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Approximate solution of some classes of hypersingular integral equations of the second kind

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Abstract. *In this work, using regularization method, we reduce the hypersingular integral equations of the exterior Neumann boundary value problem and the impedance exterior boundary value problem for the Helmholtz equation to the weakly singular integral equations. Then, after having constructed quadrature formulas for one class of curvilinear integrals, we replace the considered integral equations with the system of algebraic equations. We prove that the obtained systems of algebraic equations are uniquely solvable and the solutions of these systems converge to the solutions of the considered hypersingular integral equations.*

Keywords. Neumann boundary value problem, impedance boundary value problem, Helmholtz equation, integral equations method, curvilinear hypersingular integral, collocation method.

Mathematics Subject Classification (2010): 65R20, 45E05

1 Introduction and problem statement

As is known, in special cases (when the considered domain is a circle, a square etc.) it is possible to find an exact solution of the exterior boundary value problems for the Helmholtz equation in two-dimensional space. But, in many cases it is impossible to find an exact solution of the exterior boundary value problems for the Helmholtz equation. This generates interest for studying approximate solution of these problems. One of the methods to solve exterior boundary value problems for the Helmholtz equation is to reduce it to an integral equation of the second kind. Note that the main advantage of applying the integral equations method to exterior boundary value problems is that this method allows reducing the problem for an unbounded domain to the one for a bounded domain of lesser dimension.

Let $D \subset \mathbb{R}^2$ be a bounded domain with twice continuously differentiable boundary L, and f, g and λ be the given continuous functions on L. Consider the following boundary value problems for the Helmholtz equation:

Exterior Neumann boundary value problem. Find a function

$$u \in C^2\left(\mathbb{R}^2 \backslash \bar{D}\right) \cap C\left(\mathbb{R}^2 \backslash D\right),$$

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which has a normal derivative in the sense of uniform convergence, i.e. the limit

$$\frac{\partial u(x)}{\partial \nu(x)} = \lim_{\substack{h\to 0\\ h>0}} \left(\nu(x), grad u(x+h\nu(x))\right), \quad x\in L,$$

exists uniformly in L, satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^2 \backslash \bar{D}$, Sommerfeld radiation condition

$$\left(\frac{x}{|x|}, grad u(x)\right) - i\, k u(x) = o\left(\frac{1}{|x|^{1/2}}\right), \quad x\to\infty,$$

uniformly in all directions $x/|x|$ and the boundary condition

$$\frac{\partial u(x)}{\partial v(x)} = f(x) \text{ on } L,$$

where $\nu(x)$ is an outer unit normal at the point $x\in L$, Δ is a Laplace operator, and k is a wave number with $Im\, k \geq 0$.

Impedance exterior boundary value problem. Find a function

$$u\in C^2\left(\mathbb{R}^2\backslash \bar{D}\right)\cap C\left(\mathbb{R}^2\backslash D\right),$$

which has a normal derivative in the sense of uniform convergence, satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\bar{D}$, Sommerfeld radiation condition at infinity and the boundary condition

$$\frac{\partial u(x)}{\partial \nu(x)} + \lambda(x)\, u(x) = g\,(x) \text{ on } L,$$

where $Im\left(\bar{k}\,\lambda(x)\right)\geq 0$, $x\in L$.

Let the function $u(x)$ be a solution of the exterior Neumann boundary value problem for the Helmholtz equation. It was shown in [3, p. 103] that the unknown boundary values $\psi(x) = u(x)$, $x\in L$ satisfy the second kind integral boundary condition

$$\psi - K\psi = -Sf \tag{1.1}$$

and the hypersingular integral equation of the first kind

$$T\psi = f + \tilde{K} f, \tag{1.2}$$

where

$$(S\varphi)(x) = 2\int_L \Phi_k(x,y)\,\varphi(y)\,dl_y, \;\; x\in L,$$

$$(K\varphi)(x) = 2\int_L \frac{\partial\Phi_k(x,y)}{\partial\nu\,(y)}\varphi(y)\,dl_y, \;\; x\in L,$$

$$\left(\tilde{K}\varphi\right)(x) = 2\int_L \frac{\partial\Phi_k(x,y)}{\partial\nu\,(x)}\varphi(y)\,dl_y, \;\; x\in L,$$

$$(T\varphi)(x) = 2\frac{\partial}{\partial\nu\,(x)}\left(\int_L \frac{\partial\Phi_k(x,y)}{\partial\nu\,(y)}\varphi(y)\,dl_y\right), \;\; x\in L,$$

$\Phi(x,\, y)$ is a fundamental solution of the Helmholtz equation, i.e.

$$\Phi_k(x,\,y) = \begin{cases} \frac{1}{2\pi}\ln\frac{1}{|x-y|} & for \quad k=0, \\ \frac{i}{4}H_0^{(1)}(k\,|x-y|) & for \quad k\neq 0, \end{cases}$$

$H_0^{(1)}$ is a zero degree Hankel function of the first kind defined by the formula $H_0^{(1)}(z) = J_0(z) + i\,N_0(z)$,

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

is a Bessel function of zero degree,

$$N_0(z) = \frac{2}{\pi} \left(\ln \frac{z}{2} + C\right) J_0(z) + \sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l}\right) \frac{(-1)^{m+1}}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

is a Neumann function of zero degree, and $C = 0.57721...$ is an Euler's constant.

Lyapunov's counterexample shows ([4, p. 89–90]) that the derivative of the double-layer potential with continuous density, in general, does not exist. Consequently, the operator T is not defined in the space $C(L)$ of all functions continuous on the curve L with the norm $\|\varphi\|_\infty = \max_{x\in L} |\varphi(x)|$. Besides, in spite of solvability of the integral equations (1.1) and (1.2), the equation (1.1) has a unique solution if and only if the wave number k does not coincide with the eigenvalue of the interior Dirichlet problem, and the equation (1.2) has a unique solution if and only if the wave number k does not coincide with the eigenvalue of the interior Neumann problem. But, it was shown in [3, p. 103] that if the function $u(x)$ has a normal derivative in the sense of uniform convergence, then the hypersingular integral equation of the second kind

$$\psi - K\psi - i\eta T\psi = -Sf - i\eta \left(f + \tilde{K} f\right), \tag{1.3}$$

obtained from the linear combinations of the equations (1.1) and (1.2), is uniquely solvable in $\mathrm{N}(L)$, the linear space of of all continuous functions ψ, whose double-layer potential with the density ψ has continuous normal derivatives on both sides of L, where $\eta \neq 0$ is an arbitrary real number with $\eta\, Rek \geq 0$. Note that the exterior Neumann boundary value problem for the Helmholtz equation can be reduced to various integral equations, whose approximate solutions have been studied in [1, 5, 12, 18]. The equation (1.3) has an advantage that its solution is a boundary value of the solution of the exterior Neumann boundary value problem on L. Besides, the function

$$u(x) = \int_L \left\{\psi(y)\, \frac{\partial \Phi_k(x,y)}{\partial \nu(y)} - f(y)\, \Phi_k(x,y)\right\} dl_y, \;\; x \in \mathbb{R}^2 \backslash \bar{D},$$

is a solution of the exterior Neumann boundary value problem if $\psi \in \mathrm{N}(L)$ is a solution of the hypersingular integral equation (1.3). Also, it should be noted that the solution of the equation (1.3) is a solution of the equation of the zero field method obtained by Waterman [17] for acoustic wave scattering.

