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On embeddings of total Morrey spaces associated with Bessel operators

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Abstract. *Let* $\nu \in (0,\infty)$ *and* $\triangle_\nu := -\frac{d^2}{dx^2} - \frac{2\nu+1}{x}\frac{d}{dx}$ *be the Bessel operator on* $\mathbb{R}_+ := (0,\infty)$. *In this paper, we study some embeddings into the total Morrey space* $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$, $0 \le \lambda, \mu < 2\nu + 2$ *associated with the Bessel operator on* $\mathbb{R}_+$. *These spaces generalize the Morrey space associated with the Bessel operator on* $\mathbb{R}_+$ *so that* $L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \equiv L_{p,\lambda,\lambda}(\mathbb{R}_+, dm_\nu)$ *and the modified Morrey spaces associated with the Bessel operator on* $\mathbb{R}_+$ *so that* $\widetilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) \equiv L_{p,\lambda,0}(\mathbb{R}_+, dm_\nu)$.

Keywords. Bessel operator, total Morrey space, embedding

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1 Introduction

Let $\nu > -1/2$ and $\triangle_\nu$ be the Bessel operator defined by setting, for suitable functions f and $x \in \mathbb{R}_+$, see [5,21]. An early work concerning the Bessel operator is from Muckenhoupt and Stein [21]. They aimed to develop a theory associated to $\triangle_\nu$ which is parallel to the classical one associated to the Laplace operator $\triangle$. After that, a lot of work concerning the Bessel operators was carried out. See, for example [2,4,6–9,15,27,28] and the references therein. Among the study of $\triangle_\nu$, the properties of Riesz transforms associated to $\triangle_\nu$ defined by

$$R_{\triangle_\nu} f := \partial_x \big(\triangle_\nu\big)^{-1/2} f,$$

have been studied extensively, see for example [2,4,24]. Characterizations of function spaces associated to the Bessel operator $\triangle_\nu$ were also studied by many authors. Among these,

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we point out that the Lebesgue space associated to the Bessel operator $\triangle_\nu$ is of the form $L_p(\mathbb{R}_+, dm_\nu)$, where $1 < p < \infty$, $dm_\nu(x) := x^{2\nu+1}dx$, and dx is the standard Lebesgue measure on $\mathbb{R}_+$ (see for example [5]).

Morrey spaces, introduced by C. B. Morrey [20], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In [11] Guliyev introduce a variant of Morrey spaces called total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, see also, [1,12–14,18,19,22,23].

In the present work, we give basic properties of the total Morrey space $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$, $0 \le \lambda, \mu < 2\nu + 2$ associated with the Bessel operator on $\mathbb{R}_+$ and study some embeddings into the total Morrey space $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$. These spaces generalize the Morrey space associated with the Bessel operator on $\mathbb{R}_+$ so that $L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \equiv L_{p,\lambda,\lambda}(\mathbb{R}_+, dm_\nu)$ and the modified Morrey spaces associated with the Bessel operator on $\mathbb{R}_+$ so that $\widetilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) \equiv L_{p,\lambda,0}(\mathbb{R}_+, dm_\nu)$.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give some embeddings into the total Morrey spaces associated with the Bessel operator on $\mathbb{R}_+$.

Finally, we make some conventions on notation. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

Bessel functions were first discovered in 1732 by D. Bernoulli (1700-1782), who provided a series solution (representing a Bessel function) for the oscillatory displacements of a heavy hanging chain (see [3]). Euler later developed a series similar to that of Bernoulli, which was also a Bessel function, and Bessel's equation appeared in a 1764 article by Euler dealing with the vibrations of a circular drumhead. J. Fourier (1768-1836) also used Bessel functions in his classical treatise on heat in 1822, but it was Bessel who first recognized their special properties. Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the study of free vibrations of a circular membrane. They also occur in electromagnetic theory and numerous other areas of physics and engineering (see [3]). The more complete reference about Bessel functions is the treatise of Watson [29].

