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Commutators of maximal operator and sharp maximal operator on local Morrey-Lorentz spaces

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Abstract. *In this paper, we study the boundedness of the maximal commutator operator M_b, the commutators of the maximal operator $[b, M]$ and the commutators of the sharp maximal operator $[b, M^\sharp]$ in the local Morrey-Lorentz spaces $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$. We give necessary and sufficient conditions for the boundedness of the operators M_b, $[b, M]$ and $[b, M^\sharp]$ on local Morrey-Lorentz spaces $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ when b belongs to $BMO(\mathbb{R}^n)$ spaces, whereby some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$ spaces are obtained.*

Keywords. Maximal operator, commutator, local Morrey-Lorentz space, BMO space.

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1 Introduction

Let $0 < p, q \le \infty$ and let $0 \le \lambda \le 1$. We define the local Morrey-Lorentz spaces as the spaces of all measurable functions with finite quasinorm

$$\|f\|_{M^{loc}_{p,q;\lambda}} := \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,r)}.$$

The purpose of this paper is to give necessary and sufficient conditions for the boundedness of the maximal commutators M_b and the commutators of the maximal operator $[b, M]$ on the local Morrey-Lorentz spaces $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$. Local Morrey-Lorentz spaces $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$, which are natural generalizations of the Lorentz spaces $L^{p,q}(\mathbb{R}^n) \equiv M^{\rm loc}_{p,q;0}(\mathbb{R}^n)$ and the classical Lorentz spaces $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n) \equiv M^{\rm loc}_{p,q;1}(\mathbb{R}^n)$, were introduced and their main properties were obtained in [6], see also [7,22,25]. For $0 < q \le p < \infty$ and $0 < \lambda \le \frac{q}{p}$, the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ are equal to weak Lebesgue spaces $WL_{\frac{1}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)$. In [6] the

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basic properties of $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ were given and the boundedness of the maximal operator was proved. Generally speaking, local Morrey spaces were also introduced separately by Guliyev [21] (see also [20]) and Garcia-Cuerva and Herrero [18] (see also [4]).

Recall that the local Morrey-type spaces $LM_{p\theta,w}$ were introduced and proved the boundedness in this spaces of the fractional integral operators and singular integral operators defined on homogeneous Lie groups by Guliyev [20] in the doctoral thesis (see, also [21]) are given by

$$\|f\|_{LM_{p\theta,w}} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0,\infty)$. Some necessary and sufficient conditions for the boundedness of the maximal, fractional maximal, Riesz potential and singular integral operators in local Morrey-type space $LM_{p\theta,w}$ are given in [11–13]. We should explain that the spaces $LM_{p\theta,w}$ are closely related to the B_σ spaces (see, [34,35]).

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance [19]). For $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the maximal operator M is defined by

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the ball of radius r centered at $x \in \mathbb{R}^n$, ${}^{\complement}B(x,r)$ is its complement and $|B(x,r)| = v_n\, r^n$, $v_n = |B(0,1)|$, here $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$.

The sharp maximal operator $M^\sharp$ was introduced by Fefferman and Stein [15], which is defined as

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy,$$

where $f_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$. For a fixed $q \in (0,1)$, any suitable function h and $x \in \mathbb{R}^n$, let $M^\sharp_q h(x) = \big(M^\sharp\big(|h|^q\big)(x)\big)^{1/q}$ and $M_q h(x) = \big(M\big(|h|^q\big)(x)\big)^{1/q}$.

The maximal commutator generated by the operator M and $b \in L^1_{\rm loc}(\mathbb{R}^n)$ is defined by

$$M_b f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy.$$

The commutators generated by the operator M and a suitable function b is defined by

$$[b,M]f(x) = b(x)Mf(x) - M(bf)(x).$$

Obviously, the operators M_b and $[b,M]$ essentially differ from each other since M_b is positive and sublinear and $[b,M]$ is neither positive nor sublinear. The operators M, $[b,M]$ and M_b play an important role in real and harmonic analysis and applications (see, for instance [8,17,32,33,42,44]).

The commutator estimates have many important applications, for example, in studying the regularity and boundedness of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and in characterizing certain function spaces (see, for instance [14,19]). The nonlinear commutator of maximal function $[b,M]$ can be used in tudying the product of a function in H_1 and a function in BMO (see [10] for instance). Note that, the boundedness of the operator M_b on $L^p(\mathbb{R}^n)$ spaces was proved by Garcia-Cuerva et al. [17]. In [8] by Bastero et al. studied the necessary and sufficient condition for the boundedness of $[b,M]$ on $L^p(\mathbb{R}^n)$ spaces.

The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance [14, 27–30, 39]).

In [5, 24, 31] was obtain necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and commutators of maximal operator $[b, M]$ on the Lorentz spaces $L_{p,q}$, see also [26].