Further, in [3, p. 98] it was shown that the combination of the simple-layer and double-layer potentials

$$u(x) = \int_L \left\{\Phi_k(x,y) + i\,\eta\, \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\right\} \varphi(y)\, dl_y, \;\; x \in \mathbb{R}^2 \backslash \bar{D},$$

where $\eta \neq 0$ is an arbitrary real number with $\eta\, Rek \geq 0$, is a solution of the impedance exterior boundary value problem for the Helmholtz equation if the density φ is a solution of the hypersingular integral equation

$$(1 - i\,\eta\,\lambda)\, \varphi - \left(\tilde{K} + i\,\eta\, T + i\,\eta\,\lambda\, K + \lambda\, S\right) \varphi = -2\, g. \tag{1.4}$$

Note that in [11], the justification of the collocation method for the hypersingular integral equation of the exterior Neumann boundary value problem has been given, and in [7], the justification of the collocation method for the hypersingular integral equation of the impedance exterior boundary value problem for the Helmholtz equation has been provided in three-dimensional space. But, it is known that the fundamental solution of the Helmholtz equation in three-dimensional space has the form

$$\Phi_k(x,\,y) = \frac{\exp\left(ik\left|x-y\right|\right)}{4\pi\left|x-y\right|},\ \ x,y\in\mathbb{R}^3,\ \ x\neq y,$$

and therefore, the integral operators appearing in the equations (1.3) and (1.4) differ strictly from those appearing in the integral equations of the exterior Neumann boundary value problem and impedance exterior boundary value problem for the Helmholtz equation in three-dimensional space.

Note that in [13], the approximate solution methods for one class of hypersingular integral equations of the exterior Neumann boundary value problem for the Helmholtz equation have been studied. In that work, after discretization, the author obtains hypersingular integral equations with simpler kernels. And in our work, we explore the approximate solution methods for the hypersingular integral equations (1.3) and (1.4) by reducing them to the weakly singular integral equation, which allows to find the solution of the obtained equations in a larger space and to impose weaker conditions on the given function f.

2 Justification of collocation method for hypersingular integral equation (1.3).

As the operator T is unbounded in the space $\mathrm{N}\,(L)$ ([3, p. 62]), let's perform a regularization of the equation (1.3). Let the wave number k_0 not coincide with the eigenvalues of the interior Dirichlet or Neumann problems (for this, it suffices to choose any value of k_0 with $Im\,k_0 > 0$). In the sequel, we will assign zero index to our notations if the parameter k, involved in the operators S, $\tilde{K}$ and T, is equal to k_0. As the operator

$$A_0 = -S_0\left(I-\tilde{K}_0\right)^{-1}\left(I+\tilde{K}_0\right)^{-1} : C\,(L)\to \mathrm{N}\,(L)$$

is an inverse operator to $T_0 : \mathrm{N}\,(L) \to C\,(L)$ ([3, p. 93]), the equation (1.3) can be rewritten in the following equivalent form:

$$\psi + A\,\psi = Bf. \tag{2.1}$$

The last equation is considered in the space $C\,(L)$, where I is a unit operator in $C\,(L)$,

$$A\psi = \frac{1}{i\,\eta}A_0\left(K + i\,\eta\ (T-T_0) - I\right)\,\psi,$$

$$Bf = \frac{1}{i\,\eta}\,A_0\left(S + i\eta\,\left(I+\tilde{K}\right)\right)f.$$

It should be noted that the operators S, K and $T - T_0$ are compact in the space $C\,(L)$ (see [3, p. 61-62]), and, therefore, the operator A is also compact in $C\,(L)$ (see [3, p. 93]). But despite the invertibility of the operators $I + \tilde{K}_0$ and $I - \tilde{K}_0$, the explicit forms of the inverse operators $\left(I+\tilde{K}_0\right)^{-1}$ and $\left(I-\tilde{K}_0\right)^{-1}$ are unknown. Consequently, the explicit forms of the operators A and B are also unknown.

Remark 2.1 In [13], the solution of the equation obtained after discretization is considered in the space $C^{1,\alpha}(L)$, and the given function f satisfies the condition $f \in C^{0,\beta}(L)$, where $C^{0,\beta}(L)$ is a Hölder space with index β, and $C^{1,\alpha}(L)$ is a space of continuously differentiable functions whose derivative satisfies the Hölder condition with exponent α, with $0 < \alpha \leq \beta < 1$. As we see, the solution of the equation (2.1) is considered in the space $C(L)$ and $f \in C(L)$. This is one of the advantages of our method.

To justify the collocation method, let's first construct the quadrature formulas for $(A\psi)(x)$ and $(Bf)(x)$, $x \in L$. Assume that the curve L is defined by the parametric equation $x(t) = (x_1(t), x_2(t))$, $t \in [a, b]$. Let's divide the interval $[a, b]$ into $n > 2M_0(b-a)/d$ equal parts: $t_p = a + \frac{(b-a)\,p}{n}$, $p = \overline{0, n}$, where

$$M_0 = \max_{t \in [a,b]} \sqrt{(x_1'(t))^2 + (x_2'(t))^2} < +\infty$$

([14, p. 560]) and d is a standard radius ([16, p. 400]). As control points, we consider $x(\tau_p)$, $p = \overline{1, n}$, where $\tau_p = a + \frac{(b-a)(2p-1)}{2n}$. Then the curve L is divided into elementary parts: $L = \bigcup\limits_{p=1}^{n} L_p$, where $L_p = \{x(t) : \ t_{p-1} \leq t \leq t_p\}$.

It is known ([2]) that

(1) $\forall p \in \{1, 2, ..., n\}$: $r_p(n) \sim R_p(n)$, where

$$r_p(n) = \min\{|x(\tau_p) - x(t_{p-1})|\,,\ |x(t_p) - x(\tau_p)|\}\,,$$

$$R_p(n) = \max\{|x(\tau_p) - x(t_{p-1})|\,,\ |x(t_p) - x(\tau_p)|\}\,,$$

and $a(n) \sim b(n)$ means

$$C_1 \leq \frac{a(n)}{b(n)} \leq C_2,$$

where C_1 and C_2 are positive constants independent of n;

(2) $\forall p \in \{1, 2, ..., n\} : R_p(n) \leq d/2$;

(3) $\forall p, j \in \{1, 2, ..., n\} : r_j(n) \sim r_p(n)$;

(4) $r(n) \sim R(n) \sim \frac{1}{n}$, where $R(n) = \max\limits_{p=\overline{1,n}} R_p(n)$, $r(n) = \min\limits_{p=\overline{1,n}} r_p(n)$.

In the sequel, we will call this kind of division a division of the curve L into "regular" elementary parts.

Let $L_d(x)$ and $\Gamma_d(x)$ be parts of the curve L and tangent line $\Gamma(x)$ at the point $x \in L$, respectively, contained inside the circle $B_d(x)$ of radius d with centre at the point x. Besides, let $\tilde{y} \in \Gamma(x)$ be a projection of the point $y \in L$. Then

$$|x - \tilde{y}| \leq |x - y| \leq C_1(L)\,|x - \tilde{y}| \ \text{ and } \ mesL_d(x) \leq C_2(L)\, mes\Gamma_d(x)\,,$$

where $C_1(L)$ and $C_2(L)$ are positive constants, depending only on L (if L is a circumference, then $C_1(L) = \sqrt{2}$ and $C_2(L) = 2$).

