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Marcinkiewicz integral and its commutator on mixed Morrey spaces

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Abstract. *In this paper, we study the boundedness of the Marcinkiewicz operator μ_Ω and its commutator $\mu_{b,\Omega}$ on mixed Morrey spaces $L^{\mathbf{p},\lambda}(\mathbb{R}^n)$.*

Keywords. Mixed Morrey spaces, Marcinkiewicz operator, commutators, BMO

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1 Introduction

The classical Morrey spaces $L^{p,\lambda}$ were originally introduced by Morrey in [24] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces. In 2019, Nogayama [26] considered a new Morrey space, with the L^p norm replaced by the mixed Lebesgue norm $L^{\mathbf{p}}(\mathbb{R}^n)$, which is call mixed Morrey spaces.

For $x \in \mathbb{R}^n$, and $r > 0$, let $B(x,r)$ be the open ball centered at x with the radius r, and ${}^cB(x,r)$ be its complement. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

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(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')dx' = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_{\Omega,t} f(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t} f(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the commutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b}f(x) = \left(\int_0^\infty |F^b_{\Omega,t}f(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F^b_{\Omega,t}f(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f(y)dy.$$

On the other hand, the study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [28] considered L^p estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R^L_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. Then, Dziubański and Zienkiewicz [12] introduced the Hardy type space $H^1_L(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$, see also [1–4, 7, 15–20].

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions $\mu_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu^L_{j,\Omega}f(x) = \left(\int_0^\infty \left|\int_{B(x,t)} |\Omega(x-y)| K^L_j(x,y)f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $K^L_j(x,y) = \widetilde{K^L_j}(x,y)|x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In particular, when $V = 0$, $K^\Delta_j(x,y) = \widetilde{K^\Delta_j}(x,y)|x-y| = \frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K^\Delta_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j}\Delta^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In this paper, we write $K_j(x,y) = K^\Delta_j(x,y)$ and

$$\mu_{j,\Omega}f(x) = \left(\int_0^\infty \left|\int_{B(x,t)} |\Omega(x-y)| K_j(x,y)f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Obviously, $\mu_{j,\Omega}$ are classical Marcinkiewicz functions with rough kernel. Therefore, it will be an interesting thing to study the property of $\mu^L_{j,\Omega}$. The main purpose of this paper is to show that Marcinkiewicz operators with rough kernel associated with Schrödinger operators $\mu_{j,\Omega}^L$, $j = 1, \ldots, n$ are bounded on mixed Morrey space $L^{\mathbf{p},\lambda}(\mathbb{R}^n)$, $1 < \mathbf{p} < \infty$, $0 \le \lambda \le n$.

The commutator of the classical Marcinkiewicz function with rough kernel is defined by

$$\mu_{j,\Omega,b}f(x) = \left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)| K_j(x,y)[b(x)-b(y)]f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

The commutator $\mu_{j,\Omega,b}^L$ formed by $b \in BMO(\mathbb{R}^n)$ and the Marcinkiewicz function with rough kernel $\mu_{j,\Omega}^L$ is defined by

$$\mu_{j,\Omega,b}^L f(x) = \left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)| K_j^L(x,y)[b(x)-b(y)]f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

The well-known classical Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

where $f \in L_{loc}^1(\mathbb{R}^n)$ and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

Let T is a sublinear operator, and satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \ suppf$,

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy. \tag{1.3}$$

We point out that the condition (1.3) was first introduced by Soria and Weiss [27]. The condition (1.3) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, the Carleson's maximal operators, the Hardy-Littlewood maximal operators, the Fefferman's singular multipliers, the Fefferman's singular integrals, the Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means and so on (see [23, 27] for details).

As is well known, the commutator is also an important operator and it plays a key role in harmonic analysis. Recall that for a locally integrable function b and a integral operator T, the commutator formed by b and T is defined by $[b,T]f = bTf - T(bf)$. The commutators of the fractional maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator have been intensively studied, see [13] for more details. In this paper, the maximal commutator operator M_b under consideration is of the form

$$M_b f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x)-b(y)||f(y)|dy$$

for $f \in L_{loc}^1(\mathbb{R}^n)$.

To study a class of commutators uniformly, one can also introduce some sublinear operators with additional size conditions as before. For a function b, suppose that the commutator operator T_b represents a linear or a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin suppf$,

$$|T_b f(x)| \lesssim \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|}{|x-y|^n} |f(y)|dy. \tag{1.4}$$

The operator T_b has been studied in [14,23].

In this paper, we study the boundedness of the Marcinkiewicz operator μ_Ω and its commutator $\mu_{\Omega,b}$ on mixed Morrey spaces $L^{\mathbf{p},\lambda}(\mathbb{R}^n)$. We find the conditions with $b \in BMO(\mathbb{R}^n)$ which ensures the boundedness of the operators $\mu^L_{j,\Omega,b}$, $j = 1,\dots,n$ on mixed Morrey space $L^{\mathbf{p},\lambda}(\mathbb{R}^n)$, $1 < \mathbf{p} < \infty$, $0 \le \lambda \le n$.

