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An analog of Titchmarsh's and Younis's theorems for the q-Bessel Fourier transform in the space $\mathcal{L}^p_{q,\alpha}(\mathbb{R}^+)$

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Abstract. *In this paper by using a q-translation operator, we give an analog of Titchmarsh's theorem and Younis's theorem for q-Bessel Fourier transform satisfying q-Bessel-Lipschitz and q-Bessel-Dini-Lipschitz conditions in the space $\mathcal{L}^p_{q,\alpha}(\mathbb{R}^+)$, where $1 < p \leq 2$.*

Keywords. q-Bessel operator, q-Bessel Fourier transform, q-translation operator, q-Bessel-Lipschitz class, q-Bessel-Dini-Lipschitz class

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1 Introduction

In the recent mathematical literature one finds many articles which deal with the theory of q-Fourier analysis associated with the q-Hankel transform. This theory was elaborated first by Koornwinder and R.F.Swarttouw [13] and then by Fitouhi and Al [7]. They were interested in q-analogue of different integral transformations. In connection with q-difference Bessel operator and with the basic Bessel functions, they introduced several generalized q-Fourier transform. So, it is natural to look for the q-analogue of some well-known classical theorems.

Titchmarsh ([15], Theorem 84) characterized the set of functions in $L^p(\mathbb{R})$, $1 < p \leq 2$, satisfying the q-Bessel-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namly we have

Theorem 1.1 *[15] Let f belong to $L^p(\mathbb{R})$, $1 < p \leq 2$, such that*

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x-h)|^p dx = O(h^{\alpha p}), \quad 0 < \alpha \leq 1 \quad \text{as } h \longrightarrow 0.$$

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Then, its Fourier transform $\mathcal{F}(f)$ belong to $L^{\beta}(\mathbb{R})$ for

$$\frac{p}{p+\alpha p-1}<\beta\leq\frac{p}{p-1}.$$

On the other hand, Younis in ([16], Theorem 3.3) studied the same phenomena for the wider Dini-Lipschitz class as well as for some other allied classes of functions. More precisely

Theorem 1.2 *[16] Let $f\in L^{p}(\mathbb{R})$ with $1<p\leq 2$, such that*

$$\left(\int_{-\infty}^{+\infty}|f(x+h)-f(x)|^{p}dx\right)^{\frac{1}{p}}=O\left(\frac{h^{\alpha}}{\left(\log\frac{1}{h}\right)^{\gamma}}\right),\;\; h\longrightarrow 0,\, 0<\alpha\leq 1,\;\;\gamma>0.$$

Then $\mathcal{F}(f)\in L^{\beta}(\mathbb{R})$ for

$$\frac{p}{p+\alpha p-1}\leq\beta<p'=\frac{p}{p-1}$$

and

$$\frac{1}{\beta}<\gamma,$$

where $\mathcal{F}(f)$ stands for the Fourier transform of f.

The main aim of this paper is to generalize these theorems for the q-Bessel Fourier transform setting by means of the q-translation operator.

In recent years, these two results have been generalized in several different versions and for several different types of transform (for exemple, see [2,3,8,14]).

2 Preliminaries and auxiliary results

In the first we collect some definitions, notations and properties of the q-shifted factorials, the q-hypergeometric functions, the Jackson's q-derivative and the Jackson's q-integrals (see [10,12]). Throughout this paper, we assume that $0<q<1$ and $\alpha>-\frac{1}{2}$. we denote by

$$\mathbb{R}_q^{+}=\{q^n,n\in\mathbb{Z}\}.$$

Let $x\in\mathbb{C}$, the q-shifted factorials are defined by

$$(x;q)_0=1\;,\quad (x;q)_n=\prod_{k=0}^{n-1}(1-xq^k)\;,\, n=1,2,...,\quad (x;q)_n=\prod_{k=0}^{\infty}(1-xq^k)$$

and for $a\in\mathbb{C}$ and $n\in\mathbb{N}$ we also denote

$$[a]_q=\frac{1-q^a}{1-q}\;,\qquad [n]_q!=\frac{(q;q)_n}{(1-q)^n}.$$

The q-derivative of a function f is here defined by

$$\mathcal{D}_qf(x)=\frac{f(x)-f(qx)}{(1-q)x}\quad\text{if } x\neq 0$$

$\mathcal{D}_qf(0)=f'(0)$ provided $f'(0)$ exists.