The structure of the paper is as follows. In Section 2 we give some definitions and auxiliary results. In Section 3 we obtain necessary and sufficient conditions for the boundedness of the maximal commutator M_b on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ spaces. In Section 4 we give necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b, M]$ on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ spaces. In Section 5 we give necessary and sufficient conditions for the boundedness of the commutator of sharp maximal operator $[b, M^\sharp]$ on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Definition and some basic properties

We start with the definition of Lorentz spaces. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar L_p spaces, also they are appear to be useful in the general interpolation theory.

Suppose that f is a measurable function on $\mathbb{R}^n$, then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\},$$

where

$$d_f(s) := |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad s > 0.$$

The Lorentz space $L_{p,q} \equiv L_{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty$ is the collection of all measurable functions f on $\mathbb{R}^n$ such the quantity

$$\|f\|_{L_{p,q}} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,\infty)} \tag{2.1}$$

is finite. Clearly $L_{p,p} \equiv L_p$ and $L_{p,1} \equiv WL_p$. The functional $\|\cdot\|_{L_{p,q}}$ is a norm if and only if either $1 \leq q \leq p$ or $p = q = \infty$.

Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods. It is well known that for the classical Hardy-Littlewood maximal operator the rearrangement inequality

$$c\, f^{**}(t) \leq (Mf)^*(t) \leq C\, f^{**}(t), \;\; t \in (0, \infty)$$

holds, ([9] Chapter 3, Theorem 3.8), where $f^*(t)$ is the nonincreasing rearrangement of f and

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(t)dt.$$

Adams [1] defined the Morrey-Lorentz spaces $L_{p,q;\lambda}(\mathbb{R}^n)$ (see also [37]) by following:

Definition 2.1 *[1] The Morrey-Lorentz spaces $L_{p,q;\lambda}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$: for $1 \le p < \infty$, $0 < q < \infty$, and $0 \le \lambda \le n$, $f \in L_{p,q;\lambda}(\mathbb{R}^n)$ iff*

$$\|f\|_{L_{p,q;\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\,\chi_{B(x,t)}\|_{L_{p,q}(\mathbb{R}^n)} < \infty.$$

Here $\|\cdot\|_{L_{p,q}(\mathbb{R}^n)}$ denoted by Lorentz norm of a function.

In [37], Section 4.1, Mingione studied the boundedness of the restricted fractional maximal operator $M_{\beta,B}$:

$$M_{\beta,B}f(x) = \sup_{B(x,t)\subset B} |B(x,t)|^{\frac{\beta}{n}-1} \int_{B(x,t)} |f(y)|dy, \quad x \in \mathbb{R}^n$$

in the restricted Morrey-Lorentz spaces $L_{p,q;\lambda}(B)$, where B is any ball. Mingione derives a general non-linear version, extending a priori estimates and regularity results for possibly degenerate non-linear elliptic problems to the various spaces of Lorentz and Morrey-Lorentz type considered in [1–3].

Ragusa [41] defined the Morrey-Lorentz spaces $\mathcal{L}^{p,q;\lambda}(\mathbb{R}^n)$ and studied some embeddings between these spaces.

Definition 2.2 *[41] The Morrey-Lorentz spaces $\mathcal{L}^{p,q;\lambda}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$: for $1 \le p < \infty$, $0 < q < \infty$, and $0 \le \lambda \le n$, iff*

$$\|f\|_{\mathcal{L}^{p,q;\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{q}} \|f\,\chi_{B(x,t)}\|_{L_{p,q}(\mathbb{R}^n)} < \infty.$$

Accordingly, f belongs to
$\mathcal{L}^{p,\infty;\lambda}(\mathbb{R}^n) \equiv WL_{p,\lambda}(\mathbb{R}^n)$ iff $\|f\|_{\mathcal{L}^{p,\infty;\lambda}} = \|f\|_{WL_{p,\lambda}} < \infty$.

Note that the spaces $\mathcal{L}^{p,q;\lambda}(\mathbb{R}^n)$ and $\mathcal{L}^{p,q;\lambda\frac{q}{p}}(\mathbb{R}^n)$ defined by Adams and Ragusa respectively, coincide, thus

$$L_{p,q;\lambda}(\mathbb{R}^n) = \mathcal{L}_{p,q;\lambda\frac{q}{p}}(\mathbb{R}^n),$$
$$L_{p,1;\lambda}(\mathbb{R}^n) \subset L_{p,q;\lambda}(\mathbb{R}^n) \subset L_{p,\infty;\lambda}(\mathbb{R}^n).$$

Note that the spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ and $L_{p,q;\lambda\frac{q}{p}}(\mathbb{R}^n)$ defined by Mingione and Ragusa respectively, coincide, thus

$$\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n) = L_{p,q;\lambda\frac{q}{p}}(\mathbb{R}^n).$$

The following result completely characterizes the boundedness of the maximal operator M on Morrey-Lorentz spaces.

