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Existence and nonexistence of global solutions to the Cauchy problem for systems of semilinear Klein - Gordon equations

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Abstract. *In this paper, we study a Cauchy problem for systems of semilinear Klein-Gordon equations. Existence and absence of global solutions are proved.*

Keywords. Systems of semi-linear Klein-Gordon equations, global solutions, existence and absence, Cauchy problem

Mathematics Subject Classification (2010): 35C15, 35M10, 35M33, 35B30

1 Introduction

Among nonlinear hyperbolic equations, the Klein-Gordon equation has significant theoretical and practical importance. The nonlinear Klein-Gordon equation arises in the study of various problems in mathematical physics. For example, this equation appears in general relativity, nonlinear optics, plasma physics, fluid mechanics, radiation theory, and other fields [1–3].

Cauchy problem for the wave equation

$$u_{tt} - \Delta u + mu + u_t = f(u), \quad t > 0, \;\; x \in R^n, \tag{1.1}$$

with initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in R^n \tag{1.2}$$

has been studied by various authors and the results obtained in this area are presented in several monographs (see, for example, [4]). In particular, when $m = 0, \quad f(u) \sim |u|^p$ the existence or absence of global solutions to problem (1.1), (1.2) has been studied by various authors (see, for example, [5,6]). In [7,8], problem (1.1), (1.2) was studied in the case when $m = 0, \quad f(u) \sim |u|^p$, $1 < p \le p_c = 1 + \frac{2}{n}$ and it was proven that regardless of how smooth and small the initial data are (u_0, u_1), there is no global solution to the corresponding Cauchy problem. In [8,9], the Klein–Gordon equation was studied for $m = 0, \quad f(u) \sim |u|^p$, $p > p_c = 1 + \frac{2}{n}$. And it was proved that for sufficiently small x initial data (u_0, u_1), there is a global solution to problem (1.1), (1.2).

When $m > 0$ i.e. when there is mass in the system, posing the question this way loses significance. The main question in this case, is to determine the stability of the potential around standing waves [9–12].

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In [12], problem (1.1), (1.2) was studied for

$$p > 1, \quad n = 2$$

and

$$1 < p < \frac{n+2}{n-2}, \quad n \geq 3.$$

In [12], by studying a family of potential wells, preliminary results were obtained about the absence of a global solution to the corresponding Cauchy problem. An exponentially decline in energy standarts has also been identified, consistent with global decisions. Recently, there has been a growing interest in studying the Cauchy problem for the systems of Klein - Gordon equations (see [13–16,23]).

In this article we study problems of this type for m the Klein - Gordon equations with weak nonlinear dissipation and with different masses.

It should be noted that the term weak nonlinear connection refers to a case where the interaction between the desired functions is determined by the right-hand side of the equation, given as a product of functions.

In the domain $[0, \infty) \times R^n$ for the system

$$u_{itt} - \Delta u_i + q_i u_i + \gamma_i u_{it} = \lambda_i \prod_{\substack{j=1 \\ j \neq i}}^{m} |u_j|^{p_j+1} |u_i|^{p_i-1} u_i, \ i = 1, ..., m \tag{1.3}$$

we consider the Cauchy problem with initial conditions

$$u_i(0, x) = u_{i0}(x), \quad u_{it}(0, x) = u_{i1}(x), \quad x \in R^n, \quad i = 1, ..., m. \tag{1.4}$$

Here are $u_1, ..., u_m$ functions depending on real variables $t \in R_+, x \in R^n$,

$$p_j \geq 0, \quad n = 2 \tag{1.5}$$

and

$$\sum_{j=1}^{m} p_j \leq 1, \ n \geq 3. \tag{1.6}$$

In this work, exploring potential wells, we study the problems of the existence and nonexistance of global solutions.

As in [13], we will study the qualitative characteristics of the family of potential wells, the existence and nonexistance of global solutions, the problem of unstable standing waves, and the behavior of the energy norms of the solution at large time values. Similar problems for Klein-Gordon systems consisting of two equations were studied in [15], and for systems of m equations in the case $p_j = p, \quad j = 1, ..., m$ in [22].

In what follows $|\cdot|_q$ we denote the norm in $L_q(R^n)$. For simplicity of notation, in particular, $|\cdot|_q$ we will denote by $| \, . \, |$. The product in $L_2(R^n)$ denote by $\langle ., . \rangle$. Next, we denote the norm in Sobolev space $H^1 = W_2^1(R^n)$ as follows $\|u\| = \left[\left| |\nabla u|^2 + |u|^2 \right|\right]^{\frac{1}{2}}$, where ∇ gradient. Constants C and c, used in this article, are positive general constants that may be different in different cases.

