

Characterizations of anisotropic Lipschitz functions via the commutators of anisotropic maximal function in total anisotropic Morrey spaces

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Abstract. We shall give necessary and sufficient conditions for the boundedness of the anisotropic maximal commutators M_b^d and the commutators of the anisotropic maximal operator $[b, M^d]$ in total Morrey spaces $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ when b belongs to anisotropic Lipschitz spaces $\dot{A}_{\beta,d}(\mathbb{R}^n)$, whereby some new characterizations for certain subclasses of anisotropic Lipschitz spaces $\dot{A}_{\beta,d}(\mathbb{R}^n)$ are obtained.

Keywords. Total anisotropic Morrey spaces, anisotropic maximal function, anisotropic fractional maximal function, commutator, anisotropic Lipschitz spaces

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1 Introduction

The aim of this paper is to study anisotropic maximal commutators M_b^d and commutators of the anisotropic maximal operator $[b, M^d]$ in total anisotropic Morrey spaces $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ when b belongs to anisotropic Lipschitz spaces $\dot{A}_{\beta,d}(\mathbb{R}^n)$.

Let $\mathbb{R}^n$ be the n -dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$, S^{n-1} denotes the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $\mathcal{E}(x, r)$ denote the open ball centered at x of radius r and ${}^c\mathcal{E}(x, r)$ denote the set $\mathbb{R}^n \setminus \mathcal{E}(x, r)$. Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv (t^{d_1} x_1, \dots, t^{d_n} x_n)$. By [5, 7], the function $F(x, \rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x, \rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x - y)$ defines a distance between any

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two points $x, y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ , defines a homogeneous metric space ([5–7]). The balls with respect to ρ , centered at x of radius r , are just the ellipsoids

$$\mathcal{E}_d(x, r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \dots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x, r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let also $\Pi_d(x, r) = \{y \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelepiped, ${}^c\mathcal{E}_d(x, r) = \mathbb{R}^n \setminus \mathcal{E}_d(x, r)$ be the complement of $\mathcal{E}_d(0, r)$. If $d = \mathbf{1} \equiv (1, \dots, 1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_1(x, r) = \mathcal{E}(x, r)$. Note that in the standard parabolic case $d = (1, \dots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \quad x = (x', x_n).$$

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The anisotropic fractional maximal operator M_α^d is given by

$$M_\alpha^d f(x) = \sup_{t>0} |\mathcal{E}(x, t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}(x, t)} |f(y)| dy, \quad 0 \leq \alpha < |d|,$$

where $|\mathcal{E}(x, t)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x, t)$. If $\alpha = 0$, then $M^d \equiv M_0^d$ is the anisotropic Hardy-Littlewood maximal operator. If $d = \mathbf{1}$, then $M_\alpha \equiv M_\alpha^d$ is the fractional maximal operator and $M \equiv M^d$ is the classical Hardy-Littlewood maximal operator.

The anisotropic maximal commutator of M^d with a locally integrable function b is defined by

$$M_b^d f(x) = \sup_{t>0} |\mathcal{E}(x, t)|^{-1} \int_{\mathcal{E}(x, t)} |b(x) - b(y)| |f(y)| dy.$$

If $d = \mathbf{1}$, then $M_b \equiv M_b^d$ is the maximal commutator. The operators M_α^d and M_b^d play an important role in real and harmonic analysis (see, for example [28, 29]).

On the other hand, we can define the (nonlinear) commutator of the anisotropic maximal operator M^d with a locally integrable function b by

$$[b, M^d]f(x) = b(x)M^d f(x) - M^d(bf)(x).$$

Obviously, operators M_b^d and $[b, M^d]$ essentially differ from each other since M_b^d is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear.

The operators M , $[b, M]$ and M_b play an important role in real and harmonic analysis and applications (see, for instance [1–4, 8, 14, 16–18, 21, 22, 24, 25, 30]).

In 1978, Janson [20] gave some characterizations of the Lipschitz space $\dot{A}_{\beta, d}(\mathbb{R}^n)$ via commutator $[b, T]$ and the author proved that $b \in \dot{A}_{\beta, d}(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < n/\beta$, $1/p - 1/q = \beta/n$ and T is the classical singular integral operator (see also [26]).

