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# A new method for studying the boundary value problem for a three-dimensional elliptic equation with variable coefficients

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**Abstract.** *The paper is devoted to the study of the Fredholm property of a boundary value problem for a three-dimensional elliptic equation with variable coefficients with nonlocal boundary conditions in a bounded domain with a Lyapunov boundary. Singular necessary solvability conditions are derived, the regularization of which is carried out according to a new original scheme. Based on the regularized necessary conditions and boundary conditions, the Fredholm property of the problem is proved.*

**Keywords.** non-local boundary conditions · three-dimensional elliptic equation · variable coefficients · necessary conditions · regularization · Fredholm property.

**Mathematics Subject Classification (2010):** 35J05, 35J40

## 1 Introduction

As is known, the Fredholm property of local boundary value problems (Dirichlet, Neumann, Poincar) for partial differential equations is proved by reduction to the Fredholm integral of the second kind with a regular kernel, which is solved by the method of successive approximations. Since even-order equations are usually considered in the theory of partial differential equations, this scheme is generally accepted in this case.

In nonlocal boundary value problems for PDEs, the entire boundary is the carrier of a nonlocal boundary condition, as if "stitching" the values of the unknown function and its derivatives on different sections of the boundary. The introduction of nonlocal conditions also eliminated the misunderstanding between the O.D.E. and partial differential equations, when the number of boundary conditions does not coincide with the order of the equation.

In contrast to classical problems, for the nonlocal BVP there arose possibility of Fredholm property proof both for even and odd orders [1]-[5] by a principially new method: based on a fundamental solution of the principal part of the PDE and by means of the 2-nd Green's formula so called "necessary conditions" of solvability are obtained. The Fredholm property is investigated just from the necessary conditions.

It should be noted that for linear ordinary differential equations these necessary conditions were obtained by A.A. Dezin in the form of usual boundary conditions [6]-[7]. For partial differential equations for the first time these conditions were considered and obtained by A.V. Bitsadze in the form of singular integral equations for two-dimensional Laplace

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equation [8]. The regularization was carried out using the method of succcessive approximations. A.A.Bitsadze called them both “necessary and sufficient conditions of solvability” [8].

In three-dimansional case the singularities have a special form and are not regularized as in the general case as we arrive at a Fredholm equation of the first kind which is a dead end. A new approach [2], [5], [9]-[11] is used to regularize the obtained singularities what allows to prove the Fredholmness of the posed problem.

## 2 Problem statement

Let us consider the three-dimensional Laplace equation in a convex in the direction $x_3$ domain $D \subset R^3$ whose projection onto plane $Ox_1x_2 = Ox'$ is domain $S \subset Ox_1x_2$, $\Gamma$ is the boundary (Lyapunov surface) of the domain *D*:

$$Lu = \Delta u(x) + \sum_{k=1}^{3} a_k(x) \frac{\partial u(x)}{\partial x_k} + a(x)u(x) = 0, \ x \in D \subset R^3 \tag{2.1}$$

with non-local boundary conditions:

$$l_i u = \sum_{k=1}^{2} \sum_{j=1}^{3} \alpha_{ij}^{(k)}(x') \frac{\partial u(x)}{\partial x_j} \mid_{x_3=\gamma_k(x')}$$

$$+ \sum_{k=1}^{2} \alpha_i^{(k)}(x') u(x) \mid_{x_3=\gamma_k(x')} = \varphi_i(x'), \ i = 1,2; \ \ x' = (x_1, x_2) \in S, \tag{2.2}$$

$$u(x) = f_0(x), \ x \in \bar{\Gamma}_1 \bigcap \bar{\Gamma}_2, \tag{2.3}$$

where $S = proj_{Ox_1x_2} D$, $\Gamma_1$ and $\Gamma_2$ are the lower and the upper half surfaces of the boundary $\Gamma$ respectively defined as follows: $\Gamma_k = \{\xi = (\xi_1, \xi_2, \xi_3) : \xi_3 = \gamma_k(\xi'),$ $\xi' = (\xi_1, \xi_2) \in S\}$ where $\xi_3 = \gamma_k(\xi_1; \xi_2)\,, k = 1,\, 2,$ are equations of half surfaces $\Gamma_1$ and $\Gamma_2$, functions $\gamma_k(\xi')\,, k = 1,\, 2,$ are twice differentiable with respect of the both of the variables $\xi_1, \xi_2$; the coefficients $\alpha_{ij}^{(k)}(x')$, $\alpha_i^{(k)}(x')$ are continuous functions .

The fundamental solution for equation (2.1) is the same as for the three-dimensional Laplace equation [12]:

$$U(x-\xi) = -\frac{1}{4\pi} \frac{1}{|x-\xi|}. \tag{2.4}$$

Let us get basic relationships and necessary conditions.