For simplicity, let's take $q_j = \gamma_i = \lambda_i = 1, \ j = 1, ..., m.$

2 Study of a potential well

Consider the following system of equations

$$-\Delta u_i + u_i = \prod_{\substack{j=1 \\ j \neq i}}^{m} |u_j|^{p_j+1} |u_i|^{p_i-1} u_i, \ i = 1, ..., m. \tag{2.1}$$

Let us denote by $(\bar{\varphi}_1, ..., \bar{\varphi}_m)$ solution of system (1.1). Then it is clear that $(u_1(t,x), ..., u_m(t,x)) = (\bar{\varphi}_1, ..., \bar{\varphi}_m)$ the solution to system (1.3) satisfying the initial conditions $u_1(0,x) = \bar{\varphi}_1(x), ..., u_m(0,x) = \bar{\varphi}_m(x), \ x \in R^n$. In this case $(\bar{\varphi}_1, ..., \bar{\varphi}_m)$ called standing solution of problem (1.3), (1.4).

We define the following functionals:

$$J(\varphi_1, ..., \varphi_m) = \sum_{j=1}^{m} \frac{p_j+1}{2} \|\varphi_j\|^2 - G,$$

$$I(\varphi_1, ..., \varphi_m) = \sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m} \|\varphi_j\|^2 - G.$$

Here $G = G(\varphi_1, ..., \varphi_m) = \int_{R^n} \prod_{j=1}^{m} |\varphi_j(x)|^{p_j+1} dx$.

Lemma 2.1 *Let's* $(\varphi_1, ..., \varphi_m) \in H^1 \times ... \times H^1 \backslash \{(0, ..., 0)\}$, *then*

(i) $\lim\limits_{\lambda \to 0} J(\lambda\varphi_1, ..., \lambda\varphi_m) = 0, \quad \lim\limits_{\lambda \to +\infty} J(\lambda\varphi_1, ..., \lambda\varphi_m) = -\infty;$

(ii) There is a point $\lambda^* = \lambda^*(\varphi_1, ..., \varphi_m)$ *in the interval such* $0 < \lambda < +\infty$ *that*

$$\frac{d}{d\lambda} J(\lambda\varphi_1, ..., \lambda\varphi_m)|_{\lambda=\lambda^*} = 0;$$

(iii) $J(\lambda\varphi_1, ..., \lambda\varphi_m)$ *in the interval* $0 \leq \lambda \leq \lambda^*$ *it does not decrease, in the interval* $\lambda^* \leq \lambda < +\infty$ *it does not increase, and at the point* $\lambda = \lambda^* = \left[\frac{\sum_{\mu=1}^{m} p_\mu + m}{\sum_{j=1}^{m} (p_j+1)\|\varphi_j\|^2}\right]^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m - 2}}$ *it takes on the maximum value;*

(iv) $I(\lambda\varphi_1, ..., \lambda\varphi_m) > 0$ *in the interval* $0 < \lambda < \lambda^*$, *but* $I(\lambda\varphi_1, ..., \lambda\varphi_m) < 0$ *and* $I(\lambda^*\varphi_1, ..., \lambda^*\varphi_m) = 0$ *in the interval* $\lambda^* < \lambda < +\infty$, *and*

$$I(\lambda^*\varphi_1, ..., \lambda^*\varphi_m) = \sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m} \|\lambda^*\varphi_j\|^2 - \int_{R^n} \prod_{j=1}^{m} |\lambda^*\varphi_j(x)|^{p_j+1} dx.$$

Allow us denote by N the following set

$$N = \left\{(\varphi_1, ..., \varphi_m) : (\varphi_1, ..., \varphi_m) \in H^1 \times ... \times H^1 \backslash \{(0, ..., 0)\}, I(\varphi_1, ..., \varphi_m) = 0\right\}.$$

Let's $(\varphi_1, ..., \varphi_m) \in N$ then

$$J(\varphi_1, ..., \varphi_m) = \left(1 - \frac{2}{\sum_{\mu=1}^{m} p_\mu + m}\right) \sum_{j=1}^{m} \frac{p_j+1}{2} \|\varphi_j\|^2 > 0$$

a J function bounded below.