Morrey spaces, introduced by C. B. Morrey [23], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In [14] Guliyev introduce a variant of Morrey spaces called total Morrey spaces $L_{p, \lambda, \mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. In [1] the authors was consider the total anisotropic Morrey spaces $L_{p, \lambda, \mu}^d(\mathbb{R}^n)$, give basic properties of the spaces $L_{p, \lambda, \mu}^d(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L_{p, \lambda, \mu}^d(\mathbb{R}^n)$. Was also given necessary and sufficient conditions for the boundedness

of the anisotropic maximal commutator operator M_b^d and commutator of anisotropic maximal operator $[b, M^d]$ on $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$. Was obtained some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

The aim of this paper is to give necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator operator M_b^d and commutator of anisotropic maximal operator $[b, M^d]$ on $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ when b belongs to anisotropic Lipschitz spaces $\dot{A}_{\beta,d}(\mathbb{R}^n)$. New characterizations of some subclasses of anisotropic Lipschitz spaces $\dot{A}_{\beta,d}(\mathbb{R}^n)$ are obtained.

The structure of the paper is as follows. In Section 2 we give some theorems about the boundedness of anisotropic fractional maximal operator M_α^d on the total anisotropic Morrey spaces $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$. In Section 3 we find necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator M_b^d on $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ spaces. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ on $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

Definition 2.1 Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda}^d(\mathbb{R}^n)$ the anisotropic Morrey space, by $\tilde{L}_{p,\lambda}^d(\mathbb{R}^n)$ the modified anisotropic Morrey space [11, 13], and by $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ the total anisotropic Morrey space [1, 14] the set of all classes of locally integrable functions f with the finite norms

$$\begin{aligned} \|f\|_{L_{p,\lambda}^d} &= \sup_{x \in \mathbb{R}^n, t > 0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))}, \\ \|f\|_{\tilde{L}_{p,\lambda}^d} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))}, \\ \|f\|_{L_{p,\lambda,\mu}^d} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(\mathcal{E}(x,t))}, \end{aligned}$$

respectively.

Definition 2.2 Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak anisotropic Morrey space $WL_{p,\lambda}^d(\mathbb{R}^n)$, the weak modified anisotropic Morrey space $W\tilde{L}_{p,\lambda}^d(\mathbb{R}^n)$ [11, 13] and the weak total anisotropic Morrey space $WL_{p,\lambda,\mu}^d(\mathbb{R}^n)$ [1, 14] as the set of all locally integrable functions f with finite norms

$$\begin{aligned} \|f\|_{WL_{p,\lambda}^d} &= \sup_{x \in \mathbb{R}^n, t > 0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))}, \\ \|f\|_{W\tilde{L}_{p,\lambda}^d} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))}, \\ \|f\|_{WL_{p,\lambda,\mu}^d} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))}, \end{aligned}$$

respectively.

Lemma 2.1 *If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq \gamma$, then*

$$L_{p,\lambda,\mu}^P(\mathbb{R}^n) = L_{p,\lambda}^P(\mathbb{R}^n) \cap L_{p,\mu}^P(\mathbb{R}^n)$$

and

$$\|f\|_{L_{p,\lambda,\mu}^P(\mathbb{R}^n)} = \max \left\{ \|f\|_{L_{p,\lambda}^P}, \|f\|_{L_{p,\mu}^P} \right\}.$$

Proof. Let $f \in L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ and $0 \leq \mu \leq \lambda \leq \gamma$. Then

$$\begin{aligned} \|f\|_{L_{p,\lambda}^P} &= \|f\|_{L^{p,\lambda,\lambda}} \\ &= \max \left\{ \sup_{x \in \mathbb{R}^n, 0 < t \leq 1} t^{-\frac{\lambda}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}, \sup_{x \in \mathbb{R}^n, t > 1} t^{-\frac{\mu}{p}} t^{\frac{\mu-\lambda}{p}} \|f\|_{L^p(\mathcal{E}(x,t))} \right\} \\ &\leq \|f\|_{L_{p,\lambda,\mu}^P(\mathbb{R}^n)} \end{aligned}$$

and

$$\begin{aligned} \|f\|_{L_{p,\mu}^P} &= \|f\|_{L^{p,\mu,\mu}} \\ &= \max \left\{ \sup_{x \in \mathbb{R}^n, 0 < t \leq 1} t^{\frac{\lambda-\mu}{p}} t^{-\frac{\lambda}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}, \sup_{x \in \mathbb{R}^n, t > 1} t^{-\frac{\mu}{p}} \|f\|_{L^p(\mathcal{E}(x,t))} \right\} \\ &\leq \|f\|_{L_{p,\lambda,\mu}^P(\mathbb{R}^n)}. \end{aligned}$$

Therefore, $f \in L_{p,\lambda}^P(\mathbb{R}^n) \cap L_{p,\mu}^P(\mathbb{R}^n)$ and $\max \left\{ \|f\|_{L_{p,\lambda}^P}, \|f\|_{L_{p,\mu}^P} \right\} \leq \|f\|_{L_{p,\lambda,\mu}^P(\mathbb{R}^n)}$.