Proceeding as in the proof of Lemma 2.1 of [8], we can prove the validity of the following lemma.

Lemma 2.1 *There exist the constants $C_0' > 0$ and $C_1' > 0$, independent of n, such that the inequalities*

$$C_0'\ |y - x(\tau_p)| \leq |x(\tau_j) - x(\tau_p)| \leq C_1'\ |y - x(\tau_p)|$$

hold for $\forall p, j \in \{1,\ 2, ...,\ n\}$, $j \neq p$, and $\forall y \in L_j$.

Let

$$\Phi_k^n(x,\, y) = \frac{i}{4} H_{0,n}^{(1)}\left(k\left|x-y\right|\right), \quad x, y \in L, x \neq y,$$

where

$$H_{0,n}^{(1)}\left(z\right) = J_{0,n}\left(z\right) + i\, N_{0,n}\left(z\right), \quad J_{0,n}\left(z\right) = \sum_{m=0}^{n} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

and

$$N_{0,n}\left(z\right) = \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right) J_{0,n}\left(z\right) + \sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}}{(m!)^2}\left(\frac{z}{2}\right)^{2m}.$$

It was proved in [9] and [10] that the expressions

$$\left(S_n f\right)\left(x\left(\tau_p\right)\right) = \frac{2\left(b-a\right)}{n}\sum_{\substack{j=1\\ j\neq p}}^{n} \Phi_k^n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)\sqrt{\left(x_1'\left(\tau_j\right)\right)^2 + \left(x_2'\left(\tau_j\right)\right)^2}\, f\left(x\left(\tau_j\right)\right), \tag{2.2}$$

$$\left(K_n \psi\right)\left(x\left(\tau_p\right)\right) = \frac{2\left(b-a\right)}{n}\sum_{\substack{j=1\\ j\neq p}}^{n} \frac{\partial \Phi_k^n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2 + \left(x_2'\left(\tau_j\right)\right)^2}\, \psi\left(x\left(\tau_j\right)\right), \tag{2.3}$$

$$\left(\tilde{K}_n f\right)\left(x\left(\tau_p\right)\right) = \frac{2\left(b-a\right)}{n}\sum_{\substack{j=1\\ j\neq p}}^{n} \frac{\partial \Phi_k^n\left(x\left(\tau_p\right),\, x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_p\right)\right)}\sqrt{\left(x_1'\left(\tau_j\right)\right)^2 + \left(x_2'\left(\tau_j\right)\right)^2}\, f\left(x\left(\tau_j\right)\right) \tag{2.4}$$

and

$$\left(\left(T - T_0\right)_n \psi\right)\left(x\left(\tau_p\right)\right) = \frac{2\left(b-a\right)}{n}\sum_{\substack{j=1\\ j\neq p}}^{n} \frac{\partial}{\partial \nu\left(x\left(\tau_p\right)\right)}\left(\frac{\partial \Phi_k^n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)} - \frac{\partial \Phi_{k_0}^n\left(x\left(\tau_p\right), x\left(\tau_j\right)\right)}{\partial \nu\left(x\left(\tau_j\right)\right)}\right) \times \sqrt{\left(x_1'\left(\tau_j\right)\right)^2 + \left(x_2'\left(\tau_j\right)\right)^2}\, \psi\left(x\left(\tau_j\right)\right) \tag{2.5}$$

are the quadrature formulas for the integrals $\left(Sf\right)\left(x\right)$, $\left(K\psi\right)\left(x\right)$, $\left(\tilde{K}f\right)\left(x\right)$ and $\left(\left(T-T_0\right)\psi\right)\left(x\right)$ at the control points $x\left(\tau_p\right)$, $p = \overline{1,\, n}$, respectively, with

$$\max_{p=\overline{1,n}}\left|\left(Sf\right)\left(x\left(\tau_p\right)\right) - \left(S_n f\right)\left(x\left(\tau_p\right)\right)\right| \leq M^1\left(\omega\left(f,\, 1/n\right) + \left\|f\right\|_\infty \frac{\ln n}{n}\right),$$

$$\max_{p=\overline{1,n}}\left|\left(K\psi\right)\left(x\left(\tau_p\right)\right) - \left(K_n \psi\right)\left(x\left(\tau_p\right)\right)\right| \leq M\left(\omega\left(\psi,\, 1/n\right) + \left\|\psi\right\|_\infty \frac{\ln n}{n}\right),$$

$$\max_{p=\overline{1,n}}\left|\left(\tilde{K}f\right)\left(x\left(\tau_p\right)\right) - \left(\tilde{K}_n f\right)\left(x\left(\tau_p\right)\right)\right| \leq M\left(\omega\left(f,\, 1/n\right) + \left\|f\right\|_\infty \frac{\ln n}{n}\right)$$

[1] Hereinafter M denotes a positive constant which can be different in different inequalities.

and

$$\max_{p=\overline{1,\,n}} \left|\left(\left(T-T_0\right)\psi\right)\left(x\left(\tau_p\right)\right)-\left(\left(T-T_0\right)_n\,\psi\right)\left(x\left(\tau_p\right)\right)\right|$$
$$\leq M\left(\omega\left(\psi,\,1/n\right)+\|\psi\|_\infty\,\frac{\ln n}{n}\right),$$

where $\omega\left(\varphi,\delta\right)$ denotes the modulus of continuity of the function $\varphi\in C\left(L\right)$, i.e.

$$\omega\left(\varphi,\delta\right)=\max_{\substack{|x-y|\leq\delta\\ x,\,y\in L}}\left|\varphi(x)-\varphi(y)\right|,\ \delta>0.$$

Using the quadrature formulas (2.2), (2.3), (2.4) and (2.5), we obtain the expressions

$$\left(C_n\psi\right)\left(x\left(\tau_p\right)\right)=\sum_{j=1}^{n}c_{pj}\,\psi\left(x\left(\tau_j\right)\right) \tag{2.6}$$

and

$$\left(G_nf\right)\left(x\left(\tau_p\right)\right)=\sum_{j=1}^{n}g_{pj}\,f\left(x\left(\tau_j\right)\right)$$

are the quadrature formulas for the integrals

$$\left(C\psi\right)\,(x)=\left(K\psi\right)\,(x)+i\,\eta\,\left(\left(T-T_0\right)\,\psi\right)\,(x)-\,\psi\left(x\right)$$

and

$$\left(Gf\right)\,(x)=\left(Sf\right)\,(x)+i\,\eta\,\left(\tilde{K}f\right)\,(x)+i\,\eta\,f\left(x\right),$$

at the control points $x\left(\tau_p\right),\quad p=\overline{1,\,n}$, respectively, and the following estimates hold:

$$\max_{p=\overline{1,\,n}}\left|\left(C\psi\right)\left(x\left(\tau_p\right)\right)-\left(C_n\psi\right)\left(x\left(\tau_p\right)\right)\right|\leq M\left(\omega\left(\psi,\,1/n\right)+\|\psi\|_\infty\,\frac{\ln n}{n}\right),$$