By $A \lesssim B$, we mean that $A \le CB$ for some constant $C > 0$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Definitions and preliminaries

For any $r > 0$ and $x \in \mathbb{R}^n$, let $B(x,r) = \{y \ : \ |y-x| < r\}$ be the ball centered at x with radius r. Let $\mathrm{B} = \{B(x,r) \ : \ x \in \mathbb{R}^n, \ r > 0\}$ be the set of all such balls. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E.

Let $\mathcal{M}(\mathbb{R}^n)$ and $L^1_{loc}(\mathbb{R}^n)$ denote the class of Lebesgue measurable functions and locally integrable functions on $\mathbb{R}^n$, respectively. We also use $\mathbb{C}$ to represent all the complex numbers, and $\mathbb{N}$ to represent the collection of all integers.

Definition 2.1 *For $1 < p < \infty$, a non-negative function $w \in L_{loc}(\mathbb{R}^n)$ is said to be an $A_p(\mathbb{R}^n)$ weight if*

$$[w]_{A_p} = \sup_{B \in \mathrm{B}} \Big(\frac{1}{|B|}\int_B w(x)dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}dx\Big)^{\frac{p}{p'}} < \infty.$$

A non-negative local integrable function w is said to be an A_1 weight if

$$\frac{1}{|B|}\int_B w(y)dy \le Cw(x)\ , \ a.e.\ x \in B$$

for some constant $C > 0$. The infimum of all such C is denoted by $[w]_{A_1}$. We denote A_∞ by the union of all A_p $(1 \le p < \infty)$ functions.

Theorem 2.1 *[9] Suppose that Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L^\infty(S^{n-1})$. Then for every $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, there is a constant C independent of f such that*

$$\|\mu_\Omega f\|_{L^{p,w}} \le C\|f\|_{L^{p,w}}.$$

Theorem 2.2 *[10] Suppose that Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L^\infty(S^{n-1})$. Let also $b \in BMO(\mathbb{R}^n)$. Then for every $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, there is a constant $C > 0$ independent of f such that*

$$\big\|\mu_{\Omega,b} f\big\|_{L^{p,w}} \le C\|f\|_{L^{p,w}}.$$

Note that a nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \le C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \tag{2.1}$$

holds for every ball $x \in \mathbb{R}^n$ and $r > 0$, where $B(x,r)$ denotes the open ball centered at x with radius r; see [28]. It is worth pointing out that the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only n and the constant C in (2.1), such that $V \in B_{q+\varepsilon}$. Throughout this paper, we always assume that $0 \ne V \in B_n$.

Theorem 2.3 *[3, 15] Suppose that Ω satisfies* (1.1), (1.2) *and $V \in B_n$. If $\Omega \in L^\infty(S^{n-1})$, then the operators $\mu^L_{j,\Omega}$, $j = 1, \dots, n$ are bounded on $L^{p,w}(\mathbb{R}^n)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$.*

Theorem 2.4 *[3, 15] Suppose that Ω satisfies* (1.1), (1.2) *and $V \in B_n$. If $\Omega \in L^\infty(S^{n-1})$ and $b \in BMO(\mathbb{R}^n)$, then the operators $\mu^L_{j,\Omega,b}$, $j = 1, \dots, n$ are bounded on $L^{p,w}(\mathbb{R}^n)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$.*

We first recall the definition of mixed Lebesgue space defined in [6].

Let $\mathbf{p} = (p_1, \cdots, p_n) \in (0, \infty]^n$. Then the mixed Lebesgue norm $\|\cdot\|_{L^{\mathbf{p}}}$ or $\|\cdot\|_{L^{(p_1,\dots,p_n)}}$ is defined by

$$\begin{aligned}\|f\|_{L^{\mathbf{p}}} &= \|f\|_{L^{(p_1,\dots,p_n)}} \\ &= \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x_1, x_2, \dots, x_n)|^{p_1} dx_1\Big)^{\frac{p_2}{p_1}} dx_2\Big)^{\frac{p_3}{p_2}} \dots dx_n\Big)^{\frac{1}{p_n}}\end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function. If $p_j = \infty$ for some $j = 1, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L^{\mathbf{p}}(\mathbb{R}^n) = L^{(p_1,\cdots,p_n)}(\mathbb{R}^n)$ to be the set of all locally integrable functions f with $\|f\|_{L^{\mathbf{p}}} < \infty$.

Let $1 \le \mathbf{p} < \infty$ and $0 \le \lambda \le n$. We denote by $L^{\mathbf{p},\lambda}(\mathbb{R}^n)$ the mixed Morrey space the set of all classes of locally integrable functions f with the finite norm

$$\|f\|_{L^{\mathbf{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{n}\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\mathbf{p}}(B(x,t))}$$

Obviously, we recover the classical Morrey space $L^{\mathbf{p},\lambda}(\mathbb{R}^n)$ when $\mathbf{p} = p$. We point out that in [25, 26], the author used the cubes to define the mixed Morrey spaces. It is not hard to verify that the two definitions are equivalent.