Consider the following variation problem

$$d = \inf_{(\varphi_1, ..., \varphi_m) \in N} J(\varphi_1, ..., \varphi_m).$$

Lemma 2.2 *There is* $(\bar{\varphi}_1, ..., \bar{\varphi}_m) \in N$ *such a thing that*

1 $J(\bar{\varphi}_1, ..., \bar{\varphi}_m) = \inf\limits_{(\varphi_1,...,\varphi_m)\in N} J(\varphi_1, ..., \varphi_m) = d > 0;$

2 $(\bar{\varphi}_1, ..., \bar{\varphi}_m)$ *there is a standing solution to problem* (1.3), (1.4).

For $\delta > 0$ let's define the following functionality:

$$I_\delta(\varphi_1, ..., \varphi_m) = \delta \sum_{j=1}^{m} \frac{p_j + 1}{\sum_{\mu=1}^{m} p_\mu + m} \|\varphi_j\|^2 - \int_{R^n} \prod_{j=1}^{m} |\varphi_j(x)|^{p_j+1}\, dx. \tag{2.2}$$

Through $r(\delta)$ let's denote the following

$$r(\delta) = r(\delta, p_1, ..., p_m) = \left(\frac{\delta}{C^{\sum_{\mu=1}^{m} p_\mu + m}} \right)^{\frac{2}{\sum_{\mu=1}^{m} p_\mu + m - 2}}. \tag{2.3}$$

Here $C = \sup\limits_{\|u\|\neq 0} \frac{|u|_{L_{\sum_{\mu=1}^{m} p_\mu + m}(R^n)}}{\|u\|}$.

Lemma 2.3 *Let* $(u_1, ..., u_m) \in H^1 \times ... \times H^1 \backslash \{(0, ..., 0)\}$ *it be* $I_\delta(u_1, ..., u_m) > 0$, *then*

$$\sum_{j=1}^{m} \frac{p_j + 1}{\sum_{\mu=1}^{m} p_\mu + m} \|u_j\|^2 < r(\delta).$$

Lemma 2.4 *Let* $(u_1, ..., u_m) \in H^1 \times ... \times H^1$ *then* $I_\delta(u_1, ..., u_m) < 0$,

$$\sum_{j=1}^{m} \frac{p_j + 1}{\sum_{\mu=1}^{m} p_\mu + m} \|u_j\|^2 > r(\delta).$$

Lemma 2.5 *Let* $(u_1, ..., u_m) \in H^1 \times ... \times H^1 \backslash \{(0, ..., 0)\}$ *then* $I_\delta(u_1, ..., u_m) = 0$,

$$\sum_{j=1}^{m} \frac{p_j + 1}{\sum_{\mu=1}^{m} p_\mu + m} \|u_j\|^2 \geq r(\delta). \tag{2.4}$$

Lemma 2.6 *Let conditions* (1.5), (1.6) *be satisfied. Then*

$$d\,(\delta) \geq a\,(\delta)\; r\,(\delta)\,.$$

Here

$$d(\delta) = \delta^{\frac{2}{\sum_{\mu=1}^{m} p_\mu + m - 2}} \frac{\sum_{\mu=1}^{m} p_\mu + m - 2\delta}{\sum_{\mu=1}^{m} p_\mu + m - 2} d, \tag{2.5}$$

$$a(\delta) = \frac{\sum_{\mu=1}^{m} p_\mu + m}{2} \delta. \tag{2.6}$$

It's clear that

$$\lim_{\delta \to +0} d\,(\delta) = 0, \tag{2.7}$$

$$d \left(\frac{\sum_{\mu=1}^{m} p_\mu + m}{2} \right) = 0, \tag{2.8}$$

$$d\,(1) = d, \tag{2.9}$$

$$d'(\delta) > 0, \delta \in (0,1), \tag{2.10}$$

$$d'(\delta) < 0, \delta \in \left(1, \frac{\sum_{\mu=1}^{m} p_\mu + m}{2}\right). \tag{2.11}$$

Let us denote by $E(t)$ the following energy functional:

$$E(t) = \sum_{j=1}^{m} \frac{p_j+1}{2} \left[\left| u'_{jt}(t,\cdot) \right|^2 + \| u_j(t,\cdot) \|^2 \right] - \int_{R^n} \prod_{j=1}^{m} |u_j(t,x)|^{p_j+1} \, dx.$$