$$\max_{p=\overline{1,\,n}}\left|\left(Gf\right)\left(x\left(\tau_p\right)\right)-\left(G_nf\right)\left(x\left(\tau_p\right)\right)\right|\leq M\left(\omega\left(f,\,1/n\right)+\|f\|_\infty\,\frac{\ln n}{n}\right),$$

where

$$c_{pp}=-1\ \text{ for }\ p=\overline{1,n},$$

$$c_{pj}=\frac{2\left(b-a\right)}{n}\left(i\,\eta\frac{\partial}{\partial\nu\left(x\left(\tau_p\right)\right)}\left(\frac{\partial\left(\varPhi_k^n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)-\varPhi_{k_0}^n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)\right)}{\partial\nu\left(x\left(\tau_j\right)\right)}\right)\right.$$
$$\left.+\frac{\partial\varPhi_k^n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_j\right)\right)}\right)\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\ \text{ for }\ p\,,\,j=\overline{1\,,\,n},\ \ p\neq j,$$

and

$$g_{pp}=i\eta\ \text{ for }\ p=\overline{1\,,\,n},$$

$$g_{pj}=\frac{2\left(b-a\right)}{n}\left(\varPhi_k^n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)+i\,\eta\,\frac{\partial\varPhi_k^n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\right)$$
$$\times\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\ \text{for}\,p\,,\,j=\overline{1\,,\,n},\ p\neq j.$$

Denote by I^n the unit matrix of order n, and by C^n the space of n–dimensional vectors $z^n=\left(z_1^n,z_2^n,\ldots,z_n^n\right)^{\mathrm{T}}$, $z_l^n\in C$, $l=\overline{1,n}$, with the norm $\|z^n\|=\max\limits_{l=\overline{1,n}}|z_l^n|$, where “$a^{\mathrm{T}}$” means the transposition of the vector a. Consider the n −dimensional matrix $\tilde{K}_0^n = \left(\tilde{k}_{pj}^0\right)_{p,j=1}^n$ with the elements

$$\tilde{k}_{pj}^0 = \begin{cases} 0 \quad \text{for} \quad p = j, \\ \frac{2(b-a)}{n} \frac{\partial \Phi_{k_0}^n(x(\tau_p), x(\tau_j))}{\partial \nu(x(\tau_p))} \sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2} \quad \text{for} \quad p \neq j. \end{cases}$$

Proceeding as in [6], it is not difficult to prove the following two lemmas.

Lemma 2.2 *If $Imk_0 > 0$, then there exists an inverse matrix $\left(I^n + \tilde{K}_0^n\right)^{-1}$ such that*

$$M_1 = \sup_n \left\| \left(I^n + \tilde{K}_0^n\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,n}} \left| \left(\left(I + \tilde{K}_0\right)^{-1} g \right) (x(\tau_l)) - \sum_{j=1}^n \tilde{k}_{lj}^+ g(x(\tau_l)) \right| \leq M \left(\omega(g, 1/n) + \|g\|_\infty \frac{\ln n}{n} \right),$$

where $g \in C(L)$, and $\tilde{k}_{lj}^+$ is an element of the matrix $\left(I^n + \tilde{K}_0^n\right)^{-1}$ in the $l-$th row and $j-$th column.

Lemma 2.3 *If $Imk_0 > 0$, then there exists an inverse matrix $\left(I^n - \tilde{K}_0^n\right)^{-1}$ such that*

$$M_2 = \sup_n \left\| \left(I^n - \tilde{K}_0^n\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,n}} \left| \left(\left(I - \tilde{K}_0\right)^{-1} g \right) (x(\tau_l)) - \sum_{j=1}^n \tilde{k}_{lj}^- g(x(\tau_l)) \right| \leq M \left(\omega(g, 1/n) + \|g\|_\infty \frac{\ln n}{n} \right),$$

where $g \in C(L)$, and $\tilde{k}_{lj}^-$ is an element of the matrix $\left(I^n - \tilde{K}_0^n\right)^{-1}$ in the $l-$th row and $j-$th column.

Let

$$f_{pj}^0 = \begin{cases} 0 \quad \text{for} \quad p = j, \\ \frac{2(b-a)}{n} \Phi_{k_0}^n(x(\tau_p), x(\tau_j)) \sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2} \quad \text{for} \quad p \neq j, \end{cases} \tag{2.7}$$

and

$$a_{lj} = -\frac{1}{i\eta} \sum_{p=1}^n \left(f_{lp}^0 \left(\sum_{m=1}^n \tilde{k}_{pm}^- \left(\sum_{t=1}^n \tilde{k}_{mt}^+ c_{tj} \right) \right) \right), \quad l, j = \overline{1, n}.$$

Theorem 2.1 *The expression*

$$\left(A_n\psi\right)\left(x\left(\tau_l\right)\right)=\sum_{j=1}^{n}a_{lj}\,\psi\left(x\left(\tau_j\right)\right) \tag{2.8}$$

is a quadrature formula for $\left(A\psi\right)(x)$ *at the points* $x\left(\tau_l\right),\, l=\overline{1,n}$, *with*

$$\max_{l=\overline{1,n}}\left|\left(A\psi\right)x\left(\tau_l\right)-\left(A_n\psi\right)x\left(\tau_l\right)\right|\le M\,\left(\omega\left(\psi,1/n\right)+\|\,\psi\|_\infty\,\frac{\ln n}{n}\right).$$

Proof. As

$$\left(A_n\psi\right)x\left(\tau_l\right)=-\frac{1}{i\,\eta}\,\sum_{j=1}^{n}\left(f_{lj}^0\,\left(\sum_{p=1}^{n}\tilde{k}_{j\,p}^-\,\left(\sum_{m=1}^{n}\tilde{k}_{p\,m}^+\,\left(\sum_{t=1}^{n}c_{mt}\,\psi\left(x(\tau_t)\right)\right)\right)\right)\right),$$

the representation

$$\begin{aligned}
&\left(A\psi\right)\,\left(x\left(\tau_l\right)\right)-\left(A_n\psi\right)\left(x\left(\tau_l\right)\right)\\
&=-\frac{1}{i\,\eta}\left(S_0\,\left(I-\tilde{K}_0\right)^{-1}\left(I+\tilde{K}_0\right)^{-1}C\psi\right)\,\left(x\left(\tau_l\right)\right)\\
&-\sum_{j=1}^{n}f_{lj}^0\left(\left(I-\tilde{K}_0\right)^{-1}\left(I+\tilde{K}_0\right)^{-1}C\psi\right)\,\left(x\left(\tau_j\right)\right)\\
&-\frac{1}{i\,\eta}\sum_{j=1}^{n}f_{lj}^0\Big[\left(\left(I-\tilde{K}_0\right)^{-1}\left(I+\tilde{K}_0\right)^{-1}C\psi\right)\,\left(x\left(\tau_j\right)\right)\\
&\qquad-\sum_{p=1}^{n}\tilde{k}_{j\,p}^-\left(\left(I+\tilde{K}_0\right)^{-1}C\psi\right)\,\left(x\left(\tau_p\right)\right)\Big]\\
&-\frac{1}{i\,\eta}\sum_{j=1}^{n}f_{lj}^0\left(\sum_{p=1}^{n}\tilde{k}_{jp}^-\left[\left(\left(I+\tilde{K}_0\right)^{-1}C\psi\right)\,\left(x\left(\tau_p\right)\right)-\sum_{m=1}^{n}\tilde{k}_{pm}^+\left(C\psi\right)\,\left(x\left(\tau_m\right)\right)\right]\right)\\
&-\frac{1}{i\,\eta}\sum_{j=1}^{n}f_{lj}^0\,\left(\sum_{p=1}^{n}\tilde{k}_{jp}^-\,\left(\sum_{m=1}^{n}\tilde{k}_{pm}^+\,\left[\left(C\psi\right)\left(x\left(\tau_m\right)\right)-\sum_{t=1}^{n}c_{mt}\,\psi\left(x\left(\tau_t\right)\right)\right]\right)\right)
\end{aligned}$$