As we know, the Hardy-Littlewood maximal operator M is bounded on $L^{\mathbf{p}}(\mathbb{R}^n)$, $1 < \mathbf{p} < \infty$ (see [26]), but there is no complete boundedness results for some other operators on the mixed Lebesgue spaces. To prove the boundedness of some important operators on the mixed Lebesgue space in a uniform way, we will give the extrapolation theorems on mixed Lebesgue spaces, which have their own interest.

The extrapolation theory on mixed Lebesgue spaces relies on the classical A_p weight (see [13]).

We also need the boundedness of M on mixed norm space $L^{\mathbf{p}}(\mathbb{R}^n)$ [26].

Lemma 2.1 *[26] For $1 < \mathbf{p} < \infty$, there holds*

$$\|Mf\|_{L^{\mathbf{p}}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\mathbf{p}}(\mathbb{R}^n)}. \tag{2.2}$$

By $\mathfrak{F}$, we mean a family of pair (f, g) of non-negative measurable functions that are not identical to zero. For such a family $S, p > 0$ and a weight $w \in A_p$, the expression

$$\int_{\mathbb{R}^n} f(x)^p w(x)dx \lesssim \int_{\mathbb{R}^n} g(x)^p w(x)dx, \ (f, g) \in \mathfrak{F}$$

means that this inequality holds for all pair $(f, g) \in \mathfrak{F}$ if the left hand side is finite, and the implicated constant depends only on p and A_p.

Now we give the extrapolation theorems on the mixed Lebesgue spaces. The first one is the diagonal extrapolation theorem.

Theorem 2.5 *Let $0 < p_0 < \infty$ and $\mathbf{p} = (p_1, \cdots, p_n) \in (0, \infty)^n$. Let $f, g \in \mathcal{M}(\mathbb{R}^n)$. Suppose for every $w \in A_1$, we have*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x)dx \lesssim \int_{\mathbb{R}^n} g(x)^{p_0} w(x)dx, \ (f, g) \in \mathfrak{F}. \tag{2.3}$$

Then if $\mathbf{p} > p_0$, we have

$$\|f\|_{L^{\mathbf{p}}(\mathbb{R}^n)} \lesssim \|g\|_{L^{\mathbf{p}}(\mathbb{R}^n)}, (f, g) \in \mathfrak{F}. \tag{2.4}$$

Proof. Without loss of generality, one may assume f is a non-negative function. We use the Rubio de Francia iteration algorithm presented in [8].

Let $\bar{\mathbf{p}} = \mathbf{p}/p_0$ and $\bar{\mathbf{p}'} = \mathbf{p}'/p_0$. By the assumptions and Lemma 2.1, the maximal operator is bounded on $L^{\bar{\mathbf{p}'}}(\mathbb{R}^n)$, so there exists a positive constant B such that

$$\|Mf\|_{L^{\bar{\mathbf{p}'}}(\mathbb{R}^n)} \leq B\|f\|_{L^{\bar{\mathbf{p}'}}(\mathbb{R}^n)}.$$

For any non-negative function h, define a new operator $\mathfrak{R}h$ by

$$\mathfrak{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k B^k},$$

where for $k \geq 1, M^k$ denotes k iterations of the maximal operator, and M^0 is the identity operator. The operator $\mathfrak{R}$ satisfies

$$h(x) \leq \mathfrak{R}h(x), \tag{2.5}$$

$$\|\mathfrak{R}h\|_{L^{\bar{\mathbf{p}'}}} \leq 2\|h\|_{L^{\bar{\mathbf{p}'}}}, \tag{2.6}$$

$$\|\mathfrak{R}h\|_{A_1} \leq 2B. \tag{2.7}$$

The inequality (2.5) is straight-forward. Since

$$M(\mathfrak{R}h) \leq \sum_{k=0}^{\infty} \frac{M^{k+1}h}{2^k B^k} \leq 2B \sum_{k=1}^{\infty} \frac{M^k h}{2^k B^k} \leq 2B\mathfrak{R}h,$$

the properties (2.6) and (2.7) are consequences of Lemma 2.1 and the definition of A_1. Since the dual of $L^{\bar{\mathbf{p}}}(\mathbb{R}^n)$ is $L^{\bar{\mathbf{p}'}}(\mathbb{R}^n)$, we get

$$\begin{aligned} \|f\|_{L^{\mathbf{p}}}^{p_0} &= \|f^{p_0}\|_{L^{\bar{\mathbf{p}}}} \\ &\lesssim \sup\Big\{\int_{\mathbb{R}^n} |f(x)|^{p_0} h(x)dx : \|h\|_{L^{\bar{\mathbf{p}'}}} \leq 1, \ h \geq 0\Big\}. \end{aligned} \tag{2.8}$$

By Hölder's inequality on the mixed Lebesgue spaces and (2.5), we have

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)^{p_0} h(x)dx &\lesssim \int_{\mathbb{R}^n} f(x)^{p_0} \mathfrak{R}h(x)dx \\ &\lesssim \|f^{p_0}\|_{L^{\bar{\mathbf{p}}}} \|h\|_{L^{\bar{\mathbf{p}'}}} < \infty. \end{aligned} \tag{2.9}$$