Lemma 2.1 *[16, Lemma 3.1] Let $1 < p \le \infty$, $1 \le q \le \infty$ and $0 \le \lambda < n$. Then, for any $0 < s < p$ there is a positive constant $C = C(p,q,s,\lambda,n)$ such that*

$$\|M_s f\|_{L_{p,q;\lambda}} \lesssim \|f\|_{L_{p,q;\lambda}}.$$

Corollary 2.1 *Let $1 < p \le \infty$, $1 \le q \le \infty$ and $0 \le \lambda < n$. Then the operator M is bounded on the Morrey-Lorentz spaces $L_{p,q;\lambda}(\mathbb{R}^n)$.*

In the following we give the local Morrey spaces $LM_{p,\lambda}(0,\infty)$ which we use while proving of our main results.

Definition 2.3 *Let $1 \le p < \infty$ and $0 \le \lambda \le 1$. We denote by $LM_{p,\lambda} \equiv LM_{p,\lambda}(0,\infty)$ the local Morrey space, the space of all functions $\varphi \in L_p^{\rm loc}(0,\infty)$ with finite quasinorm*

$$\|\varphi\|_{LM_{p,\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{p}} \, \|\varphi\|_{L_p(0,r)}.$$

Also by $WLM_{p,\lambda} \equiv WLM_{p,\lambda}(0,\infty)$ we denote the weak local Morrey space of all functions $\varphi \in WL_p^{\rm loc}(0,\infty)$ for which

$$\|\varphi\|_{WLM_{p,\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{p}} \, \|\varphi\|_{WL_p(0,r)} < \infty.$$

The local Morrey-type spaces $LM_{p\theta,w}$ were introduced by Guliyev in the doctoral thesis [20], 1994, (see, also [21]) defined by

$$\|\varphi\|_{LM_{p\theta,w}} = \|w(r)\,\|\varphi\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0,\infty)$. If $\theta = \infty$, it denotes $LM_{p,w} \equiv LM_{p\infty,w}$. The boundedness of the classical operators in $LM_{p\theta,w}$ was intensively studied in [11–13,20,21], etc.

Definition 2.4 *[6] Let $0 < p,q \le \infty$ and $0 \le \lambda \le 1$. We denote by $M^{loc}_{p,q;\lambda} \equiv M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ the local Morrey-Lorentz space, the space of all measurable functions with finite quasinorm*

$$\|f\|_{M^{loc}_{p,q;\lambda}} := \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,r)}.$$

In the cases $\lambda < 0$, $\lambda > 1$ and $p = \infty$, we have $M^{loc}_{p,q;\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Also $M^{loc}_{p,q;0} = L_{p,q}$ and $M^{loc}_{p,p;\lambda} \equiv M^{loc}_{p;\lambda}$. In the limiting case $\lambda = 1$ the space $M^{loc}_{p,q;1}$ is the classical Lorentz space $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}$. For $0 < q \le p < \infty$ and $0 < \lambda \le \frac{q}{p}$, the local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}$ are equal to weak Lebesgue spaces $WL_{\frac{1}{\frac{1}{p}-\frac{\lambda}{q}}}$. Note that, in the case $q = \infty$ we have $M^{loc}_{p,\infty;\lambda} = \Lambda_{\infty,t^{\frac{1}{p}}} = WL_p$.

We denote by $WM^{loc}_{p,q;\lambda}$ the weak local Morrey-Lorentz space of all measurable functions with finite quasinorm

$$\|f\|_{WM^{loc}_{p,q;\lambda}} := \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{WL_q(0,r)}.$$

Lemma 2.2 *[6] Let $0 < q \le p < \infty$, $\frac{1}{s} = \frac{1}{p} - \frac{\lambda}{q}$ and $0 < \lambda \le \frac{q}{p}$. Then*

$$\left(\frac{q}{p}\right)^{-\frac{1}{q}} \|f\|_{WL_s} \le \|f\|_{M^{loc}_{p,q;\lambda}} \le \lambda^{-\frac{1}{q}} \|f\|_{WL_s}.$$

In particular, $\|f\|_{WL_\infty} = \|f\|_{M^{loc}_{\frac{q}{\lambda},q;\lambda}}$.

Lemma 2.3 *The inequalities*

$$(f+g)^*(t_1+t_2) \le f^*(t_1) + g^*(t_2)$$
$$(fg)^*(t_1+t_2) \le f^*(t_1)\, g^*(t_2)$$

holds for all $t_1, t_2 \ge 0$. In particular, the inequalities

$$(f+g)^*(t) \le f^*(t/2) + g^*(t/2)$$
$$(fg)^*(t) \le f^*(t/2)\, g^*(t/2)$$

holds for all $t \ge 0$.