is true. Then, taking into account the error estimates for the quadrature formulas (2.2), (2.6) and Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\left|\left(A\psi\right)\,\left(x\left(\tau_l\right)\right)-\left(A_n\psi\right)\left(x\left(\tau_l\right)\right)\right|\\
&\le M\,\Big[\left\|\left(I-\tilde{K}_0\right)^{-1}\left(I+\tilde{K}_0\right)^{-1}C\psi\right\|_\infty\frac{\ln n}{n}+\\
&\quad+\omega\left(\left(I-\tilde{K}_0\right)^{-1}\left(I+\tilde{K}_0\right)^{-1}C\psi,1/n\right)\Big]\\
&+M\,\left[\left\|\left(I+\tilde{K}_0\right)^{-1}C\psi\right\|_\infty\frac{\ln n}{n}+\omega\left(\left(I+\tilde{K}_0\right)^{-1}C\psi,1/n\right)\right]\sum_{j=1}^{n}\left|f_{lj}^0\right|
\end{aligned}$$

$$+M\left[\| C\psi\|_{\infty}\frac{\ln n}{n}+\omega\left(C\psi,1/n\right)\right]\sum_{j=1}^{n}\left(\left|f_{lj}^{0}\right|\sum_{p=1}^{n}\left|\tilde{k}_{jp}^{-}\right|\right)$$

$$+M\left[\|\psi\|_{\infty}\frac{\ln n}{n}+\omega\left(\psi,1/n\right)\right]$$

$$\times\sum_{j=1}^{n}\left(\left|f_{lj}^{0}\right|\sum_{p=1}^{n}\left(\left|\tilde{k}_{jp}^{-}\right|\sum_{m=1}^{n}\left|\tilde{k}_{pm}^{+}\right|\right)\right). \tag{2.9}$$

By the inequalities

$$\omega\left(K\,\psi\,,1/n\right)\leq M\,\|\psi\|_{\infty}\,\frac{\ln n}{n},$$

$$\omega\left(\left(T-T_{0}\right)\,\psi\,,1/n\right)\leq M\,\|\psi\|_{\infty}\,\frac{\ln n}{n},$$

we have

$$\omega\left(C\psi\,,1/n\right)\leq\omega\left(K\psi,1/n\right)+|\eta|\,\omega\left(\left(T-T_{0}\right)\psi,1/n\right)+\omega\left(\psi,1/n\right)$$

$$\leq\omega\left(\psi,1/n\right)+M\,\|\psi\|_{\infty}\,\frac{\ln n}{n}.$$

It is known ([3, p. 81]) that for every $g\in C\left(L\right)$ the equation

$$\rho+\tilde{K}_{0}\rho=g$$

has a unique solution $\rho_{*}\in C\left(L\right)$. Then we obtain

$$\omega\left(\left(I+\tilde{K}_{0}\right)^{-1}g,1/n\right)=\omega\left(\rho_{*},1/n\right)$$

$$=\omega\left(g-\tilde{K}_{0}\rho_{*},1/n\right)\leq\omega\left(g,1/n\right)+\omega\left(\tilde{K}_{0}\rho_{*},1/n\right)$$

$$\leq\omega(g,1/n)+M\,\|\rho_{*}\|_{\infty}\,\frac{\ln n}{n}=\omega\left(g,1/n\right)+M\left\|\left(I+\tilde{K}_{0}\right)^{-1}g\right\|_{\infty}\frac{\ln n}{n}$$

$$\leq\omega\left(g,1/n\right)+M\left\|\left(I+\tilde{K}_{0}\right)^{-1}\right\|\,\|g\|_{\infty}\frac{\ln n}{n}.$$

Similarly we can show that

$$\omega\left(\left(I-\tilde{K}_{0}\right)^{-1}f\,,1/n\right)\leq\omega\left(f,1/n\right)+M\,\|f\|_{\infty}\,\frac{\ln n}{n}.$$

Hence we derive a chain of inequalities

$$\omega\left(\left(I-\tilde{K}_{0}\right)^{-1}\left(I+\tilde{K}_{0}\right)^{-1}C\psi,1/n\right)\leq$$

$$\leq\omega\left(\left(I+\tilde{K}_{0}\right)^{-1}C\psi,1/n\right)+M\left\|\left(I+\tilde{K}_{0}\right)^{-1}C\psi\right\|_{\infty}\frac{\ln n}{n}\leq$$

$$\leq\omega\left(C\psi,1/n\right)+M\,\|C\psi\|_{\infty}\,\frac{\ln n}{n}+M\left\|\left(I+\tilde{K}_{0}\right)^{-1}C\psi\right\|_{\infty}\frac{\ln n}{n}\leq$$

$$\leq M\left(\omega\left(\psi,1/n\right)+\|\psi\|_{\infty}\frac{\ln n}{n}\right).$$

Proceeding as in [9], it is easy to show that the expression $\sum_{j=1}^{n}\left|f_{lj}^{0}\right|$ is a quadrature formula for the integral

$$2\int_{L}\left|\,\Phi_{k_0}\left(x,y\right)\right|\,dl_y$$

at the points $x\left(\tau_l\right),\ l=\overline{1,n}$, with

$$\max_{l=\overline{1,n}}\left|2\int_{L}\left|\,\Phi_{k_0}\left(x\left(\tau_l\right),y\right)\right|\,dl_y-\sum_{j=1}^{n}\left|f_{lj}^{0}\right|\right|\leq M\frac{\ln n}{n}.$$

Consequently,

$$\max_{l=\overline{1,n}}\sum_{j=1}^{n}\left|f_{lj}^{0}\right|\leq 2\max_{x\in L}\int_{L}\left|\,\Phi_{k_0}\left(x,y\right)\right|\,dl_y+M\frac{\ln n}{n}. \tag{2.10}$$

Besides, Lemmas 2.2 and 2.3 imply the inequalities

$$\max_{j=\overline{1,n}}\sum_{p=1}^{n}\left|\tilde{k}_{j\,p}^{+}\right|\leq M_1,\ \max_{j=\overline{1,n}}\sum_{p=1}^{n}\left|\tilde{k}_{j\,p}^{-}\right|\leq M_2. \tag{2.11}$$

So, considering the above obtained inequalities in (2.9), we get the validity of the theorem.