In view of (2.5) and $\mathfrak{R}h \in A_1$, we use (2.3) with $w = \mathfrak{R}h(x)$ to obtain

$$\int_{\mathbb{R}^n} f(x)^{p_0} h(x)dx \lesssim \int_{\mathbb{R}^n} f(x)^{p_0}\, \mathfrak{R}h(x)dx \lesssim \int_{\mathbb{R}^n} g(x)^{p_0}\, [\mathfrak{R}h(x)]\, dx.$$

Combining (2.6) with (2.9) and using Hölder's inequality on the mixed Lebesgue spaces again, we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{p_0} h(x)dx &\lesssim \|g^{p_0}\|_{L^{\bar{\mathbf{p}}}} \|\mathfrak{R}h\|_{L^{\bar{\mathbf{p}}'}} \\ &\approx \|g\|^{p_0}_{L^{\bar{\mathbf{p}}}} \|\mathfrak{R}h\|_{L^{\bar{\mathbf{p}}'}}.\end{aligned} \tag{2.10}$$

Therefore

$$\|\mathfrak{R}h\|_{L^{\bar{\mathbf{p}}'}} \lesssim \|h\|_{L^{\bar{\mathbf{p}}'}}. \tag{2.11}$$

By taking supremum over all $h \in L^{\bar{\mathbf{p}}}(\mathbb{R}^n)$ with $\|h\|_{L^{\bar{\mathbf{p}}}} \le 1$, (2.8) , (2.10) and (2.11) give us the desired conclusion (2.4) .

We point out that when $n = 2$, there are different versions of the diagonal extrapolation theorem [21] and the off-diagonal extrapolation theorem [29] on mixed Lebesgue spaces, which are different form Theorem 2.5.

By the density of smooth functions with compact support $C_c^\infty(\mathbb{R}^n)$ in the mixed Lebesgue space $L^{\mathbf{p}}(\mathbb{R}^n), 1 < \mathbf{p} < \infty$ (see [6]), one can apply Theorem 2.5 to the mapping property of some sublinear operators.

Theorem 2.6 *Suppose $0 < p_0 < \mathbf{p} < \infty$ and T is a sublinear operator such that for every $w \in A_1$,*

$$\int_{\mathbb{R}^n} |Tf(z)|^{p_0} w(z)dz \lesssim \int_{\mathbb{R}^n} |f(z)|^{p_0} w(z)dz,\ f \in C_c^\infty(\mathbb{R}^n)\ .$$

Then T can be extended to be a bounded operator on $L^{\mathbf{p}}(\mathbb{R}^n)$.

Proof. By Theorem 2.5, for any $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$\|Tf\|_{L^{\mathbf{p}}} \lesssim \|f\|_{L^{\mathbf{p}}}.$$

Since T is a sublinear operator, we have $|T(f) - T(g)| \le |T(f-g)|$, and hence, for any $f, g \in C_c^\infty(\mathbb{R}^n)$, we have

$$\|Tf - Tg\|_{L^{\mathbf{p}}} \le \|T(f-g)\|_{L^{\mathbf{p}}} \lesssim \|f-g\|_{L^{\mathbf{p}}}.$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{\mathbf{p}}(\mathbb{R}^n)$, the above inequalities guarantee that T can be extended to be a bounded operator on $L^{\mathbf{p}}(\mathbb{R}^n)$.

The following corollary is a consequence of Theorem 2.6 and the weighted boundedness of the corresponding operators.

Corollary 2.1 *Let $1 < \mathbf{p} < \infty, b \in BMO$, then $M, \mu_\Omega, \mu^L_{j,\Omega}$, $M_b, \mu_{\Omega,b}, \mu^L_{j,\Omega,b}$ are all bounded on $L^{\mathbf{p}}(\mathbb{R}^n)$.*

Proof. It is well known that $M, \mu_\Omega, \mu^L_{j,\Omega}$, $M_b, \mu_{\Omega,b}, \mu^L_{j,\Omega,b}$ are all sublinear operators, and bounded on $L^{p_0,w}(\mathbb{R}^n)$ for arbitrary $1 < p_0 < \infty$ and $w \in A_{p_0}$ (see [13] for example). Since $A_1 \subset A_p$, Theorem 2.6 implies that $M, \mu_\Omega, \mu^L_{j,\Omega}$, $M_b, \mu_{\Omega,b}, \mu^L_{j,\Omega,b}$ are all bounded on $L^{\mathbf{p}}(\mathbb{R}^n)$ for all $p_0 < \mathbf{p} < \infty$. In view of the arbitrariness of $1 < p_0 < \infty$, $M, \mu_\Omega, \mu^L_{j,\Omega}$, $M_b, \mu_{\Omega,b}, \mu^L_{j,\Omega,b}$ are also bounded on $L^{\mathbf{p}}(\mathbb{R}^n)$ for all $< \mathbf{p} < \infty$.