We also consider

$$\mathcal{D}_q^{+}f(x)=q^{-1}\mathcal{D}_qf(q^{-1}x).$$

For all $s \in \mathbb{R}$ and $x \in \mathbb{R}_q^+$, we have

$$\mathcal{D}_q x^s = [s]_q x^{s-1} \tag{2.1}$$

and

$$\mathcal{D}_q^+ x^s = q^{-1}[s]_q x^{s-1} = -[-s]_q x^{s-1}. \tag{2.2}$$

In [11], the q-Jackson integrals from 0 to a, from a to b, from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined by

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

$$\int_0^{+\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{+\infty} q^n f(q^n),$$

$$\int_{-\infty}^{+\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{+\infty} q^n [f(q^n) + f(-q^n)].$$

The q-analogue of the integration theorem by a change of variable can be stated as follows

$$\int_a^b g\left(\frac{\lambda}{r}\right) \lambda^{2\alpha+1} d_q \lambda = r^{2\alpha+2} \int_{\frac{a}{r}}^{\frac{b}{r}} g(t) t^{2\alpha+1} d_q \lambda \qquad \forall r \in \mathbb{R}_q^+. \tag{2.3}$$

The q-integration by part formulas associated with $\mathcal{D}_q$ and $\mathcal{D}_q^+$ are given by

$$\int_a^b g(x) \mathcal{D}_q f(x) d_q x = [f(b)g(b) - f(a)g(a)] - \int_a^b f(qx) \mathcal{D}_q g(x) d_q x. \tag{2.4}$$

$$\int_a^b g(q^{-1}x) \mathcal{D}_q^+ f(x) d_q x = \left[f(q^{-1}b) g(q^{-1}b) - f(q^{-1}a) g(q^{-1}a)\right] - \int_a^b f(x) \mathcal{D}_q^+ g(x) d_q x. \tag{2.5}$$

Note that for any function f we can write

$$f = f_e + f_o$$

where f_e and f_o are respectively, the even and the odd parts of f defined by

$$f_e = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o = \frac{f(x) - f(-x)}{2}$$

Now, we briefly collect the pertinent definitions and facts relevant for q-Bessel Fourier transform (see [5,7,9,13]).

In [1] the normalized third Jackson q-Bessel function of order α is defined by

$$j_\alpha(x, q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1) \Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}, \tag{2.6}$$

where Γ_q is the q-gamma function defined for $x \in \mathbb{R}_q^+$ by

$$\Gamma_q = \frac{(q,q)_\infty}{(q^x,q)_\infty} (1-q)^{1-x}.$$

The formula (6) with a simple calculation implies that

$$\lim_{x\longrightarrow 0}\frac{1-j_\alpha(x;q^2)}{x^2}=\frac{1}{[\alpha+1]_{q^2}}\left(\frac{q}{q+1}\right)^2\neq 0,$$

hence, there exists $C>0$ and $\eta>0$ satisfying

$$|x|\leq\eta\Longrightarrow|1-j_\alpha(x,q^2)|\geq Cx^2. \tag{2.7}$$

The function $x\longmapsto j_\alpha(\lambda x,q^2)$ is a solution of the following q-differential equation

$$\Lambda_{q,\alpha}f(x)=-\lambda^2f(x)$$

where $\Lambda_{q,\alpha}$ is the q-Bessel operator

$$\Lambda_{q,\alpha}f(x)=\frac{1}{x^2}\left[f(q^{-1}x)-(1+q^{2\alpha})f(x)+q^{2\alpha}f(qx)\right].$$

For $1\leq p<\infty$ we denote by $\mathcal{L}^p_{q,\alpha}$ the space of functions defined on $\mathbb{R}^+_q$ such that

$$\|f\|_{q,p,\alpha}=\left(\int_0^{+\infty}|f(x)|^px^{2\alpha+1}d_qx\right)^{\frac{1}{p}}.$$

exist.