Lemma 2.4 *Let* $0 < p, p_1, p_2, q, q_1, q_2 < \infty$, $0 \le \lambda \le 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. *Suppose that* $f \in M^{\rm loc}_{p_1,q_1;\lambda}(\mathbb{R}^n)$ *and* $f \in M^{\rm loc}_{p_2,q_2;\lambda}(\mathbb{R}^n)$. *Then*

$$\|fg\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)} \le 2^{\frac{1}{p}-\frac{1}{q}} \|f\|_{M^{\rm loc}_{p_1,q_1;\lambda}(\mathbb{R}^n)} \|g\|_{M^{\rm loc}_{p_2,q_2;\lambda}(\mathbb{R}^n)}.$$

Proof. Assume that $0 \le \lambda \le 1$, $0 < p, p_1, p_2, q, q_1, q_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $f \in M^{\rm loc}_{p_1,q_1;\lambda}(\mathbb{R}^n)$ and $f \in M^{\rm loc}_{p_2,q_2;\lambda}(\mathbb{R}^n)$. Using Lemma 2.3 and Hölder's inequality for Lebesgue spaces, we obtain

$$\begin{aligned}
\|fg\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)} &= \sup_{r>0} r^{-\frac{\lambda}{q}} \, \|t^{\frac{1}{p}-\frac{1}{q}} \, (fg)^*(t)\|_{L_q(0,r)} \\
&\le \sup_{r>0} r^{-\frac{\lambda}{q}} \, \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t/2) \, g^*(t/2)\|_{L_q(0,r)} \\
&= \sup_{r>0} r^{-\frac{\lambda}{q}} \, \|t^{\frac{1}{p_1}-\frac{1}{q_1}} f^*(t/2) \, t^{\frac{1}{p_2}-\frac{1}{q_2}} g^*(t/2)\|_{L_q(0,r)} \\
&\le \sup_{r>0} r^{-\frac{\lambda}{q}} \, \|t^{\frac{1}{p_1}-\frac{1}{q_1}} f^*(t/2)\|_{L_{q_1}(0,r)} \, \|t^{\frac{1}{p_2}-\frac{1}{q_2}} g^*(t/2)\|_{L_{q_2}(0,r)}
\end{aligned}$$

Taking $t/2 = s$, we have

$$\begin{aligned}
\|fg\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)} &\le \sup_{r>0} r^{-\frac{\lambda}{q}} \, \|(2s)^{\frac{1}{p_1}-\frac{1}{q_1}} f^*(s)\|_{L_{q_1}(0,r)} \, \|(2s)^{\frac{1}{p_2}-\frac{1}{q_2}} g^*(s)\|_{L_{q_2}(0,r)} \\
&\le 2^{\frac{1}{p}-\frac{1}{q}} \, \|f\|_{M^{\rm loc}_{p_1,q_1;\lambda}(\mathbb{R}^n)} \, \|g\|_{M^{\rm loc}_{p_2,q_2;\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, the proof is complete.

Corollary 2.2 *Let* $0 \le \lambda < 1$, $1 < p, p', q, q' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$. *Suppose that* $f \in M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$. *Then*

$$\|f\|_{L_1(B)} \le \|f\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)} \, |B|^{\frac{1}{p'}+\frac{\lambda}{q}}.$$

Proof. Let $1 < p, p', q, q' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Suppose that $f \in M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|f\|_{L_1(B)} \le \|f^*\|_{L_1(0,|B|)} &= \|t^{\frac{1}{p'}-\frac{1}{q'}} \, t^{\frac{1}{p}-\frac{1}{q}} \, f^*\|_{L_1(0,|B|)} \\
&\le \|t^{\frac{1}{p}-\frac{1}{q}} \, f^*\|_{L_q(0,|B|)} \|t^{\frac{1}{p'}-\frac{1}{q'}}\|_{L_{q'}(0,|B|)} \\
&\le \|f\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)} \, |B|^{\frac{\lambda}{q}} \, \|t^{\frac{1}{p'}-\frac{1}{q'}}\|_{L_{q'}(0,|B|)} \\
&\approx \|f\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)} \, |B|^{\frac{1}{p'}+\frac{\lambda}{q}}.
\end{aligned}$$

The following theorem is the boundedness of the maximal operator in local Morrey-Lorentz spaces $M^{loc}_{p,q;\lambda}$.

Theorem 2.1 *[22, Theorem 1.1] Let* $1 \le q \le \infty$, $0 \le \lambda < 1$ *and* $\frac{q}{q+\lambda} \le p < \infty$.

(i) *If* $\frac{q}{q+\lambda} < p < \frac{q}{\lambda}$, *then the operator* M *is bounded in the local Morrey-Lorentz space* $M^{loc}_{p,q;\lambda}$.

(ii) *If* $p = \frac{q}{q+\lambda}$, *then the operator* M *is bounded from* $M^{loc}_{p,q;\lambda}$ *to the weak space* $WM^{loc}_{p,q;\lambda}$.

3 $M^{\rm loc}_{p,q;\lambda}$-boundedness of the maximal commutator operator M_b

In this section we find necessary and sufficient conditions for the boundedness of the maximal commutator M_b on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ Morrey-Lorentz spaces.