Multiplying equation (2.1) by the fundamental solution (2.4), integrating it over the domain *D* by parts

$$\int_D Lu(x)U(x-\xi)dx = \int_D u(x)\Delta U(x-\xi)dx$$

$$+ \sum_{j=1}^{3} \int_{\Gamma} \left[ \left( \frac{\partial u(x)}{\partial x_j} U(x-\xi) - u(x) \frac{\partial U(x-\xi)}{\partial x_j} \right) \cos(\nu, x_j) dx \right]$$

$$+ \sum_{k=1}^{3} \int_D a_k(x) \frac{\partial u(x)}{\partial x_k} U(x-\xi)dx + \int_D a(x)u(x)U(x-\xi)dx$$

and taking into account that $\Delta_x U(x-\xi) = \delta(x-\xi)$ where $\delta(x-\xi)$ is the Dirac $\delta$-function we'll get **the first basic relationship**:

$$-\sum_{j=1}^{3}\int_{\Gamma}\left[\left(\frac{\partial u(x)}{\partial x_j}U(x-\xi)-u(x)\frac{\partial U(x-\xi)}{\partial x_j}\right)\cos(\nu,x_j)dx\right]$$

$$-\sum_{k=1}^{3}\int_{D}a_k(x)\frac{\partial u(x)}{\partial x_k}U(x-\xi)dx-\int_{D}a(x)u(x)U(x-\xi)dx$$

$$=\int_{D}u(x)\delta(x-\xi)dx=\begin{cases}u(\xi), & \xi\in D,\\ \frac{1}{2}u(\xi), & \xi\in\Gamma.\end{cases} \tag{2.5}$$

Here the first relationship gives the representation of the general solution of equation (2.1) and the second expression in (2.5) is the **first necessary condition.**

Taking into account $\frac{\partial U(x-\xi)}{\partial x_j}=-\frac{x_j-\xi_j}{4\pi|x-\xi|^3}=-\frac{\cos(x-\xi,x_j)}{4\pi|x-\xi|^2}$ we obtain **the 1$^{st}$ necessary condition** ($\xi\in\Gamma$) in the form

$$\frac{1}{2}u(\xi)=-\int_{\Gamma}\frac{\partial u(x)}{\partial\nu}U(x-\xi)dx-\int_{\Gamma}u(x)\frac{\cos(x-\xi,\nu_x)}{4\pi\left|x-\xi\right|^2}dx$$

$$-\sum_{k=1}^{3}\int_{D}a_k(x)\frac{\partial u(x)}{\partial x_k}U(x-\xi)dx-\int_{D}a(x)u(x)U(x-\xi)dx \tag{2.6}$$

where all the integrands have a weak singularity as the order of singularity doesn't exceed the multiplicity of the integrals.
Thus we have proved

**Theorem 2.1**. *Let a convex along the direction $x_3$ domain $D\subset R^3$ be bounded with the boundary $\Gamma$which is a Lyapunov surface. Then the obtained first necessary condition* (2.6) *is regular*.

Multiplying (2.1) by $\frac{\partial U(x-\xi)}{\partial x_i}$, $i=\overline{1,3}$, , integrating it over the domain *D* we obtain the rest of **three basic relationships**:

$$\int_{D}Lu\frac{\partial U(x-\xi)}{\partial x_i}dx=\int_{D}\Delta u(x)\frac{\partial U(x-\xi)}{\partial x_i}dx$$

$$+\int_{D}\sum_{k=1}^{3}a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{\partial U(x-\xi)}{\partial x_i}dx+\int_{D}a(x)u(x)\frac{\partial U(x-\xi)}{\partial x_i}dx,$$

whence after integrating by parts we get **the second basic relationship**

$$\int_{\Gamma}\frac{\partial u(x)}{\partial x_m}\left[\frac{\partial U(x-\xi)}{\partial x_m}\cos(\nu_x,x_i)-\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x,x_m)\right]dx$$

$$+\int_{\Gamma}\frac{\partial u(x)}{\partial x_l}\left[\frac{\partial U(x-\xi)}{\partial x_l}\cos(\nu_x,x_i)-\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x,x_l)\right]dx$$

$$-\int_{\Gamma}\frac{\partial u(x)}{\partial x_i}\frac{\partial U(x-\xi)}{\partial\nu_x}dx-\int_{D}\sum_{k=1}^{3}a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{\partial U(x-\xi)}{\partial x_i}dx$$

$$-\int_{D}a(x)u(x)\frac{\partial U(x-\xi)}{\partial x_i}dx=\begin{cases}\frac{\partial u(\xi)}{\partial\xi_i}, & \xi\in D,\\ \frac{1}{2}\frac{\partial u(\xi)}{\partial\xi_i}, & \xi\in\Gamma,\end{cases}\quad i=\overline{1,3}\,, \tag{2.7}$$

where the numbers *i, m, l* make a permutation of numbers 1,2,3. The second expressions in (2.7) are **the second necessary conditions** ( $\xi \in \Gamma,\ i = 1,2,3$ ).