We denote by $C_{q,0}(\mathbb{R}^+_q)$ the space of functions defined on $\mathbb{R}^+_q$ tending to 0 as $x\longrightarrow\infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $C_{q,0}(\mathbb{R}^+_q)$ is complete with respect to the norm

$$\|f\|_{q,\infty}=\sup_{x\in\mathbb{R}^+_q}|f(x)|.$$

The q-Bessel Fourier transform $\mathcal{F}_{q,\alpha}$ is defined by [5,7,13]

$$\mathcal{F}_{q,\alpha}f(x)=C_{q,\alpha}\int_0^{+\infty}f(t)j_\alpha(xt,q^2)t^{2\alpha+1}d_qt\qquad\forall x\in\mathbb{R}^+_q$$

where

$$C_{q,\alpha}=\frac{1}{1-q}\frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

From [5] we have the following result

Proposition 2.1 *The q-Bessel Fourier transform satisfies*

i) If $f\in\mathcal{L}^1_{q,\alpha}$, then $\mathcal{F}_{q,\alpha}f\in C_{q,0}$ and we have

$$\|\mathcal{F}_{q,\alpha}f\|_{q,\infty}\leq B_{q,\alpha}\|f\|_{q,1,\alpha} \tag{2.8}$$

where

$$B_{q,\alpha}=\frac{1}{1-q}\frac{(-q^2;q^2)_\infty(-q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

ii) For all function $f\in\mathcal{L}^p_{q,\alpha}$

$$\mathcal{F}^2_{q,\alpha}f=f. \tag{2.9}$$

iii) For all function $f\in\mathcal{L}^2_{q,\alpha}$

$$\|\mathcal{F}_{q,\alpha}f\|_{q,2,\alpha}=\|f\|_{q,2,\alpha}. \tag{2.10}$$

Proposition 2.2 *Let $f \in \mathcal{L}^p_{q,\alpha}$ where $p \geq 1$, then $\mathcal{F}_{q,\alpha} f \in \mathcal{L}^{p'}_{q,\alpha}$. Also if $1 \leq p \leq 2$, then*

$$\|\mathcal{F}_{q,\alpha} f\|_{q,p',\alpha} \leq B_{q,\alpha}^{\frac{2}{p}-1} \|f\|_{q,p,\alpha} \tag{2.11}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$

Proof. This is an immediate consequence of formulas (2.8), (2.10), the Riesz-Thorin theorem and the inversion formula (2.9).

The q-translation operator is given as follow

$$T^\alpha_{q,x} f(y) = C_{q,\alpha} \int_0^{+\infty} \mathcal{F}_{q,\alpha}(f)(t) j_\alpha(yt, q^2) j_\alpha(xt, q^2) t^{2\alpha+1} d_q t.$$

Let us now introduce

$$Q_\alpha = \left\{ q \in]0,1[\,,\ T^\alpha_{q,x} \ \ is\ positive\ for\ all\ x \in \mathbb{R}^+_q \right\}$$

the set of the positivity of $T^\alpha_{q,x}$. We recall that $T^\alpha_{q,x}$ is called positive if $T^\alpha_{q,x} \geq 0$ for $f \geq 0$. In a recent paper [6] it was proved that if $-1 < \alpha < \alpha'$ then $Q_\alpha \subset Q_{\alpha'}$. As a consequence:

- if $0 \leq \alpha$ then $Q_\alpha =]0,1[$.
- if $-\frac{1}{2} < \alpha < 0$ then $]0, q_0[\subset Q_{-\frac{1}{2}} \subset Q_\alpha \subset]0,1[\ \ , \quad q \simeq 0.43$.
- if $-1 < \alpha < -\frac{1}{2}$ then $Q_\alpha \subset Q_{-\frac{1}{2}}$. (we don't have the information if this subset is empty or not).

Proposition 2.3 *For any function $f \in \mathcal{L}^2_{q,\alpha}$ we have*

$$\mathcal{F}_{q,\alpha}(T^\alpha_{q,x} f)(\lambda) = j_\alpha(\lambda x, q^2) \mathcal{F}_{q,\alpha} f(\lambda) \quad for\ all\ \lambda,\ x \in\ \mathbb{R}^+_q. \tag{2.12}$$

For $f \in \mathcal{L}^p_{q,\alpha}$, $1 < p \leq 2$, we define the finite differences of the first order and step $h > 0$, $h \in \mathbb{R}^+_q$ by

$$\Delta_{q,h} f(x) = T^\alpha_{q,h} f(x) - f(x) = (T^\alpha_{q,h} - I) f(x)$$

where I is the unit operator in $\mathcal{L}^p_{q,\alpha}$.