Now let $f \in L_{p,\lambda}^P(\mathbb{R}^n) \cap L_{p,\mu}^P(\mathbb{R}^n)$. Then

$$\begin{aligned} \|f\|_{L_{p,\lambda,\mu}^P} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(\mathcal{E}(x,t))} \\ &= \max \left\{ \sup_{x \in \mathbb{R}^n, 0 < t \leq 1} t^{-\frac{\lambda}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}, \sup_{x \in \mathbb{R}^n, t > 1} t^{-\frac{\mu}{p}} \|f\|_{L^p(\mathcal{E}(x,t))} \right\} \\ &\leq \max \left\{ \|f\|_{L_{p,\lambda}^P}, \|f\|_{L_{p,\mu}^P} \right\}. \end{aligned}$$

Therefore, $f \in L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ and $\|f\|_{L_{p,\lambda,\mu}^P(\mathbb{R}^n)} \leq \max \left\{ \|f\|_{L_{p,\lambda}^P}, \|f\|_{L_{p,\mu}^P} \right\}$.

The following lemma is a weak version of Lemma 2.1 and is proved similarly.

Lemma 2.2 *If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq \gamma$, then*

$$WL_{p,\lambda,\mu}^P(\mathbb{R}^n) = WL_{p,\lambda}^P(\mathbb{R}^n) \cap WL_{p,\mu}^P(\mathbb{R}^n)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}^P} = \max \left\{ \|f\|_{WL_{p,\lambda}^P}, \|f\|_{WL_{p,\mu}^P} \right\}.$$

Remark 2.1 Let $0 < p < \infty$. If $\lambda < 0$ or $\mu > \gamma$, then

$$L_{p,\lambda,\mu}^P(\mathbb{R}^n) = WL_{p,\lambda,\mu}^P(\mathbb{R}^n) = \Theta(\mathbb{R}^n).$$

The following local estimate is valid (see also [12]).

Lemma 2.3 [12, Lemma 4.1] Let $0 \leq \alpha < |d|$, $1 \leq p < \frac{|d|}{\alpha}$, and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$. Then, for $p > 1$ the inequality

$$\|M_\alpha^d f\|_{L_q(\mathcal{E}(x,r))} \lesssim r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \|f\|_{L_p(\mathcal{E}(x,t))} \quad (2.1)$$

holds for all $\mathcal{E}(x, r)$ and for all $f \in L_p^{\text{loc}}(\mathbb{R}^n)$.

Moreover if $p = 1$, then the inequality

$$\|M_\alpha^d f\|_{WL_q(\mathcal{E}(x,r))} \lesssim r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \|f\|_{L_1(\mathcal{E}(x,t))} \quad (2.2)$$

holds for all $\mathcal{E}(x, r)$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

The following is Spanne's type result for fractional anisotropic maximal operators in total anisotropic Morrey spaces (see, for example, [12]).

Theorem 2.1 (Spanne type result) [18, Theorem 2.1] Let $1 \leq p < \infty$, $0 \leq \mu \leq \lambda < |d|$, $0 \leq \alpha < \frac{|d|-\lambda}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$.

1. If $p > 1$, $f \in L_{p,\lambda,\mu}^d(\mathbb{R}^n)$, then $M_\alpha^d f \in L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}^d(\mathbb{R}^n)$ and

$$\|M_\alpha^d f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}^d} \leq C_{p,q,\lambda,\mu} \|f\|_{L_{p,\lambda,\mu}^d}, \quad (2.3)$$

where $C_{p,q,\lambda,\mu}$ depends only on p, q, λ, μ and n .

2. If $p = 1$, $f \in L_{1,\lambda,\mu}^d(\mathbb{R}^n)$, then $Mf \in WL_{q,\lambda q,\mu q}^d(\mathbb{R}^n)$ and

$$\|M_\alpha^d f\|_{WL_{q,\lambda q,\mu q}^d} \leq C_{q,\lambda,\mu} \|f\|_{L_{1,\lambda,\mu}^d}, \quad (2.4)$$

where $C_{q,\lambda,\mu}$ is independent of f .

The following is Adam's type result for fractional anisotropic maximal operators in total anisotropic Morrey spaces (see, for example, [11]).

Theorem 2.2 (Adams type result) [18, Theorem 2.2] Let $1 \leq p < \infty$, $0 \leq \mu \leq \lambda < |d|$, $0 \leq \alpha < \frac{|d|-\lambda}{p}$.

1) If $1 < p < \frac{|d|-\lambda}{\alpha}$, then condition $\frac{\alpha}{|d|-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{|d|-\lambda}$ is necessary and sufficient for the boundedness of the operator M_α^d from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^d(\mathbb{R}^n)$.