Taking into account that $\frac{\partial U(x-\xi)}{\partial x_i} = -\frac{x_i-\xi_i}{4\pi|x-\xi|^3} = -\frac{\cos(x-\xi,x_i)}{4\pi|x-\xi|^2}$ and introducing the designations

$$K_{ij}(x,\xi) = \left(\cos(x-\xi,x_i)\cos(\nu_x,x_j) - \cos(x-\xi,x_j)\cos(\nu_x,x_i)\right).$$

We can rewrite **the second necessary conditions** in (2.7) as:

$$\frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_i} =$$

$$+\int_\Gamma \frac{\partial u(x)}{\partial x_m}\frac{K_{mi}(x,\xi)}{4\pi\left|x-\xi\right|^2}dx + \int_\Gamma \frac{\partial u(x)}{\partial x_l}\frac{K_{li}(x,\xi)}{4\pi\left|x-\xi\right|^2}dx\ .$$

$$-\int_\Gamma \frac{\partial u(x)}{\partial x_i}\frac{\partial U(x-\xi)}{\partial \nu_x}dx - \int_D \sum_{k=1}^{3} a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{\partial U(x-\xi)}{\partial x_i}dx -$$

$$-\int_D a(x)u(x)\frac{\partial U(x-\xi)}{\partial x_i}dx. \tag{2.8}$$

where $i = 1,2,3$ and the numbers *i,m,l* make a permutation of numbers 1,2,3.

If we disclose two first surface integrals in (2.8) ($i = 1,2,3$) over the lower and the upper half surfaces $\Gamma_k$, $k = 1,\ 2$:

$$\frac{\partial u}{\partial \xi_i}\Big|_{\xi_3=\gamma_k(\xi')} = \sum_{j=1}^{2}(-1)^{j-1}\int_S \frac{\partial u(x)}{\partial x_m}\Big|_{x_3=\gamma_j(x')}\frac{K_{mi}(x,\xi)}{2\pi\left|x-\xi\right|^2}\Bigg|_{\substack{x_3=\gamma_j(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}$$

$$+\sum_{j=1}^{2}(-1)^{j-1}\int_S \frac{\partial u(x)}{\partial x_l}\Big|_{x_3=\gamma_j(x')}\frac{K_{li}(x,\xi)}{2\pi\left|x-\xi\right|^2}\Bigg|_{\substack{x_3=\gamma_j(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}$$

$$-2\int_\Gamma \frac{\partial u(x)}{\partial x_i}\frac{\partial U(x-\xi)}{\partial \nu_x}\Big|_{\xi_3=\gamma_k(\xi')}dx - 2\int_D \sum_{p=1}^{3} a_p(x)\frac{\partial u(x)}{\partial x_p}\frac{\partial U(x-\xi)}{\partial x_i}dx$$

$$-\int_D 2a(x)u(x)\frac{\partial U(x-\xi)}{\partial x_i}dx \tag{2.9}$$

and extract only singular terms ($x_3 = \xi_3$) for $k = 1,\ 2$ then we'll get the second **necessary conditions** (2.9) ($i = 1,2,3$) in the form:

$$\frac{\partial u}{\partial \xi_i}\Big|_{\xi_3=\gamma_k(\xi')} = (-1)^k\int_S \frac{\partial u(x)}{\partial x_m}\Big|_{x_3=\gamma_k(x')}\frac{K_{mi}(x,\xi)}{2\pi\left|x-\xi\right|^2}\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}$$

$$+(-1)^{k+1}\int_S \frac{\partial u(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')}\frac{K_{li}(x,\xi)}{2\pi\left|x-\xi\right|^2}\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)} + \ldots,\ k = 1,2 \tag{2.10}$$

where three dots designate the sum of nonsingular terms.

**Remark 2.1.** *Three dots in* (2.10) *contain the derivatives* $\frac{\partial u(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')}$ , $l = 1, 2, 3;$ $k = 1, 2,$ *under the sign of integral and later we shall take it into consideration.*

Let us introduce the designations:

$$K_{ij}^{(k)}(x', \xi') = K_{ij}(x, \xi)\Bigg|_{\substack{x_3 = \gamma_k(x') \\ \xi_3 = \gamma_k(\xi')}} , \tag{2.11}$$

$$|x - \xi|^2 \Bigg|_{\substack{x_3 = \gamma_k(x') \\ \xi_3 = \gamma_k(\xi')}} = |x' - \xi'|^2 + (\gamma_k(x') - \gamma_k(\xi'))^2 = |x' - \xi'|^2 P_k(x', \xi') \tag{2.12}$$

where

$$P_k(x', \xi') = 1 + \left[\frac{\partial \gamma_k(x')}{\partial x_1} \cos(x' - \xi', x_1)\right.$$

$$\left. + \frac{\partial \gamma_k(x')}{\partial x_2} \cos(x' - \xi', x_2) + O(|x' - \xi'|)\right]^2 \neq 0, \; k = 1, 2\,.$$