Lemma 2.1 *For any function $f \in \mathcal{L}^p_{q,\alpha}$, $1 < p \leq 2$, we have*

$$\int_0^{+\infty} |1 - j_\alpha(\lambda h, q^2)|^{p'} |\mathcal{F}_{q,\alpha} f(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q\lambda \leq C_1 \|\Delta_{q,h} f\|^{p'}_{q,p,\alpha}.$$

Proof. By formula (2.12) we have

$$\begin{aligned}
\mathcal{F}_{q,\alpha}(\Delta_{q,h} f)(\lambda) &= \mathcal{F}_{q,\alpha}(T^\alpha_{q,h} f - f)(\lambda) \\
&= \mathcal{F}_{q,\alpha}(T^\alpha_{q,h} f)(\lambda) - \mathcal{F}_{q,\alpha}(f)(\lambda) \\
&= j_\alpha(\lambda h, q^2) \mathcal{F}_{q,\alpha}(f)(\lambda) - \mathcal{F}_{q,\alpha}(f)(\lambda) \\
&= \left(j_\alpha(\lambda h, q^2) - 1 \right) \mathcal{F}_{q,\alpha}(f)(\lambda).
\end{aligned}$$

Using formula (2.11) we obtain our result.

Lemma 2.2 *For any function f defined on $\mathbb{R}_q^+$, we have*

$$\mathcal{D}_q\left[\int_a^x f(t)d_qt\right]_o = f_e(x) \tag{2.13}$$

and

$$\mathcal{D}_q^+\left[\int_a^x f(t)d_qt\right]_e = q^{-1}f_o(q^{-1}x) \tag{2.14}$$

where $x \longmapsto \left[\int_a^x f(t)d_qt\right]_o$ *and* $x \longmapsto \left[\int_a^x f(t)d_qt\right]_e$ *are respectively, the odd and the even part of* $x \longmapsto \left[\int_a^x f(t)d_qt\right]$.

Proof. See Lemma 3.2 in [4].

3 Main results

Before giving our main result, we define, first, the q-Bessel-Lipschitz class.

Definition 3.1 *Let $0 < \delta < 1$. A function $f \in \mathcal{L}^p_{q,\alpha}$, $1 < p \leq 2$ is said to be in the q-Bessel-Lipschitz class, denoted q-BLip(δ, p, α) if*

$$\|\Delta_{q,h}f\|_{q,p,\alpha} = O(h^\delta) \quad \text{as } h \longrightarrow 0.$$

Theorem 3.1 *For $f \in \mathcal{L}^p_{q,\alpha}$ where $1 < p \leq 2$. If f in q-BLip(δ, p, α), then $\mathcal{F}_{q,\alpha}(f) \in \mathcal{L}^\beta_{q,\alpha}(\mathbb{R}_q^+)$ where*

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} < \beta \leq p' = \frac{p}{p-1}.$$

Proof. If $\beta = p'$ we have by the formula (2.11) that $\mathcal{F}_{q,\alpha}(f) \in \mathcal{L}^{p'}_{q,\alpha}$. And for $\alpha > -\frac{1}{2}$, $0 < \delta < 1$ we get

$$\begin{aligned}
&1 + \frac{\delta p'}{2\alpha + 2} > 1 \\
&\Longleftrightarrow \frac{p-1}{p}\left(\frac{2\alpha + 2 + \delta p'}{2\alpha + 2}\right) > \frac{p-1}{p} \\
&\Longleftrightarrow \frac{(2\alpha + 2)(p-1) + \delta p}{p(2\alpha + 2)} > \frac{1}{p'} \\
&\Longleftrightarrow \frac{p(2\alpha + 2)}{(2\alpha + 2)(p-1) + \delta p} < \beta = p' = \frac{p}{p-1} \\
&\Longleftrightarrow \frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} < \beta = p' = \frac{p}{p-1},
\end{aligned}$$

then the theorem is proved in the case where $\beta = p'$.