Trimeche [25,26] has pointed out how the theory of Bessel functions generates an harmonic analysis on the half line tied to the differential operator

$$\triangle_\nu := -\frac{d^2}{dx^2} - \frac{2\nu+1}{x}\frac{d}{dx}, \quad \nu > -1/2,$$

which is referred to as the Bessel operator of index ν.

For a real parameter $\nu \ge -1/2$, we consider the Bessel operator on $\mathbb{R}_+$:

$$\triangle_\nu := -\frac{d^2}{dx^2} - \frac{2\nu+1}{x}\frac{d}{dx}.$$

Note that $\triangle_{-1/2} = -\frac{d^2}{dx^2}$.

Let $\nu > -1/2$ be a fixed number and m_ν be the weighted Lebesgue measure on $\mathbb{R}_+$, given by

$$dm_\nu(x) := \left(2^{\nu+1}\Gamma(\nu+1)\right)^{-1} x^{2\nu+1}\, dx.$$

For every $1 \le p \le \infty$, we denote by $L_{p,\nu} = L_p(\mathbb{R}_+, dm_\nu)$ the spaces of complex-valued functions f, measurable on $\mathbb{R}_+$ such that

$$\|f\|_{p,\nu} \equiv \|f\|_{L_{p,\nu}} = \left(\int_{\mathbb{R}_+} |f(x)|^p \, dm_\nu(x) \right)^{1/p} < \infty \quad \text{if} \quad p \in [1, \infty),$$

and

$$\|f\|_{L_{\infty,\nu}} = \operatorname*{ess\,sup}_{x \in \mathbb{R}_+} |f(x)| \quad \text{if} \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\nu}$, the weak $L_{p,\nu}$ spaces defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{WL_{p,\nu}} = \sup_{r>0} r \, \left(m_\nu \{ x \in \mathbb{R}_+ \, : \, |f(x)| > r \} \right)^{1/p}.$$

Note that

$$L_{p,\nu} \subset WL_{p,\nu} \quad \text{and} \quad \|f\|_{WL_{p,\nu}} \le \|f\|_{L_{p,\nu}} \text{ for all } f \in L_{p,\nu}.$$

Let $B(x,t) = \{ y \in \mathbb{R}_+ : y \in \ (\max\{0, x-t\}, x+t) \ \} = (x-t, x+t) \cap \mathbb{R}_+$ and $t > 0$. Then $B(0,t) = (0,t)$ and $m_\nu((0,t)) = \left(2^{\nu+1} \, (\nu+1) \, \Gamma(\nu+1) \right)^{-1} t^{2\nu+2}$.

Remark 2.1 We mention that $\left(L_p(\mathbb{R}_+, dm_\nu), \|f\|_{L_{p,\nu}} \right)$ is a Banach space. Moreover, since $\left(\mathbb{R}_+, |\cdot|, dm_\nu \right)$ is a space of homogeneous type, $\left(L_p(\mathbb{R}_+, dm_\nu), \|f\|_{L_{p,\nu}} \right)$ is a Lebesgue spaces on spaces of homogeneous type in [16,17].

3 Some embeddings into the total Morrey spaces

In this section, we study some embeddings into the total Morrey space $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$, $0 \le \lambda, \mu < 2\nu + 2$ associated with the Bessel operator on $\mathbb{R}_+$.

Definition 3.1 *Let $1 \le p < \infty$, $0 \le \lambda \le 2\nu + 2$ and $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda}(\mathbb{R}_+, dm_\nu)$ Morrey space, by $\widetilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)$ the modified Morrey space and by $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ total Morrey space associated with the Bessel operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}_+$, with the finite norms*

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)} := \sup_{x \in \mathbb{R}_+, \, t > 0} \left(t^{-\lambda} \int_{B(x,t)} |f(y)|^p \, dm_\nu(y) \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)} := \sup_{x \in \mathbb{R}_+, \, t > 0} \left([t]_1^{-\lambda} \int_{B(x,t)} |f(y)|^p \, dm_\nu(y) \right)^{1/p},$$

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} := \sup_{x \in \mathbb{R}_+, \, t > 0} \left([t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{B(x,t)} |f(y)|^p \, dm_\nu(y) \right)^{1/p},$$

respectively.