3 Marcinkiewicz operator μ_Ω in mixed Morrey spaces

In this section, we investigate the boundedness of μ_Ω satisfies the conditions (1.1), (1.2) and $\Omega \in L^\infty(S^{n-1})$ on the mixed Morrey space $L^{\mathbf{p},\lambda}$.

We first prove one lemma, which give us the explicit estimates for the $L^{\mathbf{p}}(\mathbb{R}^n)$ norm of μ_Ω on a given ball $B(x_0,r)$.

Lemma 3.1 *Let Ω be satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L^\infty(S^{n-1})$.
Then for $1<\mathbf{p}<\infty$, the inequality

$$\|\mu_\Omega f\|_{L^{\mathbf{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^\infty t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\mathbf{p}}(B(x_0,t))}\, dt \tag{3.1}$$

holds for any ball $B(x_0,r)$ and all $f \in L^{\mathbf{p}}_{loc}(\mathbb{R}^n)$.

Proof. For any ball $B=B(x_0,r)$, Let $2B=B(x_0,2r)$ be the ball centered at x_0, with the radius $2r$. we represent f as $f=f_1+f_2$, where

$$f_1(y)=f\chi_{2B}(y)\ ,\ f_2(y)=f\chi_{^c(2B)}(y)\ ,\ r>0.$$

Since T is a sublinear operator, we have

$$\|\mu_\Omega f\|_{L^{\mathbf{p}}(B)} \le \|\mu_\Omega f_1\|_{L^{\mathbf{p}}(B)} + \|\mu_\Omega f_2\|_{L^{\mathbf{p}}(B)}.$$

Noting that $f_1 \in L^{\mathbf{p}}(\mathbb{R}^n)$ and μ_Ω is bounded in $L^{\mathbf{p}}(\mathbb{R}^n)$ (see Corollary 2.1), we have

$$\|\mu_\Omega f_1\|_{L^{\mathbf{p}}(B)} \le \|\mu_\Omega f_1\|_{L^{\mathbf{p}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{\mathbf{p}}(\mathbb{R}^n)} = \|f\|_{L^{\mathbf{p}}(2B)}.$$

It is clear that $x\in B, y \in {}^c(2B)$ imply $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$, which further yields

$$|\mu_\Omega f_2(x)| \lesssim \int_{^c(2B)} \frac{|f(y)|}{|x_0-y|^n} dy.$$

By Fubini's theorem, we have

$$\begin{aligned}\int_{^c(2B)} \frac{|f(y)|}{|x_0-y|^n} dy &\approx \int_{^c(2B)} |f(y)| \int_{|x_0-y|}^\infty \frac{dt}{t^{n+1}} dy \\ &\approx \int_{2r}^\infty \int_{2r\le|x_0-y|<t} |f(y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^\infty \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}.\end{aligned}$$

Applying Hölder's inequality on the mixed Lebesgue spaces (see [6]), we obtain

$$\int_{^c(2B)} \frac{|f(y)|}{|x_0-y|^n} dy \lesssim \int_{2r}^\infty \|f\|_{L^{\mathbf{p}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}. \tag{3.2}$$

Moreover, for all $1<\mathbf{p}<\infty$, we have

$$\|\mu_\Omega f_2\|_{L^{\mathbf{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^\infty \|f\|_{L^{\mathbf{p}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.$$

Therefore, one gets

$$\|\mu_\Omega f\|_{L^{\mathbf{p}}(B(x_0,r))} \lesssim \|f\|_{L^{\mathbf{p}}(2B)} + r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\mathbf{p}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.$$

On the other hand,

$$\begin{aligned} \|f\|_{L^{\mathbf{p}}(2B)} &\approx r^{\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\mathbf{p}}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}} \\ &\lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\mathbf{p}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}. \end{aligned} \tag{3.3}$$

Thus

$$\|\mu_\Omega f\|_{L^{\mathbf{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\mathbf{p}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.$$

Now we can present the first main result in this section.

Theorem 3.1 *Let Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L^\infty(S^{n-1})$. Let also $1 < \mathbf{p} < \infty$, and $0 \le \lambda \le n$. Then the operator μ_Ω is bounded on $L^{\mathbf{p},\lambda}$. Moreover,*

$$\|\mu_\Omega f\|_{L^{\mathbf{p},\lambda}} \lesssim \|f\|_{L^{\mathbf{p},\lambda}}.$$

Proof. From the inequality (3.1) we get

$$\begin{aligned} \|\mu_\Omega f\|_{L^{\mathbf{p},\lambda}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{n}\sum_{i=1}^n \frac{1}{p_i}} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\mathbf{p}}(B(x_0,t))}\, dt \\ &\lesssim \|f\|_{L^{\mathbf{p},\lambda}} \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{n}\sum_{i=1}^n \frac{1}{p_i}} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{r}^{\infty} t^{-1-\sum_{i=1}^n \frac{1}{p_i}} t^{\frac{\lambda}{n}\sum_{i=1}^n \frac{1}{p_i}}\, dt \\ &= \|f\|_{L^{\mathbf{p},\lambda}} \int_1^{\infty} t^{-1-(1-\frac{\lambda}{n})\sum_{i=1}^n \frac{1}{p_i}}\, dt \\ &\lesssim \|f\|_{L^{\mathbf{p},\lambda}}. \end{aligned}$$

By taking $\mathbf{p} = (p, \ldots, p)$ in Theorem 3.1, we obtain the boundedness of μ_Ω on the Morrey spaces.