In what follows we assume that $\beta < p'$ and $f \in q\text{-}BLip(\delta, p, \alpha)$, then we have

$$\|\Delta_{q,h}f\|_{q,p,\alpha} = O(h^\delta) \quad \text{as } h \longrightarrow 0.$$

The Lemma 2.1 yields

$$\int_0^{+\infty} |1-j_\alpha(\lambda h,q^2)|^{p'}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{2\alpha+1}d_q\lambda \le C_1\|\Delta_{q,h}f\|_{q,p,\alpha}^{p'} \le C_2 h^{\delta p'}.$$

If $0<\lambda<\dfrac{\eta}{h}$, then $0<\lambda h<\eta$ and inequality (2.7) implies that

$$|1-j_\alpha(\lambda h,q^2)| \ge C\lambda^2h^2.$$

Therefore

$$\begin{aligned}&\int_0^{\frac{\eta}{h}} h^{2p'}\lambda^{2p'}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{2\alpha+1}d_q\lambda\\ &\le \frac{1}{C^{p'}}\int_0^{\frac{\eta}{h}} |1-j_\alpha(\lambda h,q^2)|^{p'}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{2\alpha+1}d_q\lambda\\ &\le \frac{1}{C^{p'}}\int_0^{+\infty} |1-j_\alpha(\lambda h,q^2)|^{p'}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{2\alpha+1}d_q\lambda\\ &= O(h^{\delta p'}).\end{aligned}$$

Then

$$\int_0^{\frac{\eta}{h}} \lambda^{2p'}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{2\alpha+1}d_q\lambda = O\left(h^{(\delta-2)p'}\right) \quad \text{as } h\longrightarrow 0.$$

Thus

$$\int_0^{X} \lambda^{2p'}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{2\alpha+1}d_q\lambda = O\left(X^{(2-\delta)p'}\right) \quad \text{as } X\longrightarrow +\infty.$$

Let

$$\varphi(X)=\int_1^X |\lambda^2\mathcal{F}_{q,\alpha}(f)(\lambda)|^{\beta}|\lambda|^{(2\alpha+1)\beta/p'}d_q\lambda. \tag{3.1}$$

Taking into account the Hölder inequality yields

$$\begin{aligned}\varphi(X) &\le \left(\int_1^X |\lambda^2\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{(2\alpha+1)}d_q\lambda\right)^{\beta/p'}\left(\int_1^X d_q\lambda\right)^{(p'-\beta)/p'}\\ &= O\left(X^{(2-\delta)p'\times\frac{\beta}{p'}}X^{\frac{p'-\beta}{p'}}\right)\\ &= O\left(X^{2\beta-\delta\beta+1-\frac{\beta}{p'}}\right).\end{aligned}$$

Let us now estimate the next integral

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{\beta}|\lambda|^{2\alpha+1}d_q\lambda.$$

This integral is split into two

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{\beta}|\lambda|^{2\alpha+1}d_q\lambda = I_1+I_2,$$

where

$$I_1 = \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta\right]_e |\lambda|^{2\alpha+1} d_q\lambda$$

and

$$I_2 = \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta\right]_o |\lambda|^{2\alpha+1} d_q\lambda.$$

Estimate the summands I_1 and I_2 from above. It follows from formula (3.1) and Lemma 2.2 that

$$\mathcal{D}_q\varphi_o(\lambda) = |\lambda|^{2\beta+(2\alpha+1)\frac{\beta}{p'}} \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta\right]_e, \tag{3.2}$$

and

$$\mathcal{D}_q^+\varphi_e(\lambda) = q^{-1}|q^{-1}\lambda|^{2\beta+(2\alpha+1)\frac{\beta}{p'}} \left[|\mathcal{F}_{q,\alpha}(f)(q^{-1}\lambda)|^\beta\right]_o. \tag{3.3}$$

Using the formula (2.1), the q-integration by parts formula (2.4) and (3.2), we get

$$\begin{aligned}
I_1 &= \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta\right]_e |\lambda|^{2\alpha+1} d_q\lambda \\
&= \int_1^X \lambda^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \mathcal{D}_q\varphi_o(\lambda) d_q\lambda \\
&= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi_o(X) - \varphi_o(1) - \int_1^X \varphi_o(q\lambda)\mathcal{D}_q\left(\lambda^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\right) d_q\lambda \\
&= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi_o(X) - \varphi_o(1) - \left[(2\alpha+1)(1-\beta/p') - 2\beta\right]_q \\
&\quad \times \int_1^X \varphi_o(q\lambda)\lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda.
\end{aligned} \tag{3.4}$$