We should introduce the following sets:

$$W_\delta = \left\{ (u_1,...,u_m) \in H^1 \times ... \times H^1 : I_\delta(u_1,...,u_m) > 0, J(u_1,...,u_m) < \right.$$

$$\left. < d(\delta) \right\} \bigcup \{(0,...,0)\}, \ 0 < \delta < r_0;$$

$$V_\delta = \left\{ (u_1,...,u_m) \in H^1 \times ... \times H^1 : I_\delta(u_1,...,u_m) < 0, J(u_1,...,u_m) < d(\delta) \right\}, \ 0 < \delta < r_0.$$

From (2.9) and (2.10) we have that for anyone $e \in (0,d)$ equation $d(\delta) = e$ has two different δ_1, δ_2 roots such that $\delta_1 < 1 < \delta_2$.

Theorem 2.1 *Let* $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1$,$(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$, *conditions* (1.5), (1.6) *be satisfied, and for* $0 < e < d$, $\delta_1 < \delta_2$ *are the roots of the equation* $d(\delta) = e$, *then the following statements are true:*

a) If $I(u_{10},...,u_{m0}) > 0$ *or* $\|u_{i0}\| \neq 0, \ i = 1,...,m$, *then the solution to problem* (1.3), (1.4) *with the initial energy* $0 < E(0) \leq e$ *included in the set* W_δ, *those* $(u_1(t,\cdot),...,u_m(t,\cdot)) \in W_\delta$.

b) If $I(u_{10},...,u_{m0}) < 0$, *then the solution to problem* (1.3), (1.4) *with the initial energy* $0 < E(0) \leq e$ *for any* $\delta_1 < \delta < \delta_2$ *included in the set* V_δ, *i.e.* $(u_1(t,\cdot),...,u_m(t,\cdot)) \in V_\delta$.

Proof. a) Let $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1$, $(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$ and

$$0 < E(0) \leq e. \tag{2.12}$$

Assume that

$$I(u_{10},...,u_{m0}) > 0 \ \ or \ \ \|u_{i0}\| \neq 0, \ i = 1,...,m. \tag{2.13}$$

From (1.3), (1.4) we have

$$E(t) + \sum_{j=1}^{m} \frac{p_j+1}{2} \int_0^t |u_j(s,\cdot)|^2 \, ds = E(0). \tag{2.14}$$

Taking into account (2.13) and (2.14) we have $J(u_1(t,\cdot),...,u_m(t,\cdot)) < e$. On the other hand for $\delta_1 < \delta < \delta_2$ we have $e < d(\delta)$. That's why

$$J(u_1(t,\cdot),...,u_m(t,\cdot)) < d(\delta). \tag{2.15}$$

Let's a) is not fulfilled. Then taking into account (2.13) and (2.14) there is such $\bar{t} \in (0,\infty)$, what

$$I_\delta(u_1(t,\cdot),...,u_m(t,\cdot)) > 0, t \in (0,\bar{t}), \tag{2.16}$$

$$I_\delta(u_1(\bar{t},\cdot),...,u_m(\bar{t},\cdot)) = 0. \tag{2.17}$$

Thus, $(u_1(\bar{t},\cdot),...,u_m(\bar{t},\cdot)) \in N_\delta$, therefore, based on the definition $d(\delta)$ we have

$$d(\delta) \le J(u_1(\bar{t},\cdot),...,u_m(\bar{t},\cdot))$$

and this contradicts (2.4).

Now prove the statement b).

Let $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1$, $(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$, $0 < E\,(0) \le e$ and $I(u_{10},...,u_{m0}) < 0$. Similar to the proof of statement a) there is such $\bar{t} \in [0,T]$ that for anyone $t \in \left[0,\bar{t}\right)$ the inequalities are satisfied $I(u_1(t,\cdot),...,u_m(t,\cdot)) < 0$ and $I(u_1(\bar{t},\cdot),...,u_m(\bar{t},\cdot)) = 0$.

We again have the following contradiction

$$d(\delta) \le J(u_1(\bar{t},\cdot),...,u_m(\bar{t},\cdot)) \le e < d(\delta).$$

From Theorem 2.1 we have the following statement.

