
(iii) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\|(b(\cdot) - b_B)\chi_B\|_{L^{p,q}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p,q}(\mathbb{R}^n)}} \le C. \tag{3.2}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\|(b(\cdot) - b_B)\chi_B\|_{L^1(\mathbb{R}^n)}}{|B|} \le C. \tag{3.3}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Lemmas 2.2 and 3.1, we get

$$\begin{aligned}\|M_b f\|_{L^{p,q}} &\lesssim \|b\|_* \|M^2 f\|_{L^{p,q}} \\ &\lesssim \|b\|_* \|M f\|_{L^{p,q}} \\ &\lesssim \|b\|_* \|f\|_{L^{p,q}}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that M_b is bounded on $L^{p,q}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. We consider $f = \chi_B$. It is easy to compute that

$$\|\chi_B\|_{L^{p,q}} \approx r^{\frac{n}{p}}. \tag{3.4}$$

On the other hand, for all $x \in B$ we have

$$\begin{aligned}|b(x) - b_B| &\le \frac{1}{|B|} \int_B |b(x) - b(y)| dy \\ &= \frac{1}{|B|} \int_B |b(x) - b(y)|\, \chi_B(y) dy \\ &\le M_b(\chi_B)(x).\end{aligned}$$

Since M_b is bounded on $L^{p,q}(\mathbb{R}^n)$, then by (3.4) we obtain

$$\frac{\|(b-b_B)\chi_B\|_{L^{p,q}}}{\|\chi_B\|_{L^{p,q}}} \le \frac{\|M_b(\chi_B)\|_{L^{p,q}}}{\|\chi_B\|_{L^{p,q}}} \lesssim \frac{\|\chi_B\|_{L^{p,q}}}{\|\chi_B\|_{L^{p,q}}} = 1, \tag{3.5}$$

which implies that (3.2) holds since the ball $B \subset \mathbb{R}^n$ is arbitrary.

$(iii) \Rightarrow (iv)$. Assume that (3.2) holds, we will prove (3.3). For any fixed ball B, by Lemma 2.1, inequalities (3.2) and (3.4), it is easy to see

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-b(y)|dy &\lesssim \frac{1}{|B|}\,\|(b-b_B)\chi_B\|_{L^{p,q}}\,\|\chi_B\|_{L^{p',r'}}\\ &\lesssim \frac{\|(b-b_B)\chi_B\|_{L^{p,q}}}{\|\chi_B\|_{L^{p,q}}}\\ &\lesssim 1.\end{aligned}$$

$(iv) \Rightarrow (i)$. For any fixed ball B, we have

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-b_B|dy &= \frac{\|(b-b_B)\chi_B\|_{L^1}}{|B|}\\ &\le \sup_B \frac{\|(b-b_B)\chi_B\|_{L^1}}{|B|}\\ &\lesssim 1,\end{aligned}$$

which implies that $b \in BMO(\mathbb{R}^n)$. Thus the proof of the theorem is completed.

4 $L^{p,q}$-boundedness of the commutator of maximal operator $[b, M]$

In this section we obtain necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b, M]$ on $L^{p,q}(\mathbb{R}^n)$ Lorentz spaces.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M]$ and M_b are valid :

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned}&\big|[b,M]f(x)\big| = \big|b(x)Mf(x) - M(bf)(x)\big|\\ &= \big|M(b(x)f)(x) - M(bf)(x)\big| \le M(|b(x)-b|f)(x) = M_b f(x).\end{aligned}$$

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b,M]f(x)| \le M_b f(x) + 2b^-(x)\,Mf(x), \qquad x \in \mathbb{R}^n \tag{4.1}$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example [8, 14]).

Denote by $M_b f$ the local maximal function of f:

$$M_B f(x) := \sup_{B' \ni x:\, B' \subset B} \frac{1}{|B'|}\int_{B'} |f(y)|\,dy,\ x \in \mathbb{R}^n.$$

Applying Theorem 3.1, we obtain the following result.