4 Commutator of Marcinkiewicz operator $\mu_{\Omega,b}$ in mixed Morrey spaces

In this section, we investigate the boundedness of $\mu_{\Omega,b}$ conditions (1.1), (1.2) and $\Omega \in L^\infty(S^{n-1})$ on the mixed Morrey space $L^{\mathbf{p},\lambda}$. First, we review the definition of $BMO(\mathbb{R}^n)$, the bounded mean oscillation space. A function $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to the bounded mean oscillation space $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy < \infty. \tag{4.1}$$

If one regards two functions whose difference is a constant as one, then the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|.\|_{BMO}$. The John-Nirenberg ineuqalitiy for BMO yields that for any $1 < q < \infty$ and $f \in BMO(\mathbb{R}^n)$, the BMO norm of f is equivalent to

$$\|f\|_{BMO^q} = \sup_{x\in\mathbb{R}^n, r>0} \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^q dy \Big)^{\frac{1}{q}}$$

Recall that for any $\mathbf{p}=(p_1,\ \cdots,\ p_n)\in(1,\ \infty)^n$, the John-Nirenberg inequality for mixed norm space [22] shows that the BMO norm of all $f\in BMO(\mathbb{R}^n)$ is also equivalent to

$$\|f\|_{BMO^{\mathbf{p}}}=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|(f-f_{B(x,r)})\chi_{B(x,r)}\|_{L^{\mathbf{p}}}}{\|\chi_{B(x,r)}\|_{L^{\mathbf{p}}}}. \tag{4.2}$$

The following property for BMO functions is valid.

Lemma 4.1 *Let $f\in BMO(\mathbb{R}^n)$. Then for all $0<2r<t$, we have*

$$|f_{B(x,r)}-f_{B(x,t)}|\lesssim\|f\|_{BMO}\ln\frac{t}{r}. \tag{4.3}$$

We first prove one lemma, which give us the explicit estimates for the $L^{\mathbf{p}}(\mathbb{R}^n)$ norm of $\mu_{\Omega,b}$ on a given ball $B(x_0,r)$.

Lemma 4.2 *Let Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega\in L^{\infty}(S^{n-1})$. Let also $1<\mathbf{p}<\infty$ and $b\in BMO(\mathbb{R}^n)$. Then the inequality*

$$\begin{aligned}&\|\mu_{\Omega,b}f\|_{L^{\mathbf{p}}(B(x_0,r))}\\ &\lesssim\|b\|_{BMO}\,r^{\sum_{i=1}^n\frac{1}{p_i}}\int_{2r}^{\infty}\big(1+\ln\frac{t}{r}\big)\,t^{-1-\sum_{i=1}^n\frac{1}{p_i}}\|f\|_{L^{\mathbf{p}}(B(x_0,t))}dt\end{aligned} \tag{4.4}$$

holds for any ball $B(x_0,r)$ and all $f\in L^{\mathbf{p}}_{loc}(\mathbb{R}^n)$.

Proof. For any ball $B=B(x_0,r)$, Let $2B=B(x_0,2r)$. Write f as $f=f_1+f_2$, where $f_1=f\chi_{2B}$ and $f_2=f\chi_{^c(2B)}$.

Since $\mu_{\Omega,b}$ is a sublinear operator, we have

$$\|\mu_{\Omega,b}f\|_{L^{\mathbf{p}}(B)}\le\|\mu_{\Omega,b}f_1\|_{L^{\mathbf{p}}(B)}+\|\mu_{\Omega,b}f_2\|_{L^{\mathbf{p}}(B)}.$$

Noting that $f_1\in L^{\mathbf{p}}(\mathbb{R}^n)$ and $\mu_{\Omega,b}$ is bounded in $L^{\mathbf{p}}(\mathbb{R}^n)$ (see Corollary 2.1), we have

$$\|\mu_{\Omega,b}f_1\|_{L^{\mathbf{p}}(B)}\le\|\mu_{\Omega,b}f_1\|_{L^{\mathbf{p}}(\mathbb{R}^n)}\lesssim\|b\|_{BMO}\|f_1\|_{L^{\mathbf{p}}(\mathbb{R}^n)}=\|b\|_{BMO}\|f\|_{L^{\mathbf{p}}(2B)}.$$

Since $x\in B, y\in^c(2B)$ imply $\frac{1}{2}|x_0-y|\le|x-y|\le\frac{3}{2}|x_0-y|$, we get

$$\begin{aligned}|\mu_{\Omega,b}f_2(x)|&\lesssim\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|}{|x-y|^n}|f(y)|dy\\ &\approx\int_{^c(2B)}\frac{|b(x)-b(y)|}{|x_0-y|^n}|f(y)|dy.\end{aligned}$$