Definition 3.1 *We define the space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norm*

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}|dy < \infty,$$

where $f_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} f(y)dy$.

For proving our main results, we need the following estimate.

Lemma 3.1 *[32, Lemma 1] If $b \in BMO(\mathbb{R}^n)$, then for any $s \in (0,1)$, there exists a positive constant C such that*

$$M^\sharp_s\big(M_b f\big)(x) \le C\, \|b\|_*\, M^2 f(x) \tag{3.1}$$

for every $x \in \mathbb{R}^n$ and for all $f \in L^{\rm loc}_1(\mathbb{R}^n)$.

Theorem 3.1 *Let $1 \le q \le \infty$, $0 \le \lambda < 1$ and $\frac{q}{q+\lambda} < p < \infty$. The following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$.*
(ii) *The operator M_b is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$.*
(iii) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - b_B\big)\chi_B\big\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}} \le C. \tag{3.2}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - b_B\big)\chi_B\big\|_{L_1(\mathbb{R}^n)}}{|B|} \le C. \tag{3.3}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Theorem 2.1 and Lemma 3.1, we get

$$\begin{aligned}\|M_b f\|_{M^{\rm loc}_{p,q;\lambda}} &\lesssim \|M^\sharp_q\big(M_b f\big)\|_{M^{\rm loc}_{p,q;\lambda}} \\ &\lesssim \|b\|_* \|M^2 f\|_{M^{\rm loc}_{p,q;\lambda}} \\ &\lesssim \|b\|_* \|Mf\|_{M^{\rm loc}_{p,q;\lambda}} \\ &\lesssim \|b\|_* \|f\|_{M^{\rm loc}_{p,q;\lambda}}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that M_b is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. We consider $f = \chi_B$. It is easy to compute that

$$\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}} \approx |B|^{\frac{1}{p}-\frac{\lambda}{q}}. \tag{3.4}$$

On the other hand, for all $x \in B$ we have

$$\begin{aligned} \big|b(x) - b_B\big| &\le \frac{1}{|B|}\int_B |b(x)-b(y)|dy \\ &= \frac{1}{|B|}\int_B |b(x)-b(y)|\,\chi_B(y)dy \\ &\le M_b(\chi_B)(x). \end{aligned}$$

Since M_b is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$, then by (3.4) we obtain

$$\frac{\|\big(b-b_B\big)\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} \le \frac{\|M_b(\chi_B)\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} \lesssim \frac{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} = 1, \tag{3.5}$$

which implies that (3.2) holds since the ball $B \subset \mathbb{R}^n$ is arbitrary.

$(iii) \Rightarrow (iv)$. Assume that (3.2) holds, we will prove (3.3). For any fixed ball B, by Corollary 2.2, inequalities (3.2) and (3.4), it is easy to see

$$\begin{aligned} \frac{1}{|B|}\int_B |b(x)-b(y)|dy &\lesssim \frac{1}{|B|}\,\|\big(b-b_B\big)\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}\,|B|^{\frac{1}{p'}+\frac{\lambda}{q}}. \\ &\approx \frac{\|\big(b-b_B\big)\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} \\ &\lesssim 1. \end{aligned}$$

$(iv) \Rightarrow (i)$. For any fixed ball B, we have

$$\begin{aligned} \frac{1}{|B|}\int_B |b(x)-b_B|dy &= \frac{\|\big(b-b_B\big)\chi_B\|_{L_1}}{|B|} \\ &\le \sup_B \frac{\|\big(b-b_B\big)\chi_B\|_{L_1}}{|B|} \\ &\lesssim 1, \end{aligned}$$

which implies that $b \in BMO(\mathbb{R}^n)$. Thus the proof of the theorem is completed.

In the case $\lambda = 0$ from Theorem 3.1 we get the following corollary.

Corollary 3.1 *[24] Let $1 < p, q < \infty$. The following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$.*
(ii) *The operator M_b is bounded on $L_{p,q}(\mathbb{R}^n)$.*
(iii) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot)-b_B\big)\chi_B\big\|_{L_{p,q}(\mathbb{R}^n)}}{\|\chi_B\|_{L_{p,q}(\mathbb{R}^n)}} \le C.$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot)-b_B\big)\chi_B\big\|_{L_1(\mathbb{R}^n)}}{|B|} \le C.$$

4 $M^{\rm loc}_{p,q;\lambda}$-boundedness of the commutator of maximal operator $[b, M]$

In this section we obtain necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b, M]$ on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ Morrey-Lorentz spaces.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M]$ and M_b are valid :

Let b be any non-negative locally integrable function. Then for all $f \in L^{\rm loc}_1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned} &\big|[b, M]f(x)\big| = \big|b(x)Mf(x) - M(bf)(x)\big| \\ &= \big|M(b(x)f)(x) - M(bf)(x)\big| \leq M(|b(x) - b|f)(x) = M_b f(x). \end{aligned}$$

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M]f(x)| \leq M_b f(x) + 2b^-(x)\, Mf(x), \qquad x \in \mathbb{R}^n \tag{4.1}$$

holds for all $f \in L^{\rm loc}_1(\mathbb{R}^n)$ (see, for example [23,44]).