If $\min\{\lambda,\mu\}<0$ or $\max\{\lambda,\mu\}>2\nu+2$, then $L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)=\Theta(\mathbb{R}_+)$, where $\Theta(\mathbb{R}_+)$ is the set of all functions equivalent to 0 on $\mathbb{R}_+$.

Note that

$$L_{p,0,0}(\mathbb{R}_+,dm_\nu)=\widetilde{L}_{p,0}(\mathbb{R}_+,dm_\nu)=L_{p,0}(\mathbb{R}_+,dm_\nu)=L_p(\mathbb{R}_+,dm_\nu),$$
$$L_{p,\lambda,\lambda}(\mathbb{R}_+,dm_\nu)=L_{p,\lambda}(\mathbb{R}_+,dm_\nu),\ L_{p,\lambda,0}(\mathbb{R}_+,dm_\nu)=\widetilde{L}_{p,\lambda}(\mathbb{R}_+,dm_\nu)$$

$$L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)\subset\!\!\succ L_{p,\lambda}(\mathbb{R}_+,dm_\nu) \quad\text{and}\quad \|f\|_{L_{p,\lambda}(\mathbb{R}_+,dm_\nu)}\le\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}, \tag{3.1}$$

$$L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)\subset\!\!\succ L_{p,\mu}(\mathbb{R}_+,dm_\nu) \quad\text{and}\quad \|f\|_{L_{p,\mu}(\mathbb{R}_+,dm_\nu)}\le\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}. \tag{3.2}$$

Definition 3.2 *Let $1\le p<\infty$, $0\le\lambda\le 2\nu+2$. We denote by $WL_{p,\lambda}(\mathbb{R}_+,dm_\nu)$ weak Morrey space, by $W\widetilde{L}_{p,\lambda}(\mathbb{R}_+,dm_\nu)$ the modified weak Morrey space and by $WL_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)$ weak total Morrey space associated with the Bessel operator as the set of locally integrable functions $f(x)$, $x\in\mathbb{R}_+$ with finite norms*

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+,dm_\nu)}:=\sup_{r>0} r \sup_{x\in\mathbb{R}_+,\,t>0}\Big(t^{-\lambda}m_\nu\{y\in B(x,t):\ |f(y)|>r\}\Big)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mathbb{R}_+,dm_\nu)}:=\sup_{r>0} r \sup_{x\in\mathbb{R}_+,\,t>0}\Big([t]_1^{-\lambda}m_\nu\{y\in B(x,t):\ |f(y)|>r\}\Big)^{1/p},$$

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}:=\sup_{r>0} r \sup_{x\in\mathbb{R}_+,\,t>0}\Big([t]_1^{-\lambda}\,[1/t]_1^{\mu}\,m_\nu\{y\in B(x,t):\ |f(y)|>r\}\Big)^{1/p},$$

respectively.

We note that

$$L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)\subset WL_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu) \ \text{ and } \ \|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}\le\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}.$$

Remark 3.1 We mention that $\big(L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu),\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}\big)$ is a Banach space. Moreover, since $\big(\mathbb{R}_+,|\cdot|,dm_\nu\big)$ is a space of homogeneous type,

$$\big(L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu),\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}\big)$$

is a special case of Morrey spaces on spaces of homogeneous type in [10].

Lemma 3.1 *If $0<p<\infty$, $0\le\lambda\le 2\nu+2$ and $0\le\mu\le 2\nu+2$, then*

$$L_{p,2\nu+2,\mu}(\mathbb{R}_+,dm_\nu)\subset\!\!\succ L_\infty(\mathbb{R}_+,dm_\nu)\subset\!\!\succ L_{p,\lambda,2\nu+2}(\mathbb{R}_+,dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,2\nu+2}(\mathbb{R}_+,dm_\nu)}\le b_\nu^{1/p}\|f\|_{L_\infty(\mathbb{R}_+,dm_\nu)}\le\|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R}_+,dm_\nu)}.$$