2) If $p = 1 < \frac{|d|-\lambda}{\alpha}$, then condition $\frac{\alpha}{|d|-\mu} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{|d|-\lambda}$ is necessary and sufficient for the boundedness of the operator M_α^d from $L_{1,\lambda,\mu}^d(\mathbb{R}^n)$ to $WL_{q,\lambda,\mu}^d(\mathbb{R}^n)$.

3) If $\frac{|d|-\lambda}{\alpha} \leq p \leq \frac{|d|-\mu}{\alpha}$, then the operator M_α^d is bounded from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$.

3 Anisotropic maximal commutator in total anisotropic Morrey spaces

In this section, as an application of the theorems of the previous section we consider the boundedness of the anisotropic maximal commutator M_b^d on total anisotropic Morrey spaces when b belongs to an anisotropic Lipschitz space, by which some new characterizations of the anisotropic Lipschitz spaces are given. Such a characterization was given in [31] for the boundedness of M_b on Lebesgue and Morrey spaces.

Definition 3.1 Let $0 < \beta < 1$, we say a function b belongs to the anisotropic Lipschitz space $\dot{A}_{\beta,d}(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C\rho(x - y)^\beta.$$

The smallest such constant C is called the $\dot{A}_{\beta,d}(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)}$.

To prove the theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space, which is due to DeVore and Sharply [9].

Lemma 3.1 Let $0 < \beta < 1$, we have

$$\|f\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} \approx \sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{1+\beta/n}} \int_{\mathcal{E}} |f(x) - f_{\mathcal{E}}| dx,$$

where $f_{\mathcal{E}} = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} f(y) dy$.

If $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$ and $\lambda > 0$, then the function b_{λ^d} is defined by $b_{\lambda^d}(x) = b(\lambda^d x)$ is also in $\dot{A}_{\beta,d}(\mathbb{R}^n)$ and

$$\|b_{\lambda^d}\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} = \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)}. \quad (3.1)$$

See, for example, [10, Proposition 7.1.2 (6)].

Lemma 3.2 Let $0 < \beta < 1$ and $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$, then the following pointwise estimate holds:

$$M_b^d f(x) \lesssim \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} M_{\beta}^d f(x).$$

Proof. If $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$, then

$$\begin{aligned} M_b^d(f)(x) &\approx \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1} \int_{\mathcal{E}} |b(x) - b(y)| |f(y)| dy \\ &\lesssim \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1+\frac{\beta}{|d|}} \int_{\mathcal{E}} |f(y)| dy \\ &\approx \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} M_{\beta}^d f(x). \end{aligned}$$

The following is Spanne's type result for the anisotropic maximal commutator operators in total anisotropic Morrey spaces.

Theorem 3.1 (Spanne type result) Let $0 < \beta < 1$ and $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$. Let also $0 \leq \mu \leq \lambda < |d|$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{|d|}$.

If $f \in L_{p,\lambda,\mu}^d(\mathbb{R}^n)$, then $M_b^d f \in L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}^d(\mathbb{R}^n)$ and

$$\|M_b^d f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}^d} \leq C_{p,q,\lambda,\mu} \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} \|f\|_{L_{p,\lambda,\mu}^d}, \quad (3.2)$$

where $C_{p,q,\lambda,\mu}$ depends only on p, q, λ, μ, d and n .

Proof. Let $1 < p < \infty$. From Theorem 2.1 and Lemma 3.2 we get

$$\begin{aligned} \|M_b^d f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}^d} &\lesssim \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} \|M_{\beta}^d f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}^d} \\ &\lesssim \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} \|f\|_{L_{p,\lambda,\mu}^d}, \end{aligned}$$

which implies that the operator M_{α}^d is bounded from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}^d(\mathbb{R}^n)$.

The following is an Adams type result for the anisotropic maximal commutator operators in total anisotropic Morrey spaces.

Theorem 3.2 (Adams type result) *Let $0 < \beta < 1$ and $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$. Let also $0 \leq \mu \leq \lambda < |d| - \beta$ and $1 < p < \frac{|d|-\lambda}{\beta}$.*

Then condition $\frac{\beta}{|d|-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\beta}{|d|-\lambda}$ is necessary and sufficient for the boundedness of the operator M_b^d from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^d(\mathbb{R}^n)$.

Proof. Sufficiency follows from Theorem 2.2 and Lemma 3.2.

Now we will prove the necessity.

Let $1 < p < \frac{|d|-\lambda}{\alpha}$, $\frac{\alpha}{|d|-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{|d|-\lambda}$, $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$, $f \in L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ and assume that M_b^d is bounded from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^d(\mathbb{R}^n)$.