By means of the designations (2.11), (2.12) we'll rewrite the necessary conditions (2.10) for *k*=1,2 as follows:

$$\frac{\partial u}{\partial \xi_i}\Big|_{\xi_3=\gamma_k(\xi')}$$

$$= (-1)^{(k)} \int_S \frac{\partial u(x)}{\partial x_m}\Bigg|_{x_3=\gamma_k(x')} \frac{1}{2\pi |x' - \xi'|^2} \frac{K_{mi}^{(k)}(x', \xi')}{P_k(x', \xi')} \frac{dx'}{\cos(\nu_x, x_3)}$$

$$+ (-1)^{(k+1)} \int_S \frac{\partial u(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')} \frac{1}{2\pi |x' - \xi|'^2} \frac{K_{li}^{(k)}(x', \xi')}{P_k(x', \xi')} \frac{dx'}{\cos(\nu_x, x_3)} + ..., \; i = 1, 2, 3; \; k = 1, 2. \tag{2.13}$$

**Theorem 2.2.** *Under assumptions of Theorem 2.1 necessary conditions* (2.13) *are singular.*

## 3 Regularization of the necessary conditions

Let us build a linear combination of necessary conditions (2.13) for *k*=1,2 (*i*=1,2; *j*=1,2,3) with unknown yet coefficients $\beta_{ij}^{(k)}(\xi')$ and bracket the common factor $\frac{1}{2\pi|x'-\xi'|^2}$ under the sign of integral ($i = 1, 2$):

$$\sum_{j=1}^{3} \left(\beta_{ij}^{(1)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j}\Big|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j}\Big|_{\xi_3=\gamma_2(\xi')}\right)$$

$$= \int_S \frac{1}{2\pi |x' - \xi'|^2} \frac{dx'}{\cos(\nu_x, x_3)} \sum_{k=1}^{2} (-1)^k$$

$$\times \sum_{j=1}^{3} \beta_{ij}^{(k)}(\xi') \left(\frac{\partial u(x)}{\partial x_m}\Big|_{x_3=\gamma_k(x')} \frac{K_{mj}^{(k)}(x', \xi')}{P_k(x', \xi')} + \frac{\partial u(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')} \frac{K_{lj}^{(k)}(x', \xi')}{P_k(x', \xi')}\right) + \dots\,. \tag{3.1}$$

Adding and subtracting $\beta_{ij}^{(k)}(x')$ from $\beta_{ij}^{(k)}(\xi')$, $k = 1,2$ in (3.1) and suggesting that functions $\beta_{ij}^{(k)}(\xi')$ satisfy Hölder condition we get a week singularity in the integrals with $\frac{\beta_{ij}^{(k)}(\xi')-\beta_{ij}^{(k)}(x')}{|x'-\xi'|^2}$. Expanding all the coefficients at the derivatives by Taylor's formula at point $\xi' = x'$:

$$\frac{K_{ij}^{(k)}(x',\xi')}{P_k(x',\xi')} = \frac{K_{ij}^{(k)}(x',x')}{P_k(x',x')} + \frac{\partial}{\partial x_1}\left(\frac{K_{ij}^{(k)}(x',x')}{P_k(x',x')}\right)(x_1-\xi_1) + ...$$

and discarding the terms with weak singularity we obtain:

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\,|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\,|_{\xi_3=\gamma_2(\xi')}\right)$$

$$= \int_S \frac{1}{2\pi\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)}\sum_{k=1}^{2}(-1)^k$$

$$\times\sum_{j=1}^{3}\frac{\partial u(x)}{\partial x_j}\,|_{x_3=\gamma_k(x')}\left(\beta_{il}^{(k)}(x')\frac{K_{lj}^{(k)}(x',x')}{P_k(x',x')} + \beta_{im}^{(k)}(x')\frac{K_{mj}^{(k)}(x',x')}{P_k(x',x')}\right) + ... \qquad (3.2)$$

where *i*=1,2 and the numbers *j, l, m* form a permutation of numbers 1,2,3.

To regularize the integral in the right hand side of (3.2) let us impose conditions on the coefficients $\beta_{ij}^{(k)}(\xi')$, i.e. let the coefficients at the derivatives under the sign of integral (3.2) be equal to the coefficients $\alpha_{ij}^{(k)}(\xi')$ from the boundary conditions (2.2). Then we get a system of 6 equations for each *i*=1,2:

$$(-1)^k\beta_{il}^{(k)}(x')\frac{K_{lj}^{(k)}(x',x')}{P_k(x',x')} + (-1)^k\beta_{im}^{(k)}(x')\frac{K_{mj}^{(k)}(x',x')}{P_k(x',x')} = \alpha_{ij}^{(k)}(x'), \qquad (3.3)$$

*k*=1,2; *j*=1,2,3,
where the numbers *j, l, m* form a permutation of numbers 1,2,3 as we mentioned above.