Denote by $M_B f$ the local maximal function of f:

$$M_B f(x) := \sup_{B' \ni x\colon B' \subset B} \frac{1}{|B'|} \int_{B'} |f(y)|\, dy,\ x \in \mathbb{R}^n.$$

Applying Theorem 3.1, we obtain the following result.

Theorem 4.1 *Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$ and $\frac{q}{q+\lambda} < p < \infty$. The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ *and* $b^- \in L_\infty(\mathbb{R}^n)$.

(ii) *The operator $[b, M]$ is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$.*

(iii) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B\big\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}} \leq C. \tag{4.2}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B\big\|_{L_1(\mathbb{R}^n)}}{|B|} \leq C. \tag{4.3}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$ and $b^- \in L_\infty(\mathbb{R}^n)$. Combining Lemma 2.1 and Theorem 3.1, and inequality (4.1), we get

$$\begin{aligned} \|[b, M]f\|_{M^{\rm loc}_{p,q;\lambda}} &\leq \|M_b f + 2b^-\, Mf\|_{M^{\rm loc}_{p,q;\lambda}} \\ &\leq \|M_b f\|_{M^{\rm loc}_{p,q;\lambda}} + \|b^-\|_{L_\infty}\, \|Mf\|_{M^{\rm loc}_{p,q;\lambda}} \\ &\lesssim \big(\|b\|_* + \|b^-\|_{L_\infty}\big)\, \|f\|_{M^{\rm loc}_{p,q;\lambda}}. \end{aligned}$$

Thus, we obtain that $[b, M]$ is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$.

$(ii) \Rightarrow (iii)$. Assume that $[b, M]$ is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$. Let $B = B(x, r)$ be a fixed ball. Since

$$M(b\chi_B)\chi_B = M_B(b) \quad \text{and} \quad M(\chi_B)\chi_B = \chi_B,$$

we have

$$\begin{aligned}|M_B(b) - b\chi_B| &= |M(b\chi_B)\chi_B - bM(\chi_B)\chi_B| \\ &\le |M(b\chi_B) - bM(\chi_B)| = |[b, M]\chi_B|.\end{aligned}$$

Hence

$$\|M_B(b) - b\chi_B\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)} \le \|[b, M]\chi_B\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}.$$

Thus we get

$$\frac{\|\big(b - M_B(b)\big)\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} \le \frac{\|[b, M](\chi_B)\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} \lesssim \frac{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} = 1,$$

which deduces that (iii).

$(iii) \Rightarrow (iv)$. Assume that (4.2) holds, then for any fixed ball B, by Corollary 2.2, we conclude that

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x) - M_B(b)(x)|dx &\lesssim \frac{1}{|B|}\,\|\big(b - M_B(b)\big)\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}\ |B|^{\frac{1}{p'}+\frac{\lambda}{q}} \\ &\approx \frac{\|\big(b - M_B(b)\big)\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} \\ &\lesssim 1.\end{aligned}$$

$(iv) \Rightarrow (i)$. Assume that (4.3) holds, we will prove $b \in BMO(\mathbb{R}^n)$ and $b^- \in L_\infty(\mathbb{R}^n)$.

Denote by

$$E := \{x \in B : \ b(x) \le b_B\}, \quad F := \{x \in B : \ b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B|\, dt = \int_F |b(t) - b_B|\, dt,$$

in view of the inequality $b(x) \le b_B \le M_B(b)$, $x \in E$, we get

$$\begin{aligned}\frac{1}{|B|}\int_B |b - b_B| &= \frac{2}{|B|}\int_E |b - b_B| \\ &\le \frac{2}{|B|}\int_E |b - M_B(b)| \\ &\le \frac{2}{|B|}\int_B |b - M_B(b)| \lesssim c.\end{aligned}$$

Consequently, $b \in BMO(\mathbb{R}^n)$. In order to show that $b^- \in L_\infty(\mathbb{R}^n)$, note that $M_B(b) \ge |b|$. Hence

$$0 \le b^- = |b| - b^+ \le M_B(b) - b^+ + b^- = M_B(b) - b.$$

Thus

$$(b^-)_B \le c,$$

and by the Lebesgue Differentiation theorem we get that

$$0 \le b^-(x) = \lim_{|B|\to 0}\frac{1}{|B|}\int_B b^-(y)dy \le c \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Thus the proof of the theorem is completed.

In the case $\lambda = 0$ from Theorem 4.1 we get the following corollary.