Theorem 2.2 *Let* $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1$, $(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$ *and conditions* (1.5), (1.6) *are satisfied. If* $0 < E\,(0) \le e$, *a each* δ_1, δ_2 *there are solutions to the equation* $d(\delta) = e$, *then* $W_{\delta_1\delta_2} = \bigcup\limits_{\delta_1<\delta<\delta_2} W_\delta$ *and* $V_{\delta_1\delta_2} = \bigcup\limits_{\delta_1<\delta<\delta_2} V_\delta$ *are invariant along the entire trajectory with respect to the solution of problem* (1.3), (1.4).

From Theorem 2.2 we have the following result showing that between two invariant sets there is a vacuum zone (empty area).

Theorem 2.3 *Let the conditions of Theorem 2.2 be satisfied, then each solution to problem* (1.3), (1.4) *satisfies the condition* $(u_1(t,\cdot),...,u_m(t,\cdot)) \notin N_{\delta_1,\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} N_\delta$.

Now consider the cases $E(0) \le 0$.

Theorem 2.4 *Let* $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1$, $(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$ *and conditions* (1.5), (1.6) *are satisfied.*

If $E\,(0) = 0$, $\|u_{i0}\| \ne 0,\ i = 1,...,m$, *then the solution to problem* (1.3), (1.4) *satisfies the inequality*

$$\sum_{j=1}^{m} \frac{p_j+1}{2} \|u_j(t,\cdot)\|^2 \ge r_0. \tag{2.18}$$

Here

$$r_0 = \left(\frac{\sum\limits_{\mu=1}^{m} p_\mu + m}{2C^2} \right)^{\frac{\sum\limits_{\mu=1}^{m} p_\mu + m}{\sum\limits_{\mu=1}^{m} p_\mu + m - 2}}.$$

Proof. Let $(u_1(t,\cdot),...,u_m(t,\cdot))$ solution to problem (1.3), (1.4) satisfying $E\,(0) = 0$ the condition so that $\|u_{i0}\| \ne 0,\ i = 1,...,m$.

Let's assume that $T_{\max}$ is the length of the solution existence interval $(u_1(t,\cdot),...,u_m(t,\cdot))$. Using purpose $E\,(t)$ we have

$$E(t) = \sum_{j=1}^{m} \frac{p_j+1}{2} \left|u'_{jt}(t,\cdot)\right|^2 + J(u_1(t,\cdot),...,u_m(t,\cdot)) = 0. \tag{2.19}$$

From here we get

$$J(u_1(t,\cdot),...,u_m(t,\cdot)) \le 0 < d(\delta), \quad t \in [0, T_{\max}) \tag{2.20}$$

and

$$\sum_{j=1}^{m} \frac{p_j+1}{2} \left|u'_{jt}(t,\cdot)\right|^2 \le \int_{R^n} \prod_{j=1}^{m} |u_j(t,x)|^{p_j+1}\, dx.$$

On the other hand, taking into account Hölder's inequality we have

$$G = \int_{R^n} \prod_{j=1}^{m} |u_j(t,x)|^{p_j+1}\, dx \le$$

$$\le \Big(\int\limits_{R^N} |u_1(t,x)|^{\sum\limits_{\mu=1}^{m} p_\mu + m} dx\Big)^{\frac{p_1+1}{\sum\limits_{\mu=1}^{m} p_\mu + m}} \times ... \times \Big(\int\limits_{R^N} |u_m(t,x)|^{\sum\limits_{\mu=1}^{m} p_\mu + m} dx\Big)^{\frac{p_m+1}{\sum\limits_{\mu=1}^{m} p_\mu + m}}. \tag{2.21}$$

Using the embedding theorem we obtain

$$\sum_{j=1}^{m} \frac{p_j+1}{2} \left|u'_{jt}(t,\cdot)\right|^2$$

$$\le C^{\sum_{\mu=1}^{m} p_\mu + m} \left(\frac{2}{\sum_{\mu=1}^{m} p_\mu + m}\right)^{\frac{\sum_{\mu=1}^{m} p_\mu + m}{2}}$$

$$\times \left(\sum_{j=1}^{m} \frac{p_j+1}{2} \left|u'_{jt}(t,\cdot)\right|^2\right)^{\frac{\sum_{\mu=1}^{m} p_\mu + m}{2}}. \tag{2.22}$$