Proof. Let $f\in L_\infty(\mathbb{R}_+,dm_\nu)$. Then for all $x\in\mathbb{R}$ and $0<t\le 1$

$$\Big(t^{-\lambda}\int_{B(x,t)}|f(y)|^p\,dm_\nu(y)\Big)^{1/p}\le b_\nu^{1/p}\,\|f\|_{L_\infty},\quad 0\le\lambda\le 2\nu+2$$

and for all $x\in\mathbb{R}_+$ and $t\ge 1$

$$\Big(t^{-2\nu-2}\int_{B(x,t)}|f(y)|^p\,dm_\nu(y)\Big)^{1/p}\le b_\nu^{1/p}\,\|f\|_{L_\infty}.$$

Therefore $f \in L_{p,\lambda,2\nu+2}(\mathbb{R}_+, dm_\nu)$ and

$$\|f\|_{L_{p,\lambda,2\nu+2}(\mathbb{R}_+,dm_\nu)} \le b_\nu^{1/p}\|f\|_{L_\infty}.$$

Let $f \in L_{p,2\nu+2,\mu}(\mathbb{R}_+, dm_\nu)$. By the Lebesgue's differentiation theorem we have

$$\lim_{t\to 0} m_\nu(B(x,t))^{-1}\int_{B(x,t)} |f(y)|^p \, dm_\nu(y) = |f(x)|^p \quad a.e.\ x \in \mathbb{R}_+.$$

Then

$$\begin{aligned}|f(x)| &= \left(\lim_{t\to 0} m_\nu(B(x,t))^{-1}\int_{B(x,t)} |f(y)|^p \, dm_\nu(y)\right)^{1/p} \\ &\le b_\nu^{1/p}\left(t^{-2\nu-2}\int_{B(x,t)} |f(y)|^p \, dm_\nu(y)\right)^{1/p} \le b_\nu^{1/p}\, \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R}_+,dm_\nu)}.\end{aligned}$$

Therefore $f \in L_\infty(\mathbb{R}_+, dm_\nu)$ and

$$\|f\|_{L_\infty(\mathbb{R}_+,dm_\nu)} \le b_\nu^{1/p}\, \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R}_+,dm_\nu)}.$$

Corollary 3.1 *If* $0 < p < \infty$*, then*

$$L_{p,2\nu+2}(\mathbb{R}_+, dm_\nu) = \widetilde{L}_{p,2\nu+2}(\mathbb{R}_+, dm_\nu) = L_\infty(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{L_{p,2\nu+2}(\mathbb{R}_+,dm_\nu)} = \|f\|_{\widetilde{L}_{p,2\nu+2}(\mathbb{R}_+,dm_\nu)} = b_\nu^{1/p}\|f\|_{L_\infty}.$$

Lemma 3.2 *Let* $1 \le p < \infty$*,* $0 \le \lambda \le 2\nu+2$ *and* $0 \le \mu \le 2\nu+2$*. Then*

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)} = \max\left\{\|f\|_{L_{p,\lambda}(\mathbb{R}_+,dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}_+,dm_\nu)}\right\}.$$

Proof. Let $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$. Then by (3.1) and (3.2) we have

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) \subset\!\!\!\succ L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu)$$

and

$$\max\left\{\|f\|_{L_{p,\lambda}(\mathbb{R}_+,dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}_+,dm_\nu)}\right\} \le \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}.$$

Let $f \in L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu)$. Then we have

$$\begin{aligned}\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)} &= \sup_{x\in\mathbb{R}_+,t>0}\left([t]_1^{-\lambda}\,[1/t]_1^{\mu}\int_{B(x,t)} |f(y)|^p \, dm_\nu(y)\right)^{1/p} \\ &= \max\left\{\sup_{x\in\mathbb{R}_+,0<t\le 1}\left(t^{-\lambda}\int_{B(x,t)} |f(y)|^p \, dm_\nu(y)\right)^{1/p},\right. \\ &\qquad \left.\sup_{x\in\mathbb{R}_+,t>1}\left(t^{-\mu}\int_{B(x,t)} |f(y)|^p \, dm_\nu(y)\right)^{1/p}\right\} \\ &\le \max\left\{\|f\|_{L_{p,\lambda}(\mathbb{R}_+,dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}_+,dm_\nu)}\right\}.\end{aligned}$$

Therefore, $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ and the embedding $L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu) \subset\!\!\succ L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ is valid.