Theorem 4.1 *Let $p, q \in (1, \infty)$. The following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*
(ii) *The operator $[b, M]$ is bounded on $L^{p,q}(\mathbb{R}^n)$.*
(iii) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B\big\|_{L^{p,q}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p,q}(\mathbb{R}^n)}} \le C. \tag{4.2}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B\big\|_{L^1(\mathbb{R}^n)}}{|B|} \le C. \tag{4.3}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$. Combining Lemma 2.2 and Theorem 3.1, and inequality (4.1), we get

$$\begin{aligned}
\|[b, M]f\|_{L^{p,q}} &\le \|M_b f + 2b^- \, Mf\|_{L^{p,q}} \\
&\le \|M_b f\|_{L^{p,q}} + \|b^-\|_{L^\infty} \, \|Mf\|_{L^{p,q}} \\
&\lesssim \big(\|b\|_* + \|b^-\|_{L^\infty}\big) \, \|f\|_{L^{p,q}}.
\end{aligned}$$

Thus, we obtain that $[b, M]$ is bounded on $L^{p,q}(\mathbb{R}^n)$.

$(ii) \Rightarrow (iii)$. Assume that $[b, M]$ is bounded on $L^{p,q}(\mathbb{R}^n)$. Let $B = B(x, r)$ be a fixed ball. Since

$$M(b\chi_B)\chi_B = M_B(b) \quad \text{and} \quad M(\chi_B)\chi_B = \chi_B,$$

we have

$$\begin{aligned}
|M_B(b) - b\chi_B| &= |M(b\chi_B)\chi_B - bM(\chi_B)\chi_B| \\
&\le |M(b\chi_B) - bM(\chi_B)| = |[b, M]\chi_B|.
\end{aligned}$$

Hence

$$\|M_B(b) - b\chi_B\|_{L^{p,q}(\mathbb{R}^n)} \le \|[b, M]\chi_B\|_{L^{p,q}(\mathbb{R}^n)}.$$

Thus we get

$$\frac{\|\big(b - M_B(b)\big)\chi_B\|_{L^{p,q}}}{\|\chi_B\|_{L^{p,q}}} \le \frac{\|[b, M](\chi_B)\|_{L^{p,q}}}{\|\chi_B\|_{L^{p,q}}} \lesssim \frac{\|\chi_B\|_{L^{p,q}}}{\|\chi_B\|_{L^{p,q}}} = 1,$$

which deduces that (iii).

$(iii) \Rightarrow (iv)$. Assume that (4.2) holds, then for any fixed ball B, by Lemma 2.1, we conclude that

$$\begin{aligned}
\frac{1}{|B|}\int_B |b(x) - M_B(b)(x)| dx &\lesssim \frac{1}{|B|} \, \|\big(b - M_B(b)\big)\chi_B\|_{L^{p,q}} \, \|\chi_B\|_{L^{p',q'}} \\
&\lesssim \frac{\|\big(b - M_B(b)\big)\chi_B\|_{L^{p,q}}}{\|\chi_B\|_{L^{p,q}}} \\
&\lesssim 1.
\end{aligned}$$

$(iv) \Rightarrow (i)$. Assume that (4.3) holds, we will prove $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.
Denote by

$$E := \{x \in B : \, b(x) \le b_B\}, \quad F := \{x \in B : \, b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B|\, dt = \int_F |b(t) - b_B|\, dt,$$

in view of the inequality $b(x) \le b_B \le M_B(b)$, $x \in E$, we get

$$\begin{aligned} \frac{1}{|B|}\int_B |b - b_B| &= \frac{2}{|B|}\int_E |b - b_B| \\ &\le \frac{2}{|B|}\int_E |b - M_B(b)| \\ &\le \frac{2}{|B|}\int_B |b - M_B(b)| \lesssim c. \end{aligned}$$

Consequently, $b \in BMO(\mathbb{R}^n)$. In order to show that $b^- \in L^\infty(\mathbb{R}^n)$, note that $M_B(b) \ge |b|$. Hence

$$0 \le b^- = |b| - b^+ \le M_B(b) - b^+ + b^- = M_B(b) - b.$$

Thus

$$(b^-)_B \le c,$$

and by the Lebesgue Differentiation theorem we get that

$$0 \le b^-(x) = \lim_{|B|\to 0} \frac{1}{|B|}\int_B b^-(y)dy \le c \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Thus the proof of the theorem is completed.

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