If $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1$, $(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$ and $\|u_{i0}\| \ne 0,\ i = 1,...,m$, then there is such a half-interval $[0, t_1)$, what is in this half-interval $\|u_i(t,\cdot)\| \ne 0,\ i = 1,...,m$. Taking this fact into account, from (2.9) we obtain the following inequality

$$\sum_{j=1}^{m} \frac{p_j+1}{2} \|u_j(t,\cdot)\|^2 \ge \left(\frac{\sum_{\mu=1}^{m} p_\mu + m}{2C^2}\right)^{\frac{\sum_{\mu=1}^{m} p_\mu + m}{\sum_{\mu=1}^{m} p_\mu + m - 2}} = r_0, \quad t \in [0, t_1)\,. \tag{2.23}$$

From here we have $\|u_i(t,\cdot)\| \ne 0,\ i = 1,...,m$, therefore (2.21) is true in the half-interval $[t_1, 2t_1)$ etc.

Thus, (2.17) is satisfied for half - interval $[0, T_{\max})$.

Theorem 2.5 *Let* $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1 \setminus \{0,...,0\}$, $(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$ *and conditions* (1.5), (1.6) *are satisfied. If* $E(0) < 0$ *or* $E(0) = 0$ *and* $(u_{10},...,u_{m0}) \ne (0,...,0)$, *then for anyone* $t \in [0, T_{\max})$ *and* $0 < \delta < \frac{\sum_{\mu=1}^{m} p_\mu + m}{2}$ *we have* $(u_1(t,\cdot),...,u_m(t,\cdot)) \in V_\delta$.

Proof. If $E(0) < 0$ from (2.9) we have

$$J(u_1(t,\cdot),...,u_m(t,\cdot)) \leq E(0) < 0 < d(\delta). \tag{2.24}$$

On the other side

$$J(u_1(t,\cdot),...,u_m(t,\cdot)) = \frac{\sum_{\mu=1}^m p_\mu + m - 2\delta}{\sum_{\mu=1}^m p_\mu + m} \sum_{j=1}^m \frac{p_j+1}{2} \|u_j(t,\cdot)\|^2 + I_\delta(u_1(t,\cdot),...,u_m(t,\cdot)).$$

Therefore, if $0 < \delta < \frac{\sum_{\mu=1}^m p_\mu + m}{2}$, then

$$I_\delta(u_1(t,\cdot),...,u_m(t,\cdot)) < 0, \quad t \in [0, T_{\max}). \tag{2.25}$$

If $E(0) = 0$, then taking into account Theorem 2.4 from (2.22), (2.23) we have that (2.24) is also true for the values δ such that $0 < \delta < \frac{\sum_{\mu=1}^m p_\mu + m}{2}$.

Thus, if $0 < \delta < \frac{\sum_{\mu=1}^m p_\mu + m}{2}$, then $I_\delta(u_1(t,\cdot),...,u_m(t,\cdot)) \in V_\delta$.
From Theorems 2.3-2.5 we obtain the following result.

Theorem 2.6 *Let $E(0) < d$ then and W_1 and V_1 invariant with respect to the dynamic system generated by the problem* (1.3)*,* (1.4)*.*

3 Existence and asymptotic behavior of the global solution

From Theorem 2.6 we obtain the following theorem on the global solvability of solutions.

Theorem 3.1 *Let $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1$, $(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$, $E(0) < d$ conditions* (1.5) *and* (1.6) *be satisfied.*

If at any time $t_0 \in [0, T_{\max})$ $(u_1(t_0,\cdot),...,u_m(t_0,\cdot)) \in W_1$, then $T_{\max} = +\infty$ and $(u_1(t,\cdot),...,u_m(t,\cdot))$ the following a priori estimate is true:

$$\sum_{j=1}^m (p_j+1)\left[\left|u'_{jt}(t,\cdot)\right|^2 + \|u_j(t,\cdot)\|^2\right] \leq \frac{2d(\sum_{\mu=1}^m p_\mu + m)}{\sum_{\mu=1}^m p_\mu + m - 2}, \quad t \in [0, T_{\max}). \tag{3.1}$$

Proof. From Theorem 2.6 we have $(u_1(t,\cdot),...,u_m(t,\cdot)) \in W_1, t \in [0, T_{\max})$. Therefore, $I(u_1(t,\cdot),...,u_m(t,\cdot)) > 0, \quad 0 < t < T_{\max}$.

Thus, from (2.18) we obtain that $0 \leq t < T_{\max}$ the a priori estimate (3.1) is correct in the domain. Therefore $T_{\max} = +\infty$, i.e. problem (1.3), (1.4) has a global solution.