Thus $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu)$ and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)} = \max\left\{\|f\|_{L_{p,\lambda}(\mathbb{R}_+,dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}_+,dm_\nu)}\right\}.$$

Corollary 3.2 *If* $0 < p < \infty$, $0 \le \lambda \le 2\nu + 2$, *then*

$$\widetilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) = L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_p(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}_+,dm_\nu)} = \max\left\{\|f\|_{L_{p,\lambda}(\mathbb{R}_+,dm_\nu)}, \|f\|_{L_p(\mathbb{R}_+,dm_\nu)}\right\}.$$

From Lemmas 3.1 and 3.2 for $1 \le p < \infty$ we have

$$\widetilde{L}_{p,2\nu+2}(\mathbb{R}_+, dm_\nu) = L_\infty(\mathbb{R}_+, dm_\nu) \cap L_p(\mathbb{R}_+, dm_\nu). \tag{3.3}$$

Lemma 3.3 *Let* $0 < p < \infty$, $0 \le \lambda \le 2\nu + 2$ *and* $0 \le \mu \le 2\nu + 2$. *Then*

$$WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = WL_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap WL_{p,\mu}(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)} = \max\left\{\|f\|_{WL_{p,\lambda}(\mathbb{R}_+,dm_\nu)}, \|f\|_{WL_{p,\mu}(\mathbb{R}_+,dm_\nu)}\right\}.$$

Corollary 3.3 *If* $0 < p < \infty$, $0 \le \lambda \le 2\nu + 2$, *then*

$$W\widetilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) = WL_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap WL_p(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mathbb{R}_+,dm_\nu)} = \max\left\{\|f\|_{WL_{p,\lambda}(\mathbb{R}_+,dm_\nu)}, \|f\|_{WL_p(\mathbb{R}_+,dm_\nu)}\right\}.$$

Remark 3.2 If $0 < p < \infty$, and $\min\{\lambda, \mu\} < 0$ or $\max\{\lambda, \mu\} > 2\nu + 2$, then

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = \Theta(\mathbb{R}_+).$$

Lemma 3.4 *If* $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le 2\nu + 2$ *and* $0 \le \mu_1 \le \mu_2 \le 2\nu + 2$, *then*

$$L_{p,\lambda_1,\mu_1}(\mathbb{R}_+, dm_\nu) \subset\!\!\succ L_{p,\lambda_2,\mu_2}(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R}_+,dm_\nu)} \le \|f\|_{L_{p,\lambda_1,\mu_1}(\mathbb{R}_+,dm_\nu)}.$$

Proof. Let $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$, $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le 2\nu + 2$, $0 \le \mu_1 \le \mu_2 \le 2\nu + 2$. Then

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R}_+,dm_\nu)} = \max\left\{ \sup_{x\in\mathbb{R}_+,\, 0<t\le 1} \left(t^{\lambda_1-\lambda_2} t^{-\lambda_1} \int_{B(x,t)} |f(y)|^p \, dm_\nu(y)\right)^{1/p}, \right.$$
$$\left. \sup_{x\in\mathbb{R}_+,\, t\ge 1} \left(t^{\mu_1-\mu_2} t^{-\mu_1} \int_{B(x,t)} |f(y)|^p \, dm_\nu(y)\right)^{1/p} \right\} \le \|f\|_{L^d_{p,\lambda_1,\mu_1}}.$$

On the total Morrey spaces associated with the Bessel operator the following embedding is valid.

Lemma 3.5 *Let* $0 \le \lambda < 2\nu+2$, $0 \le \mu < 2\nu+2$, $0 \le \beta_1 < 2\nu+2-\lambda$ *and* $0 \le \beta_2 < 2\nu+2-\mu$. *Then for* $\frac{2\nu+2-\lambda}{\beta_1} \le p \le \frac{2\nu+2-\mu}{\beta_2}$

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) \subset\!\succ L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+, dm_\nu)$$

and for $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ *the following inequality*

$$\|f\|_{L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+,dm_\nu)} \le b_\nu^{\frac{1}{p'}} \, \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}$$

is valid.