Similarly we can prove the following one:

Theorem 2.2 *The expression*

$$\left(B_n f\right)\left(x\left(\tau_l\right)\right)=\sum_{j=1}^{n}b_{l\,j}\,f\left(x\left(\tau_j\right)\right) \tag{2.12}$$

is a quadrature formula for $\left(Bf\right)\left(x\right)$ *at the points* $x\left(\tau_l\right),\ l=\overline{1,n}$*, with*

$$\max_{l=\overline{1,n}}\left|\left(Bf\right)x\left(\tau_l\right)-\left(B_n f\right)x\left(\tau_l\right)\right|\leq M\left(\omega\left(f,1/n\right)+\|\,f\|_{\infty}\frac{\ln n}{n}\right),$$

where

$$b_{lj}=-\frac{1}{i\,\eta}\sum_{p=1}^{n}\left(f_{lp}^{0}\left(\sum_{m=1}^{n}\tilde{k}_{p\,m}^{-}\left(\sum_{t=1}^{n}\tilde{k}_{m\,t}^{+}\,g_{tj}\right)\right)\right),\quad l,j=\overline{1,n}.$$

Now let's give the justification of the collocation method for the equation (2.1). Using the quadrature formulas (2.8) and (2.12), we replace the equation (2.1) with the system of algebraic equations with respect to z_l^n, approximate values of $\psi\left(x\left(\tau_l\right)\right),\ l=\overline{1,n}$, stated as

$$\left(I^n+A^n\right)z^n=B^n f^n, \tag{2.13}$$

where $A^n=\left(a_{lj}\right)_{l,j=1}^{n}$, $B^n=\left(b_{lj}\right)_{l,j=1}^{n}$, $f^n=p^n f$, and $p^n:C\left(L\right)\to C^n$ is a linear bounded operator defined by the formula

$$p^n f=\left(f\left(x\left(\tau_1\right)\right),\ f\left(x\left(\tau_2\right)\right),\ldots,f\left(x\left(\tau_n\right)\right)\right)^{\mathrm{T}}$$

and called a simple restriction operator.

Theorem 2.3 *The equations* (2.1) *and* (2.13) *have the unique solutions* $\psi_* \in C(L)$ *and* $z_*^n \in C^n$*, respectively, with* $\| z_*^n - p^n\psi_*\| \to 0$ *as* $n \to \infty$ *and the convergence rate estimate*

$$\| z_*^n - p^n\psi_*\| \leq M \left(\omega(f, 1/n) + \frac{\ln n}{n}\right).$$

Proof. Note that here we will use Vainikko's convergence theorem for linear operator equations ([15]), and we will use notations, definitions and statements from [15]. Let's verify the fulfilment of conditions of Theorem 4.2 of [15]. It was shown in [3, p.104] that $Ker\ (I + A) = \{0\}$. Obviously, the operators $I^n + A^n$ are Fredholm operators of index 0 and the system of simple restriction operators $P = \{p^n\}$ is a connecting system for the spaces $C(L)$ and C^n ([15, p. 676]). Then, by Definition 1.1 of [15] and Theorem 2.2, we obtain $B^n f^n \xrightarrow{P} Bf$. Now let's show that $I^n + A^n \xrightarrow{PP} I + A$. Taking into account the way the curve L has been divided into "regular" elementary parts and Lemma 2.1, it is not difficult to show that the expression

$$F^n(x(\tau_m)) = \sum_{\substack{t=1 \\ t\neq m}}^{n} |c_{m\,t}|$$

is a quadrature formula for the weakly singular integral

$$F(x) = 2\int_L \left| \frac{\partial \varPhi_k(x,y)}{\partial \nu\ (y)} + i\eta \frac{\partial}{\partial \nu\ (x)} \left(\frac{\partial\left(\varPhi_k(x,y) - \varPhi_{k_0}(x,y)\right)}{\partial \nu\ (y)} \right) \right| dl_y, \quad x \in L,$$

at the points $x(\tau_m)$, $m = \overline{1, n}$, with

$$\max_{m=\overline{1,n}} |F(x(\tau_m)) - F^n(x(\tau_m))| \leq M \frac{\ln n}{n}.$$

Consequently,

$$\max_{m=\overline{1,n}} \sum_{t=1}^{n} |c_{m\,t}| = 1 + \max_{m=\overline{1,n}} F^n(x(\tau_m)) \leq$$

$$\leq 1 + \max_{m=\overline{1,n}} |F(x(\tau_m)) - F^n(x(\tau_m))| + \max_{x\in L} F(x) \leq M. \tag{2.14}$$

Taking into account the inequalities (2.10), (2.11) and (2.14), we arrive at the estimate

$$\| A^n z^n\| = \max_{l=\overline{1,n}} \left| \sum_{j=1}^{n} a_{lj}\, z_j^n \right| \leq M \|z^n\|, \quad \forall\, z^n \in C^n.$$

Let $\psi_n \xrightarrow{P} \psi$. Then, by Theorem 2.1, we obtain

$$\| (I^n + A^n)\psi_n - p^n((I + A)\psi) \| \leq \| \psi_n - p^n\psi \| + M\ \| \psi_n - p^n\psi \| +$$

$$+ \|A^n(p^n\psi) - p^n(A\psi)\| \to 0 \text{ as } n \to \infty.$$

Consequently, by Definition 2.1 of [15] we have $I^n + A^n \xrightarrow{PP} I + A$.

As $I^n \to I$ stably by Definition 3.2 of [15], then, by Proposition 3.5 and Definition 3.3 of [15], it remains to verify the compactness condition, which, due to Proposition 1.1 of [15], is equivalent to the following condition: $\forall \{z^n\}$, $z^n \in C^n$, $\|z^n\| \leq M$, there exists a relatively compact sequence $\{A_n z^n\} \subset C(L)$ such that

$$\| A^n z^n - p^n(\ A_n z^n) \| \to 0 \ \text{ as } \ n \to \infty.$$

As $\{A_n z^n\}$, we choose the sequence

$$(A_n z^n)(x) = \sum_{j=1}^{n} a_j(x)\, z_j^n, \quad x \in L,$$

where

$$a_j(x) = -\frac{1}{i\,\eta} \sum_{p=1}^{n} \left(f_p^0(x) \left(\sum_{m=1}^{n} \tilde{k}_{pm}^{-} \left(\sum_{t=1}^{n} \tilde{k}_{m\,t}^{+}\, c_{tj} \right) \right) \right), \quad j = \overline{1,\, n},$$

$$f_p^0(x) = 2 \int_{L_p} \Phi_{k_0}(x, y)\, dl_y, \;\; x \in L, \;\; p = \overline{1,\, n}.$$

Taking into account the way the curve L has been divided into "regular" elementary parts and Lemma 2.1, we have

$$\sum_{p=1}^{n} \left| f_{lp}^0 - f_p^0(x(\tau_l)) \right| \le 2 \sum_{\substack{p=1 \\ p \ne l}}^{n} \int_{L_p} \left| \Phi_{k_0}(x(\tau_l), y) - \Phi_{k_0}(x(\tau_l), x(\tau_p)) \right| dl_y +$$

$$+2 \int_{L_l} \left| \Phi_{k_0}(x(\tau_l), y) \right| dl_y \le M\, \frac{\ln n}{n}, l = \overline{1,\, n}. \tag{2.15}$$