Corollary 4.1 *[24] Let $1 < p, q < \infty$. The following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L_\infty(\mathbb{R}^n)$.*
(ii) *The operator $[b, M]$ is bounded on $L_{p,q}(\mathbb{R}^n)$.*
(iii) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\left\| \big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B \right\|_{L_{p,q}(\mathbb{R}^n)}}{\|\chi_B\|_{L_{p,q}(\mathbb{R}^n)}} \le C.$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\left\| \big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B \right\|_{L_1(\mathbb{R}^n)}}{|B|} \le C.$$

5 $M^{\rm loc}_{p,q;\lambda}$-boundedness of the commutator of sharp maximal operator $[b, M^\sharp]$

In this section we obtain necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b, M^\sharp]$ on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ Morrey-Lorentz spaces.

Next, our third result is as follows.

Theorem 5.1 *Let $1 \le q \le \infty$, $0 \le \lambda < 1$ and $\frac{q}{q+\lambda} < p < \infty$. The following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L_\infty(\mathbb{R}^n)$.*
(ii) *The operator $[b, M^\sharp]$ is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$.*
(iii) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\left\| \big(b(\cdot) - 2M^\sharp\big(b\chi_B\big)(\cdot)\big)\chi_B \right\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}} \le C. \tag{5.1}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\left\| \big(b(\cdot) - 2M^\sharp\big(b\chi_B\big)(\cdot)\big)\chi_B \right\|_{L_1(\mathbb{R}^n)}}{|B|} \le C. \tag{5.2}$$

Proof. We only need to prove $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$, $(3) \Rightarrow (4)$ and $(4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$. Since $b \in BMO(\mathbb{R}^n)$ and $b^- \in L_\infty(\mathbb{R}^n)$, then for any locally integrable function f and a.e. $x \in \mathbb{R}^n$

$$
\begin{aligned}
&\big|[b, M^\sharp]f(x)\big| = \Big| \sup_{B \ni x} \frac{b(x)}{|B|} \int_B |f(y) - f_B| dy \\
&- \sup_{B \ni x} \frac{1}{|B|} \int_B |b(y)f(y) - (bf)_B| dy \Big| \\
&\le \sup_{B \ni x} \frac{1}{|B|} \int_B \big|(b(y) - b(x))f(y) + b(x)f_B - (bf)_B\big| dy \\
&\le \sup_{B \ni x} \Big(\frac{1}{|B|} \int_B |b(y) - b(x)| \, |f(y)| + \big|b(x)f_B - (bf)_B\big| \Big) \\
&\lesssim \|b\|_* \, M_b f(x) + \sup_{B \ni x} \Big| \frac{b(x)}{|B|} \int_B f(z)dz - \frac{1}{|B|} \int_B b(z)f(z)dz \Big| \\
&\lesssim \|b\|_* \, M_b f(x) + \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(z)||f(z)| dz \\
&\lesssim \|b\|_* \, M_b f(x).
\end{aligned}
$$

Then, it follows from Theorem 2.1 that $[b, M^\sharp]$ is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$.

$(2) \Rightarrow (3)$. Assume $[b, M^\sharp]$ is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$, we will prove (5.1). For any fixed ball B, we have (see [8, page 3333] or [44, page 1383] for details)

$$
M^\sharp\big(\chi_{_B}\big)(x) = \frac{1}{2} \quad \text{for all} \quad x \in B.
$$

Then, for all $x \in B$,

$$
\begin{aligned}
&b(x) - 2M^\sharp\big(b\,\chi_{_B}\big)(x) = 2\Big(\frac{b(x)}{2} - M^\sharp\big(b\,\chi_{_B}\big)(x)\Big) \\
&= 2\Big(b(x)M^\sharp\big(\chi_{_B}\big)(x) - M^\sharp\big(b\,\chi_{_B}\big)(x)\Big) = [b, M^\sharp]\big(\chi_{_B}\big)(x).
\end{aligned}
$$

Since $[b, M^\sharp]$ is bounded on $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$, then by applying (3.4), we have

$$
\begin{aligned}
&\sup_B \frac{\big\|\big(b(\cdot) - 2M^\sharp\big(b\chi_{_B}\big)(\cdot)\big)\chi_{_B}\big\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}}{\|\chi_{_B}\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}} \\
&\le \sup_B \frac{\big\|[b, M^\sharp]\big(\chi_{_B}\big)\big\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}}{\|\chi_{_B}\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}} \\
&\le \sup_B \frac{\|\chi_{_B}\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}}{\|\chi_{_B}\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}} \lesssim 1,
\end{aligned}
$$

which implies (5.1).

$(3) \Rightarrow (4)$: We deduce (5.1) from (5.2). Assume (5.1) holds, then for any fixed ball B, it follows from Corollary 2.2 and (3.4) that

$$
\begin{aligned}
&|B|^{-1} \left\| b(\cdot) - 2M^\sharp(b\chi_B)(\cdot) \right\|_{L^1(B)} \\
&\leq |B|^{-1} \left\| b(\cdot) - 2M^\sharp(b\chi_B)(\cdot) \right\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)} |B|^{\frac{1}{p'}+\frac{\lambda}{q}} \\
&\approx \frac{\|(b-b_B)\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}}} \lesssim 1.
\end{aligned}
$$

where the constant C is independent of B. So we obtain (5.2).