Furthemore, it follows from (2.2), the q-integration by parts formula (2.5), (2.3) and (3.3) that

$$\begin{aligned}
I_2 &= \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta\right]_o |\lambda|^{2\alpha+1} d_q\lambda \\
&= \int_q^{qX} q^{-1} \left[|\mathcal{F}_{q,\alpha}(f)(q^{-1}\lambda)|^\beta\right]_o |q^{-1}\lambda|^{2\alpha+1} d_q\lambda \\
&= \int_q^{qX} \left(q^{-1}\lambda\right)^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \mathcal{D}_q^+\varphi_e(\lambda) d_q\lambda \\
&= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi_e(X) - \varphi_e(1) - \int_q^{qX} \varphi_e(\lambda)\mathcal{D}_q^+\left(\lambda^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\right) d_q\lambda \\
&= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi_e(X) - \varphi_e(1) + \left[2\beta - (2\alpha+1)(1-\beta/p')\right]_q \\
&\quad \times \int_q^{qX} \varphi_e(\lambda)\lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda
\end{aligned}$$

$$= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi_e(X) - \varphi_e(1) + \left[2\beta - (2\alpha+1)(1-\beta/p')\right]_q$$
$$q^{-(2\beta-(2\alpha+1)(1-\beta/p'))} \times \int_1^X \varphi_e(q\lambda)\lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda$$
$$= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi_e(X) - \varphi_e(1) - \left[(2\alpha+1)(1-\beta/p') - 2\beta\right]_q$$
$$\times \int_1^X \varphi_e(q\lambda)\lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda. \tag{3.5}$$

Hence, combining the formula (3.4) and (3.5), we conclude that

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{2\alpha+1} d_q\lambda$$
$$= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi(X) - \left[(2\alpha+1)(1-\beta/p') - 2\beta\right]_q$$
$$\times \int_1^X \varphi(q\lambda)\lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda$$
$$= O\left(X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\delta\beta+\beta\left(\frac{p+1}{p}\right)}\right) - \left[(2\alpha+1)(1-\beta/p') - 2\beta\right]_q$$
$$\times \int_1^X \varphi(q\lambda)\lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda$$
$$= O\left(X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\delta\beta+\beta\left(\frac{p+1}{p}\right)}\right)$$
$$+O\left(\int_1^X \lambda^{1-\delta\beta+\beta\left(\frac{p+1}{p}\right)+2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda\right)$$
$$= O\left(X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\delta\beta+\beta\left(\frac{p+1}{p}\right)}\right)$$

and this is bounded as $X \longrightarrow \infty$ if

$$-2\beta - (2\alpha+1)\frac{\beta}{p'} + 2\alpha + 2 - \delta\beta + \beta\left(\frac{p+1}{p}\right) < 0,$$

that is

$$\beta > \frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2}$$

and this ends the proof.

In the rest of this paper, we give our second main result which is a generalization of Younis's theorem 1.2.

For this objective, we need to define the q-Bessel-Dini-Lipschitz class.

Definition 3.2 *Let* $0 < \delta < 1,\ \gamma > 0$. *A function* $f \in \mathcal{L}^p_{q,\alpha},\quad 1 < p \leq 2$ *is said to be in the q-Bessel-Dini-Lipschitz class, denoted D-q-BLip*$(\delta, \gamma, p, \alpha)$ *if*

$$\|\Delta_{q,h} f\|_{q,p,\alpha} = O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right) \quad \text{as } h \longrightarrow 0.$$

Theorem 3.2 *Let* $f \in \mathcal{L}^p_{q,\alpha}$ *whis* $1 < p \leq 2$. *If* f *belong to D-q-BLip*$(\delta, \gamma, p, \alpha)$, *then* $\mathcal{F}_{q,\alpha}(f)$ *belong to* $\mathcal{L}^\beta_{q,\alpha}(\mathbb{R}^+_q)$, *where*

$$\frac{2p\alpha+2p}{2p+2\alpha(p-1)+\delta p-2}<\beta\leq p'=\frac{p}{p-1}\ \ \textit{and}\ \ \ \beta>\frac{1}{\gamma}.$$