We assume that system (3.3) has the unique solution $\beta_{ij}^{(k)}(x')$, $i,k = 1,2;\ j = 1,2,3$.

**Remark 3.1.** *As the system* (3.3) *is linear the obtained functions* $\beta_{ij}^{(k)}(x')$, $i,k = 1,2;$ $j = 1,2,3$, *are linear with respect to the given functions* $\alpha_{ij}^{(k)}(x')$, $i,k = 1,2;\ j = 1,2,3$, *and*, *therefore*, *satisfy a Hlder condition*.

Then for the further regularization we replace the expression under the integral sign in the right-hand side of (3.3) using boundary conditions (2.2):

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\,|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\,|_{\xi_3=\gamma_2(\xi')}\right)$$

$$= \int_S \frac{\varphi_i(x')}{2\pi\,|x'-\xi'|^2}\frac{dx'}{\cos(\nu_x,x_3)}$$

$$-\int_S \frac{1}{2\pi\,|x'-\xi'|^2}\left[\sum_{k=1}^{2}\alpha_i^{(k)}(x')u(x',\gamma_k(x'))\right]\frac{dx'}{\cos(\nu_x,x_3)}..... \qquad (3.4)$$

From necessary condition (2.6) for $u(\xi)$ on $\Gamma_k$ , $k = 1$ , 2, discarding the term with the normal derivative $\frac{\partial u}{\partial \nu_x}$ in the integrand and leaving only weakly singular terms we have:

$$u(\xi)\left|_{\xi_3=\gamma_k(\xi')}\right.$$

$$= -\int_S \frac{u(x)\left|_{x_3=\gamma_k(x')}\right.}{2\pi\left|x'-\xi'\right|^2} \frac{\cos(x-\xi,\nu_x)\left|\begin{matrix}\xi_3=\gamma_k(\xi')\\ x_3=\gamma_k(x')\end{matrix}\right.}{P_k(x',\xi')} \frac{dx'}{\cos(\nu_x,x_3)} + .... \tag{3.5}$$

Substituting necessary conditions (3.5) into (3.4) and changing the order of integration we'll obtain regularized relationships:

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right. + \beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right.\right)$$

$$= -\sum_{k=1}^{2}\int_S \frac{u(\zeta)\left|_{\zeta_3=\gamma_k(\zeta')}\right. d\zeta'}{\cos(\nu_\zeta,\zeta_3)}\int_S \alpha_i^{(k)}(x')\frac{\cos(\zeta-\xi,\nu_\zeta)\left|\begin{matrix}\zeta_3=\gamma_k(\zeta')\\ x_3=\gamma_k(x')\end{matrix}\right.}{2\pi\left|x'-\zeta'\right|^2\left|x'-\xi'\right|^2 P_k(x',\zeta')}\frac{dx'}{\cos(\nu_x,x_3)}$$

$$+\int_S \frac{\varphi_i(x')}{2\pi\left|x'-\xi'\right|^2}\frac{dx'}{\cos(\nu_x,x_3)} + \dots , \tag{3.6}$$

as the interior integral in the first term in the RHS of (3.6) doesn't contain the unknown and the second term is convergent if functions $\varphi_i(x'),\ i = 1, 2$, are continuously differentiable in S and vanish on the boundary $\partial S = \bar{S}\backslash S$.

Thus we have established the following

**Theorem 3.1**. *Let the conditions of Theorem* 2.1 *hold true. If system* (3.3) *is uniquely resolved, the conditions* (2.2) *are linear independent, the coefficients* $\alpha_{ij}^{(k)}(x')$ *for* $i = 1, 2;\ j = \overline{1, 3};\ k = 1, 2$, *belong to some Hlder class and the rest of the coefficients and kernels are continuous functions, functions* $\varphi_i(x'),\ i = 1, 2$, *are continuously differentiable and vanish on the boundary* $\partial S = \bar{S}\backslash S$ *then the relationships* (3.6) *are regular.*

## 4 Fredholm property of the problem

It is well-known from the calculus that

$$\frac{\partial}{\partial x_p}u(x_1,x_2,\gamma_k(x_1,x_2))$$

$$= \frac{\partial u(x)}{\partial x_p}\left|_{x_3=\gamma_k(x')}\right. + \frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_k(x')}\right. \frac{\partial \gamma_k(x')}{\partial x_p},\ k = 1, 2; p = 1, 2, \tag{4.1}$$

whence we have

$\frac{\partial u(x)}{\partial x_p}\left|_{x_3=\gamma_k(x')}\right. = \frac{\partial u(x',\gamma_k(x'))}{\partial x_p} - \frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_k(x')}\right. \frac{\partial \gamma_k(x')}{\partial x_p}$, *p*=1,2; *k*=1,2. (4.1)

So, the derivatives $\frac{\partial u(x)}{\partial x_1}\left|_{x_3=\gamma_k(x')}\right.$ and $\frac{\partial u(x)}{\partial x_2}\left|_{x_3=\gamma_k(x')}\right.$ are defined through the derivative $\frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_k(x')}\right.$. Then we have two unknown quantities: the boundary values of the unknown function $u(x', \gamma_1(x'))$ and $u(x', \gamma_2(x'))$.