Further, it is obvious that

$$\sum_{p=1}^{n} \left| f_p^0(x) \right| \le 2 \int_{L} \left| \Phi_{k_0}(x, y) \right| dl_y \le M. \tag{2.16}$$

As

$$(A_n z^n)(x) = -\frac{1}{i\,\eta} \sum_{j=1}^{n} \left(f_j^0(x) \left(\sum_{p=1}^{n} \tilde{k}_{jp}^{-} \left(\sum_{m=1}^{n} \tilde{k}_{pm}^{+} \left(\sum_{t=1}^{n} c_{mt}\, z_t^n \right) \right) \right) \right),$$

taking into account the condition $\|z^n\| \le M$ and the inequalities (2.11), (2.14) and (2.15), we obtain

$$\| A^n\, z^n - p^n\,(\;A_n z^n) \| \to 0 \;\text{ as }\; n \to \infty.$$

Consider arbitrary points x', $x'' \in L$ such that $|x' - x''| = \delta < d/2$. Then, taking into account the inequalities (2.11), (2.14) and proceeding as in [10], we can show that

$$\left| (A_n z^n)(x') - (A_n z^n)(x'') \right| \le M\, \|z^n\| \sum_{j=1}^{n} \left| f_j^0(x') - f_j^0(x'') \right| \le$$

$$\le M \|z^n\| \int_{L} \left| \Phi_{k_0}(x', y) - \Phi_{k_0}(x'', y) \right| dl_y \le M\, \|z^n\| \int_{L_{\delta/2}(x')} \left| \Phi_{k_0}(x', y) \right| dl_y +$$

$$+M\, \|z^n\| \int_{L_{\delta/2}(x'')} \left| \Phi_{k_0}(x'', y) \right| dl_y + M\, \|z^n\| \int_{L_{\delta/2}(x')} \left| \Phi_{k_0}(x'', y) \right| dl_y +$$

$$+M\, \|z^n\| \int_{L_{\delta/2}(x'')} \left| \Phi_{k_0}(x', y) \right| dl_y +$$

$$+M\left\|z^{n}\right\|\int_{L\backslash\left(L_{\delta/2}\left(x'\right)\cup L_{\delta/2}\left(x''\right)\right)}\left|\Phi_{k_0}\left(x',y\right)-\Phi_{k_0}\left(x'',y\right)\right|dl_y\leq$$
$$\leq M\left\|z^{n}\right\|\delta\left|\ln\delta\right|, \tag{2.17}$$

and, consequently, $\{ A_n z^n\}\subset C(L)$.

Relative compactness of the sequence $\{A_n z^n\}$ follows from the Arzela theorem. In fact, the uniform boundedness follows directly from the inequalities (2.11), (2.14), (2.16) and the condition $\|z^n\|\leq M$, and the equicontinuity follows from the estimate (2.17). Then, applying Theorem 4.2 of [15], we see that the equations (2.1) and (2.13) have unique solutions $\psi_*\in C(L)$ and $z_*^n\in C^n$, respectively, with

$$m_3\delta_n\leq\|\, z_*^n-p^n\psi_*\|\leq\, M_3\delta_n,$$

where

$$m_3\,=1/\sup_n\,\|\,I^n+A^n\|>0, M_3\,=\sup_n\,\left\|\left(I^n+A^n\right)^{-1}\right\|<\infty,$$
$$\delta_n=\|\,\left(I^n+A^n\right)\,\left(p^n\psi_*\right)-B^n f^n\,\|\,.$$

By Theorems 2.1 and 2.2, we obtain

$$\delta_n=\max_{l=\overline{1,n}}\left|\psi_*\left(x\left(\tau_l\right)\right)+\sum_{j=1}^{n}a_{lj}\,\psi_*\left(x\left(\tau_j\right)\right)-\sum_{j=1}^{n}b_{lj}\,f\left(x\left(\tau_j\right)\right)\right|=$$
$$=\max_{l=\overline{1,n}}\left|\left(\left(Bf\right)\left(x\left(\tau_l\right)\right)-\sum_{j=1}^{n}b_{lj}\,f\left(x\left(\tau_j\right)\right)\right)+\right.$$
$$\left.+\left(\sum_{j=1}^{n}a_{lj}\,\psi_*\left(x\left(\tau_j\right)\right)-\left(A\psi_*\right)\left(x\left(\tau_j\right)\right)\right)\right|\leq$$
$$\leq M\,\left(\omega\left(f,1/n\right)+\omega\left(\psi_*,1/n\right)+\left(\|f\|_\infty+\|\,\psi_*\|_\infty\right)\frac{\ln n}{n}\right).$$

As $\psi_*=(I+A)^{-1}Bf$, we have

$$\|\,\psi_*\|_\infty\leq\left\|\,(I+A)^{-1}\right\|\,\|B\|\,\|f\|_\infty\,.$$

Besides, by the estimate

$$\omega\left(F_0\rho,1/n\right)\leq M\,\;\|\rho\|_\infty\,\frac{\ln n}{n},$$

we have

$$\omega\left(Bf,1/n\right)\leq M\,\;\|f\|_\infty\,\frac{\ln n}{n},\;\;\omega\left(A\psi_*,1/n\right)\leq M\,\;\|f\|_\infty\,\frac{\ln n}{n}.$$

Consequently,

$$\omega\left(\psi_*,1/n\right)=\omega\left(Bf-A\psi_*,1/n\right)\leq$$
$$\leq\omega\left(Bf,1/n\right)+\omega\left(A\psi_*,1/n\right)\leq M\,\;\|f\|_\infty\,\frac{\ln n}{n},$$

which completes the proof of the theorem.

Remark 2.2 As seen, if $f\in C(L)\backslash C^\beta(L)$, then the method proposed in [13] does not allow to treat the solution of the integral equation obtained after discretization of the initial equation. Moreover, if $f\in C^\beta(L)$, then the convergence rate of this method is $\omega(f,1/n)+\frac{\ln n}{n}\approx\frac{1}{n^\beta}$, while in [13] the convergence rate of the method is $\frac{\ln n}{n^{\beta-\alpha}}$, i.e. in this case the convergence rate of our method is higher than the one of the method in [13], where $0<\alpha\leq\beta<1$.