Note that

$$M_b^d f_{t^d}(x) = M_{b_{\frac{1}{t^d}}}^d f(tx),$$

$$\begin{aligned} \|M_b^d f_{t^d}\|_{L_{q,\lambda,\mu}^d} &= \sup_{x \in \mathbb{R}^n, r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|M_{b_{\frac{1}{t^d}}}^d f(t^d \cdot)\|_{L_q(\mathcal{E}(x,r))} \\ &= t^{-\frac{n}{q}} \sup_{r > 0} \left(\frac{[tr]_1}{[r]_1} \right)^{\lambda/q} \sup_{r > 0} \left(\frac{[1/r]_1}{[1/(tr)]_1} \right)^{\mu/q} \sup_{x \in \mathbb{R}^n, r > 0} [tr]_1^{-\frac{\lambda}{p}} [1/(tr)]_1^{\frac{\mu}{p}} \|M_{b_{\frac{1}{t^d}}}^d f\|_{L_q(\mathcal{E}(t^d x, tr))} \\ &= t^{-\frac{n}{q}} [t]_{1,+}^{\frac{\lambda}{q}} [1/t]_{1,+}^{-\frac{\mu}{q}} \|M_{b_{\frac{1}{t^d}}}^d f\|_{L_{q,\lambda,\mu}^d}. \end{aligned}$$

By the boundedness of M_b^d from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^d(\mathbb{R}^n)$ and from the equality (3.1) we get

$$\begin{aligned} \|M_b^d f\|_{L_{q,\lambda,\mu}^d} &= t^{\frac{|d|}{q}} [t]_{1,+}^{-\frac{\lambda}{q}} [1/t]_{1,+}^{\frac{\mu}{q}} \|M_{b_{\frac{1}{t^d}}}^d f_t\|_{L_{q,\lambda,\mu}^d} \\ &\lesssim t^{\frac{|d|}{q}} [t]_{1,+}^{-\frac{\lambda}{q}} [1/t]_{1,+}^{\frac{\mu}{q}} \|f_{t^d}\|_{L_{p,\lambda,\mu}^d} \\ &= t^{\frac{|d|}{q} - \frac{n}{p}} [t]_{1,+}^{\frac{\lambda}{p} - \frac{\lambda}{q}} [1/t]_{1,+}^{-\frac{\mu}{p} + \frac{\mu}{q}} \|f\|_{L_{p,\lambda,\mu}^d} \\ &= [t]_{1,+}^{-\frac{|d|-\lambda}{p} + \frac{|d|-\lambda}{q}} [1/t]_{1,+}^{\frac{|d|-\mu}{p} - \frac{|d|-\mu}{q}} \|f\|_{L_{p,\lambda,\mu}^d}. \end{aligned}$$

Since $L_{p,\lambda,\mu}^d(\mathbb{R}^n) = L_{p,\mu,\lambda}^d(\mathbb{R}^n)$, we can assume that $\lambda < \mu$, and then $\mu = \lambda$, $\lambda = \mu$.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{|d|-\lambda}$, then by letting $t \rightarrow 0$ we have $\|M_{b,\alpha}^d f\|_{L_{q,\lambda,\mu}^d} = 0$ for all $f \in L_{p,\lambda,\mu}^d(\mathbb{R}^n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{|d|-\mu}$, then at $t \rightarrow \infty$ we obtain $\|M_{b,\alpha}^d f\|_{L_{q,\lambda,\mu}^d} = 0$ for all $f \in L_{p,\lambda,\mu}^d(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{|d|-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{|d|-\lambda}$.

Theorem 3.3 *Let $0 < \beta < 1$ and $b \in L_1^{\text{loc}}(\mathbb{R}^n)$. Let also $0 \leq \mu \leq \lambda < |d|$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{\beta}{|d|-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\beta}{|d|-\lambda}$. Then, the following statements are equivalent:*

- (i) $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$.
- (ii) The operator M_b^d is bounded from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^d(\mathbb{R}^n)$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$. Combining Lemma 3.2 and Theorems 2.2, we get

$$\|M_b^d f\|_{L_{q,\lambda,\mu}^d} \lesssim \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} \|M_\beta^d f\|_{L_{q,\lambda,\mu}} \lesssim \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} \|f\|_{L_{p,\lambda,\mu}^d}.$$

(ii) $\Rightarrow$ (i). Assume that M_b^d is bounded from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^d(\mathbb{R}^n)$. Let $\mathcal{E} = \mathcal{E}(x, r)$ be a fixed ball. We consider $f = \chi_{\mathcal{E}}$. It is easy to compute that

$$\begin{aligned} \|\chi_{\mathcal{E}}\|_{L_{p,\lambda,\mu}^d} &\approx \sup_{y \in \mathbb{R}^n, t > 0} \left([t]_1^{-\lambda} [1/t]_1^\mu \int_{\mathcal{E}(y,t)} \chi_{\mathcal{E}}(z) dz \right)^{\frac{1}{p}} \\ &= \sup_{y \in \mathbb{R}^n, t > 0} \left(|\mathcal{E}(y,t) \cap \mathcal{E}| [t]_1^{-\lambda} [1/t]_1^\mu \right)^{\frac{1}{p}} \\ &= \sup_{\mathcal{E}(y,t) \subseteq \mathcal{E}} \left(|\mathcal{E}(y,t)| [t]_1^{-\lambda} [1/t]_1^\mu \right)^{\frac{1}{p}} = r^{\frac{|d|}{p}} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}}. \end{aligned} \quad (3.3)$$