From Theorem 3.1 we obtain the following statement.

Theorem 3.2 *Let*

a) $(u_{10},...,u_{m0}) \in H^1 \times ... \times H^1$, $(u_{11},...,u_{m1}) \in L_2(R^n) \times ... \times L_2(R^n)$;
b) conditions (1.5), (1.6) *are satisfied;*
c) $0 < E(0) < d$;
d) $I_{\delta_2}(u_{10},...,u_{m0}) > 0$ *or* $\|u_{i0}\| = 0,\ i = 1,...,m$.

Then problem (1.3)*,* (1.4) *has a unique solution*

$$(u_1(\cdot),...,u_m(\cdot)) \in C\left([0,\infty); H^1 \times ... \times H^1\right) \bigcap C^1\left([0,\infty); L_2(R^n) \times ... \times L_2(R^n)\right),$$

such that $(u_1(t,\cdot),...,u_m(t,\cdot)) \in W_\delta$, $\delta_1 < \delta < \delta_2, \quad 0 \leq t < +\infty$. Here $\delta_1 < \delta_2$ roots of the equation $d(\delta) = E(0)$.

Proof. From the conditions of the theorem it is clear $I(u_{10},...,u_{m0})>0$. That indeed, in otherwise there is $I_{\bar{\delta}}(u_{10},...,u_{m0})=0$, such as $\bar{\delta}\in[1,\delta_2)$. In that case $J(u_{10},...,u_{m0})\leq d(\delta)$. This is for $\delta_1<\delta<\delta_2$ contradicts the condition $J(u_{10},...,u_{m0})\leq E(0)<d(\delta)$.

If $(u_{10},...,u_{m0})\in H^2\times...\times H^2$, $(u_{11},...,u_{m1})\in H^1\times...\times H^1$, then for solutions $(u_1(t,x),...,u_m(t,x))$ of problem (1.3), (1.4) we have

$$\begin{gathered}I(u_1,...,u_m)=(\textstyle\sum_{\mu=1}^m p_\mu+m)^{-1}\left\{\sum_{j=1}^m(p_j+1)\left|u'_{jt}(t,\cdot)\right|^2-\right.\\ \left.-\frac{d}{dt}\textstyle\sum_{j=1}^m(p_j+1)\left[\left\langle u_j(t,\cdot),u'_{jt}(t,\cdot)\right\rangle+\frac{1}{2}\left|u_j(t,\cdot)\right|^2\right]\right\}\end{gathered}\tag{3.2}$$

and the following inequality is true

$$I(u_1,...,u_m)>(1-\delta_1)\sum_{j=1}^m\frac{p_j+1}{\sum_{\mu=1}^m p_\mu+m}\left\|u_j(t,\cdot)\right\|^2.\tag{3.3}$$

Here δ_1 smallest root of equation $d\left(\delta\right)=E\left(0\right)$.

From Theorem 3.2 it follows

Theorem 3.3 *Let*

a) $(u_{10},...,u_{m0})\in H^1\times...\times H^1$, $(u_{11},...,u_{m1})\in L_2(R^n)\times...\times L_2(R^n)$;
b) $0<E\left(0\right)<d$;
c) $I(u_{10},...,u_{m0})>0$ *or* $\|u_{i0}\|=0,\ i=1,...,m$;
d) conditions (1.5)*,* (1.6) *are satisfied.*

Then there are such $K>0$ and $k>0$ that

$$E\left(t\right)\leq Ke^{-kt},t\geq0.$$

4 Existence of a global solution and instability of stagnant waves

Let us prove the following theorems on the absence of global solutions.

Theorem 4.1 *Let $s>\frac{n}{2}$, $(u_{10},...,u_{m0})\in H^s\times...\times H^s$, conditions* (1.5)*,* (1.6) *are satisfied and one of the following conditions is satisfied:*

a) $E(0)<0$;
b) $0\leq E(0)<d$, $I(u_{10},...,u_{m0})<0$ *and* $0\leq\gamma<\lambda_1\sum_{\mu=1}^m p_\mu$.