By the generalized Minkowski's inequality on mixed Lebesgue spaces (see [6]), we have

$$\begin{aligned}\|\mu_{\Omega,b}f_2\|_{L^{\mathbf{p}}(B))}&\lesssim\Big\|\int_{^c(2B)}\frac{|b(\cdot)-b(y)|}{|x_0-y|^n}|f(y)|dy\Big\|_{L^{\mathbf{p}}(B(x_0,r))}\\ &\lesssim\Big\|\int_{^c(2B)}\frac{|b_B-b(y)|}{|x_0-y|^n}|f(y)|dy\Big\|_{L^{\mathbf{p}}(B(x_0,r)}\\ &+\Big\|\int_{^c(2B)}\frac{|b(\cdot)-b_B|}{|x_0-y|^n}|f(y)|dy\Big\|_{L^{\mathbf{p}}(B(x_0,r)}\\ &=\ I_1+I_2.\end{aligned}$$

For the term I_1, we have

$$\begin{aligned}
I_1 &\approx r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{^c(2B)} \frac{|b_B - b(y)|}{|x_0 - y|^n} |f(y)| dy \\
&\approx r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{^c(2B)} |b(y) - b_B||f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \int_{2r\le|x_0-y|<t} |b(y) - b_B||f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_B||f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality and by (4.2), (4.3), we get

$$\begin{aligned}
I_1 &\lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)| dy \frac{dt}{t^{n+1}} \\
&+ r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b_{B(x_0,r)} - b_{B(x_0,t)}||f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} \|(b(\cdot) - b_{B(x_0,t)})\chi_{B(x_0,t)}\|_{L^{\mathbf{p}'}} \|f\|_{L^{\mathbf{p}}(B(x_0,t)} dy \frac{dt}{t^{n+1}} \\
&+ r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L^{\mathbf{p}}(B(x_0,t)} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}} \\
&\lesssim \|b\|_{BMO} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \big(1 + \ln \frac{t}{r}\big) \|f\|_{L^{\mathbf{p}}(B(x_0,t)} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.
\end{aligned}$$

In order to estimate I_2, note that

$$I_2 = \int_{^c(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \cdot \|b(\cdot) - b_B\|_{L^{\mathbf{p}}(B(x_0,r)}.$$

It follows from (4.2) that

$$I_2 \lesssim \|b\|_{BMO} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{^c(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

Thus by (3.2), we get

$$I_2 \lesssim \|b\|_{BMO} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\mathbf{p}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.$$

Summing up I_1 and I_2, we get

$$\|\mu_{\Omega,b} f_2\|_{L^{\mathbf{p}}(B)} \le \|b\|_{BMO} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \big(1 + \ln \frac{t}{r}\big) \|f\|_{L^{\mathbf{p}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.$$

Therefore, by (3.3), there holds

$$\begin{aligned}
\|\mu_{\Omega,b} f_2\|_{L^{\mathbf{p}}(B)} &\lesssim \|b\|_{BMO} \|f\|_{L^{\mathbf{p}}(2B)} \\
&+ \|b\|_{BMO} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \big(1 + \ln \frac{t}{r}\big) t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\mathbf{p}}(B(x_0,t))} dt \\
&\lesssim \|b\|_{BMO} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \big(1 + \ln \frac{t}{r}\big) t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\mathbf{p}}(B(x_0,t))} dt.
\end{aligned}$$

We are done.

Now we give the boundedness of $\mu_{\Omega,b}$ on the mixed Morrey space.

Theorem 4.1 *Let Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L^{\infty}(S^{n-1})$. Let also $1 < \mathbf{p} < \infty$, $b \in BMO(\mathbb{R}^n)$, and $0 \le \lambda \le n$. Then the operator $\mu_{\Omega,b}$ is bounded on $L^{\mathbf{p},\lambda}$. Moreover,*

$$\|\mu_{\Omega,b} f\|_{L^{\mathbf{p},\lambda}} \lesssim \|b\|_{BMO} \, \|f\|_{L^{\mathbf{p},\lambda}}.$$

Proof. From the inequality (4.4) we get

$$\begin{aligned}
&\|\mu_{\Omega,b} f\|_{L^{\mathbf{p},\lambda}} \lesssim \|b\|_{BMO} \\
&\times \sup_{x\in\mathbb{R}^n, r>0} r^{(1-\frac{\lambda}{n})\sum_{i=1}^{n}\frac{1}{p_i}} \int_{2r}^{\infty} \big(1+\ln\frac{t}{r}\big)\, t^{-1-\sum_{i=1}^{n}\frac{1}{p_i}} \, \|f\|_{L^{\mathbf{p}}(B(x_0,t))}\, dt \\
&\lesssim \|b\|_{BMO} \, \|f\|_{L^{\mathbf{p},\lambda}} \sup_{x\in\mathbb{R}^n, r>0} r^{(1-\frac{\lambda}{n})\sum_{i=1}^{n}\frac{1}{p_i}} \int_{r}^{\infty} \big(1+\ln\frac{t}{r}\big)\, t^{-1-(1-\frac{\lambda}{n})\sum_{i=1}^{n}\frac{1}{p_i}}\, dt \\
&= \|b\|_{BMO} \, \|f\|_{L^{\mathbf{p},\lambda}} \sup_{x\in\mathbb{R}^n, r>0} \int_{1}^{\infty} (1+\ln t)\, t^{-1-(1-\frac{\lambda}{n})\sum_{i=1}^{n}\frac{1}{p_i}}\, dt \\
&\lesssim \|b\|_{BMO} \, \|f\|_{L^{\mathbf{p},\lambda}}.
\end{aligned}$$