After substituting (4.1) for $\frac{\partial u(x)}{\partial x_1}\left|_{x_3=\gamma_k(x')}\right.$ and $\frac{\partial u(x)}{\partial x_2}\left|_{x_3=\gamma_k(x')}\right.$ into (2.2) and grouping the unknown quantities we get a system:

$$l_i u=\sum_{m=1}^{2}\left[\sum_{j=1}^{2}\alpha_{ij}^{(m)}(x')\left(\frac{\partial u(x',\gamma_m(x'))}{\partial x_j}-\frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_m(x')}\right.\frac{\partial\gamma_m(x')}{\partial x_j}\right)\right.$$

$$\left.+\alpha_{i3}^{(m)}(x')\frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_m(x')}\right.\right]$$

$$+\sum_{m=1}^{2}\alpha_i^{(m)}(x')u(x',\gamma_m(x'))=\varphi_i(x'),x'\in S,\ i=1,\ 2. \qquad (4.2)$$

Introducing the designations:

$$A_{ij}(x')=\alpha_{i3}^{(j)}(x')-\sum_{m=1}^{2}\alpha_{im}^{(j)}(x')\frac{\partial\gamma_j(x')}{\partial x_m},\ i,j=1,2\,,$$

system (4.2) can be rewritten in the form:

$$A_{i1}(x')\frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_1(x')}\right.+A_{i2}(x')\frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_2(x')}\right.=F_i(x')\,,\ i=1,2, \qquad (4.3)$$

where the right-hand sides of system (4.3) have the form:

$$\mathrm{F_i}(x')=\varphi_i(x')-\sum_{j=1}^{2}\sum_{m=1}^{2}\alpha_{ij}^{(m)}(x')\frac{\partial u(x',\gamma_m(x'))}{\partial x_j}+$$

$$+\sum_{k=1}^{2}\alpha_i^{(k)}(x')u(x',\gamma_k(x')),\ x'\in S,\ i=1,\ 2. \qquad (4.4)$$

**Remark 4.1.** *Note that the right hand sides* $\mathrm{F_i}(x')$ *of system* (4.3) *are, in the virtue of* (4.4), *functionals of* $u\left|_{\Gamma_1}\right.,u\left|_{\Gamma_2}\right.$ and $\frac{\partial u|_{\Gamma_k}}{\partial x_j},\ k,j=1,\ 2$:

$$F_i(x')=\mathrm{F_i}(x',\ u\left|_{\Gamma_1}\right.,u\left|_{\Gamma_2}\right.,\frac{\partial u\left|_{\Gamma_1}\right.}{\partial x_1},\frac{\partial u\left|_{\Gamma_1}\right.}{\partial x_2},\frac{\partial u\left|_{\Gamma_2}\right.}{\partial x_1},\frac{\partial u\left|_{\Gamma_2}\right.}{\partial x_2}),\ i=1,2. \qquad (4.5)$$

If to assume that

$$\Delta(x')=\begin{vmatrix}A_{11}(x') & A_{12}(x')\\ A_{21}(x') & A_{22}(x')\end{vmatrix}\neq 0 \qquad (4.6)$$

the system (4.3) can be reduced to a normal form:

$$\frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_1(x')}\right.=\frac{1}{\Delta(x')}\begin{vmatrix}F_1(x') & A_{12}(x')\\ F_2(x') & A_{22}(x')\end{vmatrix},$$

$$\frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_2(x')}\right.=\frac{1}{\Delta(x')}\begin{vmatrix}A_{11}(x') & F_1(x')\\ A_{21}(x') & F_2(x')\end{vmatrix}.$$

whence, in the virtue of (4.5), we have (from boundary conditions (2.2) and (4.1) :

$$\frac{\partial u(x)}{\partial x_3}\left|_{x_3=\gamma_k(x')}\right.=\Phi_k(u\left|_{\Gamma_1}\right.,u\left|_{\Gamma_2}\right.,\frac{\partial u\left|_{\Gamma_1}\right.}{\partial x_1},\frac{\partial u\left|_{\Gamma_1}\right.}{\partial x_2},\frac{\partial u\left|_{\Gamma_2}\right.}{\partial x_1},\frac{\partial u\left|_{\Gamma_2}\right.}{\partial x_2}),k=1,2. \qquad (4.7)$$