On the other hand, since

$$M_b^d(\chi_{\mathcal{E}})(x) \gtrsim \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}| dz \quad \text{for all } x \in B,$$

we have

$$\begin{aligned} \|M_b^d(\chi_{\mathcal{E}})\|_{L_{q,\lambda,\mu}^d} &\approx \sup_{\mathcal{E}(y,t) \subseteq \mathcal{E}} \left([t]_1^{-\lambda} [1/t]_1^\mu \int_{\mathcal{E}(y,t)} |M_b(\chi_{\mathcal{E}})(z)|^q dz \right)^{\frac{1}{q}} \\ &\gtrsim r^{\frac{n}{q}} [r]_1^{-\frac{\lambda}{q}} [1/r]_1^{\frac{\mu}{q}} \frac{1}{|\mathcal{E}|} \int_B |b(z) - b_B| dz \\ &= r^{\beta + \frac{n}{q}} [r]_1^{-\frac{\lambda}{q}} [1/r]_1^{\frac{\mu}{q}} \frac{1}{|\mathcal{E}|^{1+\beta/|d|}} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}| dz. \end{aligned} \quad (3.4)$$

Since $L_{p,\lambda,\mu}^d(\mathbb{R}^n) = L_{p,\mu,\lambda}^d(\mathbb{R}^n)$, we can assume that $\lambda < \mu$, and then $\mu = \lambda, \lambda = \mu$.

On the other hand, by assumption

$$\|M_b^d(\chi_{\mathcal{E}})\|_{L_{q,\lambda,\mu}^d} \lesssim \|\chi_{\mathcal{E}}\|_{L_{p,\lambda,\mu}^d},$$

by (3.3) and (3.4), we get that

$$\begin{aligned} \frac{1}{|\mathcal{E}|^{1+\beta/|d|}} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}| dz &\lesssim r^{-\beta - \frac{|d|}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \|M_b^d(\chi_{\mathcal{E}})\|_{L_{q,\lambda,\mu}^d} \\ &\lesssim r^{-\beta - \frac{|d|}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \|\chi_{\mathcal{E}}\|_{L_{p,\lambda,\mu}^d} \\ &\lesssim r^{-\beta - \frac{|d|}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} r^{\frac{|d|}{p}} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \\ &\lesssim [r]_1^{-\beta + \frac{|d|-\lambda}{p} - \frac{|d|-\lambda}{q}} [1/r]_1^{\beta - \frac{|d|-\mu}{p} + \frac{|d|-\mu}{q}} \\ &\lesssim 1. \end{aligned}$$

From Theorem 3.3 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3.1 [31] Let $0 < \beta < 1$ and $b \in L_1^{\text{loc}}(\mathbb{R}^n)$. Let also $0 \leq \lambda < |d| - \beta$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{|d|-\lambda}$. Then, the following statements are equivalent:

- (i) $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$.
- (ii) The operator M_b^d is bounded from $L_{p,\lambda}^d(\mathbb{R}^n)$ to $L_{q,\lambda}^d(\mathbb{R}^n)$.

Corollary 3.2 *Let $0 < \beta < 1$ and $b \in L_1^{\text{loc}}(\mathbb{R}^n)$. Let also $0 \leq \lambda < |d| - \beta$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{\beta}{|d|} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\beta}{|d|-\lambda}$. Then, the following statements are equivalent:*

- (i) $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$.
- (ii) *The operator M_b^d is bounded from $\tilde{L}_{p,\lambda}^d(\mathbb{R}^n)$ to $\tilde{L}_{q,\lambda}^d(\mathbb{R}^n)$.*

Remark 3.1 Note that Corollary 3.2 is new.

4 Commutator of anisotropic maximal operator in total anisotropic Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ in the $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ spaces.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha^d]$ and $M_{b,\alpha}^d$ are valid:

Let b be any non-negative locally integrable function. Then for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned} |[b, M^d]f(x)| &= |b(x)M^d f(x) - M^d(bf)(x)| \\ &= |M^d(b(x)f)(x) - M^d(bf)(x)| \\ &\leq M^d(|b(x) - b|f)(x) = M_b^d f(x). \end{aligned} \quad (4.1)$$

Applying Theorem 3.3, we obtain the following result.