By taking $\mathbf{p} = (p, \ldots, p)$ in Theorem 4.1, we obtain the boundedness of $\mu_{\Omega,b}$ on the Morrey spaces.

5 Marcinkiewicz operator $\mu^{L}_{j,\Omega}$ and its commutator $\mu^{L}_{j,\Omega,b}$ in mixed Morrey spaces

In this section, we prove the boundedness of the Marcinkiewicz operator $\mu^{L}_{j,\Omega}$ and its commutator $\mu^{L}_{j,\Omega,b}$ on mixed Morrey space $L^{\mathbf{p},\lambda}$.

For $x \in \mathbb{R}^n$, the function $\rho(x)$ is defined by

$$\rho(x) = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V\,(y)dy \ \le 1 \right\}.$$

Lemma 5.1 *[28] Let $V \in B_q$ with $q \ge n/2$. Then there exists $l_0 > 0$ such that*

$$\frac{l}{C}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l_0} \le \frac{\rho(y)}{\rho(x)} \le C\left(1+\frac{|x-y|}{\rho(x)}\right)^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \sim \rho(y)$ if $|x-y| < C\rho(x)$.

Lemma 5.2 *[28] Let $V \in B_q$ with $q \ge n/2$. For any $l > 0$, there exists $C_l > 0$ such that*

$$\left|K^{L}_{j}(x,y)\right| \le \frac{C_l}{\left(1+\frac{|x-y|}{\rho(x)}\right)^{l}} \frac{1}{|x-y|^{n-1}},$$

and

$$\left|K^{L}_{j}(x,y) - K_j(x-y)\right| \le \ C\frac{\rho(x)}{|x-y|^{n-2}}.$$

Analogously proof of Lemma 3.1 and Theorem 3.1 the following results is valid.

Lemma 5.3 *Let Ω be satisfies the conditions* (1.1), (1.2), $\Omega \in L^\infty(S^{n-1})$ *and* $V \in B_n$. *Then for* $1 < \mathbf{p} < \infty$, *the inequality*

$$\|\mu^L_{j,\Omega} f\|_{L^{\mathbf{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^\infty t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\mathbf{p}}(B(x_0,t))}\, dt$$

holds for any ball $B(x_0,r)$ *and all* $f \in L^{\mathbf{p}}_{loc}(\mathbb{R}^n)$.

Theorem 5.1 *Let Ω be satisfies the conditions* (1.1), (1.2), $\Omega \in L^\infty(S^{n-1})$ *and* $V \in B_n$. *Let also* $1 < \mathbf{p} < \infty$, *and* $0 \le \lambda \le n$. *Then the operator* $\mu^L_{j,\Omega}$ *is bounded on* $L^{\mathbf{p},\lambda}$. *Moreover,*

$$\|\mu^L_{j,\Omega} f\|_{L^{\mathbf{p},\lambda}} \lesssim \|f\|_{L^{\mathbf{p},\lambda}}.$$

Analogously proof of Lemma 4.2 and Theorem 4.1 the following results is valid.

Lemma 5.4 *Let Ω be satisfies the conditions* (1.1), (1.2), $\Omega \in L^\infty(S^{n-1})$ *and* $V \in B_n$. *Then for* $1 < \mathbf{p} < \infty$ *and* $b \in BMO(\mathbb{R}^n)$, *the inequality*

$$\|\mu^L_{j,\Omega,b} f\|_{L^{\mathbf{p}}(B(x_0,r))} \lesssim \|b\|_{BMO}\, r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)\, t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\mathbf{p}}(B(x_0,t))}\, dt$$

holds for any ball $B(x_0,r)$ *and all* $f \in L^{\mathbf{p}}_{loc}(\mathbb{R}^n)$.

Theorem 5.2 *Let Ω be satisfies the conditions* (1.1), (1.2), $\Omega \in L^\infty(S^{n-1})$ *and* $V \in B_n$. *Let also* $1 < \mathbf{p} < \infty$, $b \in BMO(\mathbb{R}^n)$, *and* $0 \le \lambda \le n$. *Then the operator* $\mu^L_{j,\Omega,b}$ *is bounded on* $L^{\mathbf{p},\lambda}$. *Moreover,*

$$\|\mu^L_{j,\Omega,b} f\|_{L^{\mathbf{p},\lambda}} \lesssim \|b\|_{BMO}\, \|f\|_{L^{\mathbf{p},\lambda}}.$$

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