Proof. If $\beta=p'$. From formula (2.11) we have that $\mathcal{F}_{q,\alpha}(f)\in\mathcal{L}^{p'}_{q,\alpha}$. And for $\alpha>-\frac{1}{2}$, $0<\delta<1$ we get

$$\begin{aligned}
&1+\frac{\delta p'}{2\alpha+2}>1\\
&\Longleftrightarrow\frac{p-1}{p}\left(\frac{2\alpha+2+\delta p'}{2\alpha+2}\right)>\frac{p-1}{p}\\
&\Longleftrightarrow\frac{(2\alpha+2)(p-1)+\delta p}{p(2\alpha+2)}>\frac{1}{p'}\\
&\Longleftrightarrow\frac{p(2\alpha+2)}{(2\alpha+2)(p-1)+\delta p}<\beta=p'=\frac{p}{p-1}\\
&\Longleftrightarrow\frac{2p\alpha+2p}{2p+2\alpha(p-1)+\delta p-2}<\beta=p'=\frac{p}{p-1}.
\end{aligned}$$

So, we asume that $\beta<p'$ and $f\in$ D-q-BLip(δ,γ,p,α).
By analogy with the proof of Theorem 3.1, we can establish the following result

$$\int_0^{\eta/h}\lambda^{2p'}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{2\alpha+1}d_q\lambda=O\left(\frac{h^{(\delta-2)p'}}{\left(\log\frac{1}{h}\right)^{\gamma p'}}\right)\quad\text{as } h\longrightarrow 0.$$

Thus

$$\int_0^{X}\lambda^{2p'}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'}|\lambda|^{2\alpha+1}d_q\lambda=O\left(\frac{X^{(2-\delta)p'}}{(\log X)^{\gamma p'}}\right)\quad\text{as } h\longrightarrow 0.$$

Set

$$\varphi(x)=\int_1^{X}|\lambda^2\mathcal{F}_{q,\alpha}(f)(\lambda)|^{\beta}|\lambda|^{(2\alpha+1)\beta/p'}d_q\lambda.$$

We us the Hölder inequality we obtain

$$\varphi(X)=O\left(\frac{X^{2\beta-\delta\beta+1-\frac{\beta}{p'}}}{(\log X)^{\gamma\beta}}\right)\quad\text{as } X\longrightarrow\infty.$$

Let us estimate the next integral

$$\int_1^{X}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{\beta}|\lambda|^{2\alpha+1}d_q\lambda.$$

We write

$$\int_1^{X}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{\beta}|\lambda|^{2\alpha+1}d_q\lambda=I_1+I_2,$$

where

$$I_1=\int_1^{X}\left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^{\beta}\right]_e|\lambda|^{2\alpha+1}d_q\lambda$$

and

$$I_2 = \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta\right]_o |\lambda|^{2\alpha+1} d_q\lambda.$$

Similary as in the proof of Theorem 3.1 we have

$$I_1 = X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi_o(X) - \varphi_o(1) - [(2\alpha+1)(1-\beta/p') - 2\beta]_q \times \int_1^X \varphi_o(q\lambda)\lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda \tag{3.6}$$

and

$$I_2 = X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1}\varphi_e(X) - \varphi_e(1) - [(2\alpha+1)(1-\beta/p') - 2\beta]_q \times \int_1^X \varphi_e(q\lambda)\lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q\lambda. \tag{3.7}$$

Combining (3.6) and (3.7) we conclude that

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{2\alpha+1} d_q\lambda = O\left(\frac{X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\delta\beta+\beta\left(\frac{p+1}{p}\right)}}{(\log X)^{\gamma\beta}}\right).$$

and this is bounded as $X \longrightarrow \infty$ if

$$-2\beta - (2\alpha+1)\frac{\beta}{p'} + 2\alpha + 2 - \delta\beta + \beta\left(\frac{p+1}{p}\right) < 0 \quad \text{and} \quad -\gamma\beta < -1.$$

Hence

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} < \beta \le p' = \frac{p}{p-1}.$$

Then, the theorem is proved.

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