Theorem 4.1 *Let $0 < \beta < 1$ and $b \in L_1^{\text{loc}}(\mathbb{R}^n)$. Let also $0 \leq \mu \leq \lambda < |d|$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{\beta}{|d|-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\beta}{|d|-\lambda}$. Then, the following statements are equivalent:*

- (i) $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$ and $b \geq 0$.
- (ii) *The operator $[b, M^d]$ is bounded from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^d(\mathbb{R}^n)$.*
- (iii) *There exists a constant $C > 0$ such that*

$$\sup_{\mathcal{E}} |\mathcal{E}|^{-\frac{\beta}{|d|}} \frac{\|(b - M_{\mathcal{E}}^d(b)) \chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}^d}}{\|\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}^d}} \leq C. \quad (4.2)$$

Proof. (i) $\Rightarrow$ (ii). Suppose that $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$. Combining Theorems 2.2 and 3.3, and inequality (4.1), we get

$$\|[b, M^d]f\|_{L_{q,\lambda,\mu}^d} \leq \|M_b^d f\|_{L_{q,\lambda,\mu}^d} \lesssim \|b\|_{\dot{A}_{\beta,d}(\mathbb{R}^n)} \|M_{\beta}^d f\|_{L_{q,\lambda,\mu}^d} \lesssim \|f\|_{L_{p,\lambda,\mu}^d}.$$

(ii) $\Rightarrow$ (iii). Assume that $[b, M^d]$ is bounded from $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^d(\mathbb{R}^n)$. For a given ellipsoid $\mathcal{E}$, we define the following local maximal function:

$$M_{\mathcal{E}}^d f(x) := \sup_{\mathcal{E} \supseteq \mathcal{E}' \ni x} |\mathcal{E}'|^{-1} \int_{\mathcal{E}'} |f(y)| dy,$$

where the supremum is taken over all ellipsoid $\mathcal{E}'$ such that $x \in \mathcal{E}' \subseteq \mathcal{E}$.

Since

$$M^d(b\chi_{\mathcal{E}})\chi_{\mathcal{E}} = M_{\mathcal{E}}^d(b)\chi_{\mathcal{E}} \quad \text{and} \quad M^d(\chi_{\mathcal{E}})\chi_{\mathcal{E}} = M_{\mathcal{E}}^d\chi_{\mathcal{E}} = \chi_{\mathcal{E}},$$

we have

$$\begin{aligned} (b - M_{\mathcal{E}}^d(b))\chi_{\mathcal{E}} &= (b - M_{\mathcal{E}}^d(b))\chi_{\mathcal{E}} \\ &= (bM^d(\chi_{\mathcal{E}}) - M^d(b\chi_{\mathcal{E}}))\chi_{\mathcal{E}} = [b, M^d](\chi_{\mathcal{E}}). \end{aligned} \quad (4.3)$$

By this and $[b, M_{\alpha}^d] : L_{p,\lambda,\mu}^d(\mathbb{R}^n) \rightarrow L_{q,\lambda,\mu}^d(\mathbb{R}^n)$, we obtain

$$\|(b - M_{\mathcal{E}}^d(b))\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}} \leq \|[b, M^d]\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}^d(\mathbb{R}^n)} \lesssim \|\chi_{\mathcal{E}}\|_{L_{p,\lambda,\mu}^d(\mathbb{R}^n)}.$$

Thus from (3.3) we get

$$\begin{aligned} |\mathcal{E}|^{-\frac{\beta}{|d|}} \frac{\|(b - M_{\mathcal{E}}^d(b))\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}}} &\leq |\mathcal{E}|^{-\frac{\beta}{|d|}} \frac{\|\chi_{\mathcal{E}}\|_{L_{p,\lambda,\mu}^d(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}^d(\mathbb{R}^n)}} \\ &\approx r^{-\beta + \frac{|d|}{p} - \frac{n}{q}} [r]_1^{\frac{\lambda}{q} - \frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p} - \frac{\mu}{q}} \\ &\approx [r]_1^{-\beta + \frac{|d|-\lambda}{p} - \frac{|d|-\lambda}{q}} [1/r]_1^{\beta - \frac{|d|-\mu}{p} + \frac{|d|-\mu}{q}} \\ &\lesssim 1. \end{aligned}$$

(iii) $\Rightarrow$ (i). Assume that (4.2) is valid.