Now substituting (4.1) for $\frac{\partial u(x)}{\partial x_j}\left|_{x_3=\gamma_k(x')}\right.$ , $j,\ k=1,2$, into regular relationships (3.6) and grouping the terms we'll get a system:

$$\sum_{m=1}^{2}\beta_{im}(\xi')\frac{\partial u(\xi)}{\partial \xi_3}\left|_{\xi_3=\gamma_m(\xi')}\right. = B_i(\xi')\ ,\ i=1,\ 2, \tag{4.8}$$

where the RHSs of (4.8) are as follows

$$B_i(\xi')=-\sum_{j=1}^{2}\sum_{m=1}^{2}\beta_{ij}^{(m)}(\xi')\frac{\partial u(\xi',\gamma_m(\xi'))}{\partial \xi_j}$$

$$-\sum_{k=1}^{2}\int_S\frac{u(\zeta)\left|_{\zeta_3=\gamma_k(\zeta')}\right. d\zeta'}{\cos(\nu_\zeta,\zeta_3)}\int_S\alpha_i^{(k)}(x')\frac{\cos(\zeta-\xi,\nu_\zeta)\Big|_{\substack{\zeta_3=\gamma_k(\zeta')\\ x_3=\gamma_k(x')}}}{2\pi\left|x'-\zeta'\right|^2 P_k(x',\zeta')}\frac{dx'}{\cos(\nu_x,x_3)}$$

$$+\int_S\frac{\varphi_i(x')}{2\pi\left|x'-\xi'\right|^2}\frac{dx'}{\cos(\nu_x,x_3)}+....,\ \xi'\in S,\ \ i=1,\ 2.$$

In the virtue of remark 2.1 system (4.8), is a system of integral Fredholm equations of the second kind with respect to $\frac{\partial u(\xi)}{\partial \xi_3}\left|_{\xi_3=\gamma_k(\xi')}\right.$ , $k=1,2$. Consequently, the system (4.8) has the unique solution

$$\frac{\partial u(\xi)}{\partial \xi_3}\left|_{\xi_3=\gamma_k(\xi')}\right. = \Psi_k(u\left|_{\Gamma_1}\right.,\ u\left|_{\Gamma_2}\right.,\frac{\partial u\left|_{\Gamma_1}\right.}{\partial \xi_1},\ \frac{\partial u\left|_{\Gamma_2}\right.}{\partial \xi_1},\ \frac{\partial u\left|_{\Gamma_1}\right.}{\partial \xi_2},\ \frac{\partial u\left|_{\Gamma_2}\right.}{\partial \xi_2}), \tag{4.9}$$

as, evidently, $B_i(\xi')$, $j=1,2$, are linear functionals of $u(\xi',\gamma_k(\xi'))$, $\frac{\partial u(\xi',\gamma_k(\xi'))}{\partial \xi_j}$, $j=1,2$, $k=1,2$.

The functionals $\Phi_k,\ \Psi_k\ ,k=1,\ 2,$ from (4.7) and (4.9) are linear with respect to the unknown values $u\left|_{\Gamma_1}\right.$ , $u\left|_{\Gamma_2}\right.$ , $\frac{\partial u|_{\Gamma_1}}{\partial \xi_j}$, $\frac{\partial u|_{\Gamma_2}}{\partial \xi_j}$, $j=1,\ 2$:

$$\Phi_k=\sum_{i=1}^{2}a_i^{(k)}(\xi')u\left|_{\Gamma_i}\right.+\sum_{i,j=1}^{2}b_{ij}^{(k)}(\xi')\frac{\partial u\left|_{\Gamma_i}\right.}{\partial \xi_j}+\ \sum_{i=1}^{2}\int_S c_i^{(k)}(\zeta')u\left|_{\Gamma_i}\right. d\zeta$$

$$+\sum_{i,j=1}^{2}\int_S d_{ij}^{(k)}(\zeta')\frac{\partial u\left|_{\Gamma_i}\right.}{\partial \zeta_j}d\zeta+e_k(\xi'),\ k=1,\ 2, \tag{4.10}$$

$$\Psi_k\ =\sum_{i=1}^{2}a_i^{(l)}(\xi')u\left|_{\Gamma_i}\right.+\sum_{i,j=1}^{2}b_{ij}^{(l)}(\xi')\frac{\partial u\left|_{\Gamma_i}\right.}{\partial \xi_j}+\sum_{i=1}^{2}\int_S c_i^{(l)}(\zeta')u\left|_{\Gamma_i}\right. d\zeta$$

$$+\sum_{i,j=1}^{2}\int_S d_{ij}^{(l)}(\zeta')\frac{\partial u\left|_{\Gamma_i}\right.}{\partial \zeta_j}d\zeta+e_l(\xi'),\ \ l=3,4;\ \ k=1,\ 2. \tag{4.11}$$