3 Justification of collocation method for hypersingular integral equation (1.4)**.**

Let's first perform a regularization of the equation (1.4). Let the wave number k_0 not coincide with the eigenvalues of the interior Dirichlet or Neumann problems (for this, it suffices to choose any value of k_0 with $Im\, k_0 > 0$). We will assign zero index to our notations if the parameter k, involved in the operators S, $\tilde{K}$ and T, is equal to k_0. As the operator

$$A_0 = -S_0 \left(I - \tilde{K}_0\right)^{-1} \left(I + \tilde{K}_0\right)^{-1}$$

is an inverse operator to T_0, the equation (1.4) can be rewritten in the following equivalent form ([3, p. 98]):

$$\varphi + \tilde{A}\,\varphi = \tilde{B}g. \tag{3.1}$$

The last equation is considered in the space $C\,(L)$, where

$$\tilde{A}\varphi = -\frac{1}{i\,\eta} A_0 \left[(1 - \,i\,\eta\,\lambda)\; I - \left(\tilde{K} + i\,\eta\;(T - T_0) + i\,\eta\,\lambda\,K + \lambda\,S\right)\right]\;\varphi,$$

$$\tilde{B}g = \frac{2}{i\,\eta}\,A_0 g.$$

To justify the collocation method, let's first construct the quadrature formulas for the integrals $\left(\tilde{A}\varphi\right)(x)$ and $\left(\tilde{B}g\right)(x), x \in L$. Let's divide L into "regular" elementary parts $L = \bigcup\limits_{j=1}^{n} L_j$. Taking into account the quadrature formulas (2.2), (2.3), (2.4) and (2.5) constructed for the integrals S, K, $\tilde{K}$ and $T - T_0$, respectively, and the error estimates for them, it is not difficult to show that the expression

$$\left(\tilde{C}_n\varphi\right)(x\,(\tau_l)) = \sum_{j=1}^{n} \tilde{c}_{lj}\,\varphi\,(x\,(\tau_j))$$

is a quadrature formula for the integral

$$\left(\tilde{C}\varphi\right)\;(x) = (1 - \,i\,\eta\,\lambda\,(x))\;\varphi\,(x) -$$

$$-\left(\left(\tilde{K}\varphi\right)\;(x) + i\,\eta\;((T - T_0)\,\varphi)\;(x) + i\,\eta\,\lambda\,(x)\;(K\varphi)\;(x) + \lambda\,(x)\;(S\varphi)\;(x)\right)$$

at the control points $x\,(\tau_l)\,,\;\; l = \overline{1,\,n}$, where

$$\tilde{c}_{ll} = 1 - \,i\,\eta\,\lambda\,(x\,(\tau_l)) \;\;\text{for}\;\; l = \overline{1, n},$$

$$\tilde{c}_{lj} = -\frac{2\,(b - a)}{n}\,\sqrt{\left(x_1'\,(\tau_j)\right)^2 + \left(x_2'\,(\tau_j)\right)^2}\left(\frac{\partial \Phi_k^n\,(x\,(\tau_l)\,, x\,(\tau_j))}{\partial \nu\,(x\,(\tau_l))} + \right.$$

$$+ i\eta\frac{\partial}{\partial \nu\,(x\,(\tau_l))}\left(\frac{\partial\left(\Phi_k^n\,(x\,(\tau_l)\,, x\,(\tau_j)) - \Phi_{k_0}^n\,(x\,(\tau_l)\,, x\,(\tau_j))\right)}{\partial \nu\,(x\,(\tau_j))}\right) +$$

$$\left. + i\,\eta\,\lambda\,(x\,(\tau_l))\;\frac{\partial \Phi_k^n\,(x\,(\tau_l)\,, x\,(\tau_j))}{\partial \nu\,(x\,(\tau_j))} + \lambda\,(x\,(\tau_l))\;\Phi_k^n\,(x\,(\tau_l)\,, x\,(\tau_j))\right)$$

for $l, j = \overline{1, n}, l \neq j$, with

$$\max_{l = \overline{1, n}} \left| \left(\tilde{C}\varphi\right)(x\,(\tau_l)) - \left(\tilde{C}_n\varphi\right)(x\,(\tau_l))\right| \leq M \left(\omega\,(\varphi, 1/n) + \|\varphi\|_\infty\,\frac{\ln n}{n}\right).$$

Let the elements f^0_{lj}, $l,j=\overline{1,n}$, be defined by the formula (2.7), $\tilde{k}^-_{lj}$ be an element of the matrix $\left(I^n-\tilde{K}^n_0\right)^{-1}$ in the $l-$th row and $j-$th column, $\tilde{k}^+_{lj}$ be an element of the matrix $\left(I^n+\tilde{K}^n_0\right)^{-1}$ in the $l-$th row and $j-$th column, and

$$\tilde{a}_{lj}=\frac{1}{i\,\eta}\sum_{p=1}^{n}\left(f^0_{lp}\left(\sum_{m=1}^{n}\tilde{k}^-_{p\,m}\left(\sum_{t=1}^{n}\tilde{k}^+_{m\,t}\,\tilde{c}_{tj}\right)\right)\right),\quad l\,,\,j=\overline{1\,,\,n},$$

$$\tilde{b}_{lj}=-\frac{2}{i\,\eta}\sum_{p=1}^{n}\left(f^0_{lp}\left(\sum_{m=1}^{n}\tilde{k}^-_{p\,m}\,\tilde{k}^+_{m\,j}\right)\right),\quad l\,,\,j=\overline{1\,,\,n}.$$

Proceeding as in the proof of Theorem 2.1, we can show that the expressions

$$\left(\tilde{B}_n g\right)(x\,(\tau_l))=\sum_{j=1}^{n}\tilde{b}_{l\,j}\,g\,(x\,(\tau_j)) \tag{3.2}$$

and

$$\left(\tilde{A}_n\varphi\right)(x\,(\tau_l))=\sum_{j=1}^{n}\tilde{a}_{l\,j}\,\varphi\,(x\,(\tau_j)) \tag{3.3}$$

are the quadrature formulas for $\left(\tilde{B}g\right)(x)$ and $\left(\tilde{A}\varphi\right)(x)$ at the points $x\,(\tau_l)$, $l=\overline{1,n}$, respectively, with

$$\max_{l=\overline{1,\,n}}\left|\left(\tilde{B}g\right)(x\,(\tau_l))-\left(\tilde{B}_n g\right)(x\,(\tau_l))\right|\le M\left(\omega\,(g,1/n)+\|\,g\,\|_\infty\,\frac{\ln n}{n}\right),$$

$$\max_{l=\overline{1,\,n}}\left|\left(\tilde{A}\varphi\right)(x\,(\tau_l))-\left(\tilde{A}_n\varphi\right)(x\,(\tau_l))\right|\le$$

$$\le M\left(\omega\,(\varphi,1/n)+\|\,\varphi\,\|_\infty\,\omega\,(\lambda,1/n)+\|\,\varphi\,\|_\infty\,\frac{\ln n}{n}\right).$$

Using the quadrature formulas (3.2) and (3.3), we replace the equation (3.1) with the system of algebraic equations with respect to z^n_l, approximate values of $\varphi\,(x\,(\tau_l))\,,\,l=\overline{1,n}$, stated as

$$\left(I^n+\tilde{A}^n\right)\,z^n=\tilde{B}^n g^n, \tag{3.4}$$

where $\tilde{A}^n=(\tilde{a}_{lj})^n_{l,j=1}$, $\tilde{B}^n=\left(\tilde{b}_{lj}\right)^n_{l,j=1}$ and $g^n=p^n g$.

Proceeding as in the proof of Theorem 2.3, we can show the validity of the main result of this section:

Theorem 3.1 *The equations* (3.1) *and* (3.4) *have the unique solutions* $\varphi_*\in C\,(L)$ *and* $z^n_*\in C^n$*, respectively, with* $\|\,z^n_*-p^n\varphi_*\|\to 0$ *as* $n\to\infty$ *and the convergence rate estimate*

$$\|\,z^n_*-p^n\varphi_*\|\le M\left(\omega\,(g,1/n)+\omega\,(\lambda,1/n)+\frac{\ln n}{n}\right).$$

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