Now, let us prove $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$ and $b \geq 0$. For any ellipsoid $\mathcal{E}$, let $E = \{y \in \mathcal{E} : b(y) \leq b_B\}$ and $F = \{y \in \mathcal{E} : b(y) > b_B\}$. The following equality is true (see [4, page 3331]):

$$\int_E |b(y) - b_{\mathcal{E}}| dy = \int_F |b(y) - b_{\mathcal{E}}| dy.$$

Since $b(y) \leq b_{\mathcal{E}} \leq |b_{\mathcal{E}}| \leq M_{\mathcal{E}}^d(b)(y)$ for any $y \in E$, we obtain

$$|b(y) - b_{\mathcal{E}}| \leq |b(y) - M_{\mathcal{E}}^d(b)(y)|, \quad y \in E.$$

Then from Hölder's inequality and (4.2) we have

$$\begin{aligned} \frac{1}{|\mathcal{E}|^{1+\beta/|d|}} \int_{\mathcal{E}} |b(y) - b_{\mathcal{E}}| dy &= \frac{2}{|\mathcal{E}|^{1+\beta/|d|}} \int_E |b(y) - b_{\mathcal{E}}| dy \\ &\leq \frac{2}{|\mathcal{E}|^{1+\beta/|d|}} \int_E |b(y) - M_{\mathcal{E}}^d(b)(y)| dy \leq \frac{2}{|\mathcal{E}|^{1+\beta/|d|}} \int_{\mathcal{E}} |b(y) - M_{\mathcal{E}}^d(b)(y)| dy \\ &\lesssim \frac{2}{|\mathcal{E}|^{1+\beta/|d|}} \|b - M_{\mathcal{E}}^d(b)\|_{L_q(\mathcal{E})} |\mathcal{E}|^{\frac{1}{q'}} \\ &\lesssim |\mathcal{E}|^{-\frac{1}{q} - \frac{\beta}{|d|}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \|(b - M_{\mathcal{E}}^d(b))\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}^d(\mathbb{R}^n)} \\ &\lesssim r^{-\frac{|d|}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} \|\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}^d} \\ &\lesssim r^{-\frac{|d|}{q}} [r]_1^{\frac{\lambda}{q}} [1/r]_1^{-\frac{\mu}{q}} r^{\frac{|d|}{q}} [r]_1^{-\frac{\lambda}{q}} [1/r]_1^{\frac{\mu}{q}} \approx 1. \end{aligned}$$

From Theorem 4.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 4.1 [31] Let $0 < \beta < 1$ and $b \in L_1^{\text{loc}}(\mathbb{R}^n)$. Let also $0 \leq \lambda < |d| - \beta$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{|d|-\lambda}$. Then, the following statements are equivalent:

- (i) $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$ and $b \geq 0$.
- (ii) The operator $[b, M^d]$ is bounded from $L_{p,\lambda}^d(\mathbb{R}^n)$ to $L_{q,\lambda}^d(\mathbb{R}^n)$.
- (iii) There exists a constant $C > 0$ such that

$$\sup_{\mathcal{E}} \frac{\|(b - M_{\mathcal{E}}^d(b)) \chi_{\mathcal{E}}\|_{L_{q,\lambda}^d}}{\|\chi_{\mathcal{E}}\|_{L_{p,\lambda}^d}} \leq C.$$

Corollary 4.2 Let $0 < \beta < 1$ and $b \in L_1^{\text{loc}}(\mathbb{R}^n)$. Let also $0 \leq \lambda < |d| - \beta$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{\beta}{|d|} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\beta}{|d|-\lambda}$. Then, the following statements are equivalent:

- (i) $b \in \dot{A}_{\beta,d}(\mathbb{R}^n)$ and $b \geq 0$.
- (ii) The operator $[b, M^d]$ is bounded from $\tilde{L}_{p,\lambda}^d(\mathbb{R}^n)$ to $\tilde{L}_{q,\lambda}^d(\mathbb{R}^n)$.
- (iii) There exists a constant $C > 0$ such that

$$\sup_{\mathcal{E}} \frac{\|(b - M_{\mathcal{E}}^d(b)) \chi_{\mathcal{E}}\|_{\tilde{L}_{q,\lambda}^d}}{\|\chi_{\mathcal{E}}\|_{\tilde{L}_{p,\lambda}^d}} \leq C.$$

Remark 4.1 Note that Corollaries 4.1 and 4.2 are new. In the case of $d = \mathbf{1} \equiv (1, \dots, 1)$ the Theorem 4.1 were proven in [15], see also [16, 19, 24].

5 Conclusion

The paper gives necessary and sufficient conditions for the boundedness of anisotropic maximal commutators M_b^d and commutators of the anisotropic maximal operator $[b, M^d]$ in total anisotropic Morrey spaces $L_{p,\lambda,\mu}^d(\mathbb{R}^n)$, when b belongs to the anisotropic Lipschitz spaces $\dot{A}_{\beta,d}(\mathbb{R}^n)$. As an application, new characterizations of some subclasses of anisotropic Lipschitz spaces $\dot{A}_{\beta,d}(\mathbb{R}^n)$ are obtained.

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