Excluding $\frac{\partial u(\xi)}{\partial \xi_3}\left|_{\xi_3=\gamma_k(\xi')}\right.$ , $k=1,2$, from system (4.7), (4.9) and taking into account (4.10), (4.11) we'll obtain a system of linear integro-differential Fredholm equations of the second kind with respect to $u(\xi',\gamma_k(\xi'))$, $k=1,\ 2$:

$$\sum_{i=1}^{2}A_i^{(k)}(\xi')u\left|_{\Gamma_i}\right.+\sum_{i,j=1}^{2}B_{ij}^{(k)}(\xi')\frac{\partial u\left|_{\Gamma_i}\right.}{\partial \xi_j}+\sum_{i=1}^{2}\int_S C_i^{(k)}(\zeta')u\left|_{\Gamma_i}\right. d\zeta$$

$$+\sum_{i,j=1}^{2}\int_S D_{ij}^{(k)}(\zeta')\frac{\partial u\,|_{\Gamma_i}}{\partial\zeta_j}d\zeta+g_k(\xi')=0,\quad k=1,2, \tag{4.12}$$

where

$$A_i^{(k)}(\xi')=a_i^{(k)}(\xi')-a_i^{(k+2)}(\xi')\,,\ B_{ij}^{(k)}(\xi')=b_{ij}^{(k)}(\xi')-b_{ij}^{(k+2)}(\xi'),$$

$$C_i^{(k)}(\zeta')=c_i^{(k)}(\zeta')-c_i^{(k+2)}(\zeta'),D_{ij}^{(k)}(\zeta')=d_{ij}^{(k)}(\zeta')-d_{ij}^{(k+2)}(\zeta'),$$

$$g_k(\xi')=e_k(\xi')-e_{k+2}(\xi'),k=1,\ 2.$$

Thus, we have come to a two-dimensional system (4.12) of linear integro-differential equations of the first order with Dirichlet's conditions (2.3) on $\partial S=\bar{\Gamma}_1\bigcap\bar{\Gamma}_2$ . As the boundary $\partial S$ is one-dimensional then this Dirichlet's condition doesn't restrict the generality because its dimension is two units less than the dimension of the domain D.

Thus, we have established the following

**Theorem 4.1.** *If the assumptions of Theorem* 3.1 *and conditions* (4.6) *hold true and system* (4.8) *is uniquely resolved then boundary-value problem* (2.1)-(2.2) *is reduced to a two-dimensional system of linear integro-differential equations* (4.12) *with Dirichlet's condition* (2.3) *on the boundary* $\partial S=\overline{S}\backslash S$.

When solution $u\,|_{\Gamma_k}$ , $\frac{\partial u|_{\Gamma_k}}{\partial\xi_j}$, $k,\,j=1,\,2$, is obtained from system (4.12), then solution $\frac{\partial u(x)}{\partial x_3}\,\big|_{x_3=\gamma_k(x')}$ , $k=1,\,2$, is derived from (4.7), or (4.9). Then we find $\frac{\partial u(x)}{\partial x_1}\,\big|_{x_3=\gamma_k(x')}$ and $\frac{\partial u(x)}{\partial x_2}\,\big|_{x_3=\gamma_k(x')}$ from (4.1).

The solution to problem (2.1)-(2.3) is obtained from the $1^{st}$ and $2^{nd}$ basic relationships (2.5) and (2.7) as from a system of Fredholm equations for $u(\xi)$, $\frac{\partial u(\xi)}{\partial\xi_i}$, $\xi\in D$, $i=1,2,3$:

$$u(\xi)=-\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\Gamma_k}\left(\frac{\partial u(x)}{\partial x_j}U(x-\xi)-u(x)\frac{\partial U(x-\xi)}{\partial x_j}\right)\cos(\nu_x,x_j)dx$$

$$-\sum_{m=1}^{3}\int_D a_m(x)\frac{\partial u(x)}{\partial x_m}U(x-\xi)dx-\int_D a(x)u(x)U(x-\xi)dx,$$

$$\frac{\partial u(\xi)}{\partial\xi_i}=\int_\Gamma\frac{\partial u(x)}{\partial x_m}\left[\frac{\partial U(x-\xi)}{\partial x_m}\cos(\nu_x,x_i)-\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x,x_m)\right]dx$$

$$+\int_\Gamma\frac{\partial u(x)}{\partial x_l}\left[\frac{\partial U(x-\xi)}{\partial x_l}\cos(\nu_x,x_i)-\frac{\partial U(x-\xi)}{\partial x_i}\cos(\nu_x,x_l)\right]dx$$

$$-\int_\Gamma\frac{\partial u(x)}{\partial x_i}\frac{\partial U(x-\xi)}{\partial\nu_x}dx-\int_D\sum_{k=1}^{3}a_k(x)\frac{\partial u(x)}{\partial x_k}\frac{\partial U(x-\xi)}{\partial x_i}dx$$

$$-\int_D a(x)u(x)\frac{\partial U(x-\xi)}{\partial x_i}dx,\ i=1,2,3.$$

Finally, there has been established

**Theorem 4.2.** *If the assumptions of Theorem* 4.1 *hold true then boundary value problem* (2.1), (2.2), (2.3) *has Fredholm property.*

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