$(4) \Rightarrow (1)$. We first prove $b \in BMO(\mathbb{R}^n)$. For any fixed ball B, we have (see (2) in [8] for details)

$$
|b_B| \leq 2M^\sharp(b\chi_B)(x), \quad \text{for any} \quad x \in B. \tag{5.3}
$$

For any ball B, let $E = \{y \in B : b(y) \leq b_B\}$ and $F = \{y \in B : b(y) > b_B\}$. The following equality is true (see [8, page 3331]):

$$
\int_E |b(y) - b_B| dy = \int_F |b(y) - b_B| dy.
$$

Since $b(y) \leq b_B \leq |b_B| \leq 2M^\sharp(b\chi_B)(y)$ for any $y \in E$, we obtain

$$
|b(y) - b_B| \leq |b(y) - 2M^\sharp(b\chi_B)(y)|, \ y \in E.
$$

Then we have

$$
\begin{aligned}
&\frac{1}{|B|}\int_B |b(y) - b_B| dy = \frac{2}{|B|}\int_E |b(y) - b_B| dy \\
&\leq \frac{2}{|B|}\int_E |b(y) - 2M^\sharp(b\chi_B)(y)| dy \\
&\leq \frac{2}{|B|}\int_B |b(y) - 2M^\sharp(b\chi_B)(y)| dy.
\end{aligned}
$$

Applying (5.2) we get $b \in BMO(\mathbb{R}^n)$.

In order to show that $b^- \in L_\infty(\mathbb{R}^n)$, note that $|b_B| \leq 2M^\sharp(b\chi_B)(x)$ for any $x \in B$. Then, for all $x \in B$,

$$
2M^\sharp(b\chi_B)(x) - b(x) \geq |b_B| - b(x) = |b_B| - b^+(x) + b^-(x).
$$

Therefore

$$
\begin{aligned}
&|b_B| - \frac{1}{\mu(B)}\int_B b^+(x)dx + \frac{1}{\mu(B)}\int_B b^-(x)dx \\
&= \frac{1}{\mu(B)}\int_B \left(|b_B| - b^+(x) + b^-(x)\right)dx \\
&\leq \frac{1}{\mu(B)}\int_B \left(2M^\sharp(b\chi_B)(x) - b(x)\right)dx \\
&\leq \frac{1}{\mu(B)}\int_B |b(x) - 2M^\sharp(b\chi_B)(x)| dx.
\end{aligned} \tag{5.4}
$$

Then from Corollary 2.2 and (3.4) we get

$$
\begin{aligned}
&\frac{1}{\mu(B)}\int_B \big|b(x)-2M^\sharp\big(b\,\chi_B\big)(x)\big|dx \\
&\frac{1}{\mu(B)}\Big\|\big(b(\cdot)-2M^\sharp(b\,\chi_B)\big)\chi_B\Big\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}\ |B|^{\frac{1}{p'}+\frac{\lambda}{q}} \\
&\le \frac{\big\|\big(b(\cdot)-2M^\sharp(b\,\chi_B)\big)\chi_B\big\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}}{\|\chi_B\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}} \le C\,,
\end{aligned}
\tag{5.5}
$$

From (5.4) abd (5.5) we obtain

$$
\big|b_B\big|-\frac{1}{\mu(B)}\int_B b^+(x)dx+\frac{1}{\mu(B)}\int_B b^-(x)dx\le C.
$$

By the Lebesgue differentiation theorem we get that

$$
2|b^-(x)|=2b^-(x)=|b(x)|-b^+(x)-b^-(x)\le C.
$$

This implies that $b^-\in L_\infty(\mathbb{R}^n)$.

Thus the proof of the theorem is completed.

In the case $\lambda=0$ from Theorem 5.1 we obtain a new consequence.

Corollary 5.1 *Let $1<p,q<\infty$. The following assertions are equivalent:*

(i) *$b\in BMO(\mathbb{R}^n)$ and $b^-\in L_\infty(\mathbb{R}^n)$.*

(ii) *The operator $[b,M^\sharp]$ is bounded on $L_{p,q}(\mathbb{R}^n)$.*

(iii) *There exist a constant $C>0$ such that*

$$
\sup_B\frac{\big\|\big(b(\cdot)-2M^\sharp\big(b\chi_B\big)(\cdot)\big)\chi_B\big\|_{L_{p,q}(\mathbb{R}^n)}}{\|\chi_B\|_{L_{p,q}(\mathbb{R}^n)}}\le C.
$$

(iv) *There exist a constant $C>0$ such that*

$$
\sup_B\frac{\big\|\big(b(\cdot)-2M^\sharp\big(b\chi_B\big)(\cdot)\big)\chi_B\big\|_{L_1(\mathbb{R}^n)}}{|B|}\le C.
$$

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