Proof. Let $0 < \lambda < 2\nu+2$, $0 \le \mu < 2\nu+2$, $0 < \beta_1 < 2\nu+2-\lambda$, $0 < \beta_2 < 2\nu+2-\mu$, $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ and $\frac{2\nu+2-\lambda}{\beta_1} \le p \le \frac{2\nu+2}{\beta_2}$. By the Hölder's inequality we have

$$\|f\|_{L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+,dm_\nu)} = \sup_{x\in\mathbb{R}_+,\, t>0} [t]_1^{\beta_1-2\nu-2} \, [1/t]_1^{-\beta_2+2\nu+2} \int_{B(x,t)} |f(y)| \, dm_\nu(y)$$

$$\le b_\nu^{\frac{1}{p'}} \sup_{x\in\mathbb{R}_+,\, t>0} \Big([t]_1 \, t^{-1}\Big)^{-(2\nu+2)/p'} [t]_1^{\beta_1-\frac{2\nu+2-\lambda}{p}} \Big([t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{B(x,t)} |f(y)|^p \, dm_\nu(y)\Big)^{1/p}$$

$$\le b_\nu^{\frac{1}{p'}} \, \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)} \sup_{t>0} \big([t]_1 \, t^{-1}\big)^{\frac{2\nu+2-\mu}{p}-\beta_2} \, [t]_1^{\beta_1-\frac{2\nu+2-\lambda}{p}}.$$

Note that

$$\sup_{t>0} \big([t]_1 \, t^{-1}\big)^{\frac{2\nu+2-\mu}{p}-\beta_2} \, [t]_1^{\beta_1-\frac{2\nu+2-\lambda}{p}} = \max\{\sup_{0<t\le1} t^{\beta_1-\frac{2\nu+2-\lambda}{p}}, \sup_{t>1} t^{\beta_2-\frac{2\nu+2-\mu}{p}}\} < \infty$$

$$\text{if and only if} \quad \frac{2\nu+2-\lambda}{\beta_1} \le p \le \frac{2\nu+2-\mu}{\beta_2}.$$

Therefore $f \in L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+, dm_\nu)$ and

$$\|f\|_{L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+,dm_\nu)} \le b_\nu^{\frac{1}{p'}} \, \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+,dm_\nu)}.$$

Corollary 3.4 *Let* $0 \le \lambda < 2\nu+2$ *and* $0 \le \beta < 2\nu+2-\lambda$. *Then for* $p = \frac{2\nu+2-\lambda}{\beta}$

$$L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \subset\!\succ L_{1,2\nu+2-\beta}(\mathbb{R}_+, dm_\nu)$$

and for $f \in L_{p,\lambda}(\mathbb{R}_+, dm_\nu)$ *the following inequality*

$$\|f\|_{L_{1,2\nu+2-\beta}(\mathbb{R}_+,dm_\nu)} \le b_\nu^{\frac{1}{p'}} \, \|f\|_{L_{p,\lambda}(\mathbb{R}_+,dm_\nu)}$$

is valid.

Corollary 3.5 *Let* $0 \le \lambda < 2\nu+2$ *and* $0 \le \beta < 2\nu+2-\lambda$. *Then for* $\frac{2\nu+2-\lambda}{\beta} \le p \le \frac{2\nu+2-\mu}{\beta}$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) \subset\!\succ \widetilde{L}_{1,2\nu+2-\beta}(\mathbb{R}_+, dm_\nu)$$

and for $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)$ *the following inequality*

$$\|f\|_{\widetilde{L}_{1,2\nu+2-\beta}(\mathbb{R}_+,dm_\nu)} \le b_\nu^{\frac{1}{p'}} \, \|f\|_{L_{p,\lambda}(\mathbb{R}_+,dm_\nu)}$$

is valid.

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