Here $\lambda_1=\frac{1}{c_0}$, a c_0 is the norm of the embedding operator $\mathrm{W}_2^1(R^n)\subset L_2(R^n)$*.*
Then

$$T_{\max}<+\infty\ \textit{and}\ \lim_{t\to T_{\max}}\sum_{j=1}^m\left\|u_j(t,\cdot)\right\|^2=+\infty.$$

Proof.

a) If $E(0)<0$, then, similar to the proof from [22], we can obtain the required result.
b) Let $0\leq E(0)<d$, $I(u_{10},...,u_{m0})<0$ and $0\leq\gamma^2<\lambda_1(\sum_{\mu=1}^m p_\mu+m-2)$, $\lambda_1=\frac{1}{c_0}$.

Let's denote

$$F(t)=\sum_{j=1}^{m}(p_j+1)\left|u_j(t,\cdot)\right|^2, \quad t\in[0,T_{\max}),$$

we get

$$F'(t)=2\sum_{j=1}^{m}(p_j+1)\left\langle u_j(t,\cdot),u'_{jt}(t,\cdot)\right\rangle, \quad t\in[0,T_{\max}). \tag{4.1}$$

Let us assume that the statement of Theorem 4.1 is false, i.e. $T_{\max}=+\infty$.

If $(u_{10},...,u_{m0})\in H^s\times...\times H^s$ and $(u_{11},...,u_{m1})\in H^{s-1}\times...\times H^{s-1}$, $s>\frac{N}{2}$, that $(u_1(t,x),...,u_m(t,x))\in C\left([0,\infty);H^s\times...\times H^s\right)\bigcap C^1\left([0,\infty);H^{s-1}\times...\times H^{s-1}\right)$, and it's clear that $F''(t)\in C\,[0,\infty)$.

Using simple operations, taking into account (1.3), we obtain

$$\frac{d}{dt}\left[e^{\gamma t}F'(t)\right]$$

$$=2\gamma e^{\gamma t}\sum_{j=1}^{m}(p_j+1)\left\langle u_j(t,\cdot),u'_{jt}(t,\cdot)\right\rangle+2e^{\gamma t}\sum_{j=1}^{m}(p_j+1)\left[\left|u'_{jt}(t,\cdot)\right|^2-\left\|u_j(t,\cdot)\right\|^2\right.$$

$$\left.-\gamma\left\langle u_j(t,\cdot),u'_{jt}(t,\cdot)\right\rangle\right]+2(\sum_{\mu=1}^{m}p_\mu+m)e^{\gamma t}\int_{R^n}\prod_{j=1}^{m}\left|u_j(t,x)\right|^{p_j+1}dx$$

$$=2e^{\gamma t}\sum_{j=1}^{m}(p_j+1)\left|u'_{jt}(t,\cdot)\right|^2+2(\delta-1)e^{\gamma t}\sum_{j=1}^{m}(p_j+1)\left\|u_j(t,\cdot)\right\|^2$$

$$-2e^{\gamma t}I_\delta(u_1(t,\cdot),...,u_m(t,\cdot)). \tag{4.2}$$

From $E(0)<d$ we have these δ_1,δ_2, $\delta_1<1<\delta_2$ as

$$d(\delta_{\mathrm{i}})=E(0)\,,\ \ i=1,2.$$

In (3.2) we take $\delta=\delta_2$. Based on Theorem 2.5

$$I_{\delta_2}(u_1(t,\cdot),...,u_m(t,\cdot))\le 0. \tag{4.3}$$

Therefore, from (4.1), (4.2) we have

$$\frac{d}{dt}\left[e^{\gamma t}F'(t)\right]\ge 2(\delta_2-1)e^{\gamma t}\sum_{j=1}^{m}(p_j+1)\left\|u_j(t,\cdot)\right\|^2. \tag{4.4}$$

On the other hand, applying Lemma 2.4 we get

$$\sum_{j=1}^{m}\frac{p_j+1}{\sum_{\mu=1}^{m}p_\mu+m}\left\|u_j(t,\cdot)\right\|^2>r(\delta_2). \tag{4.5}$$

From (4.4) and (4.5) it follows

$$\frac{d}{dt}\left[e^{\gamma t}F'(t)\right]\ge e^{\gamma t}c(\delta_2). \tag{4.6}$$

Here

$$c(\delta_2) = 2(\delta_2 - 1)r(\delta_2)(\sum_{\mu=1}^{m} p_\mu + m).$$

From (4.6) for sufficiently large t_0 we have

$$F'(t) \geq \frac{C(\delta_2)}{2\lambda}, \quad t \geq t_0. \tag{4.7}$$

Thus $\lim\limits_{t\to+\infty} F(t) = +\infty$.

On the other side

$$F''(t) = 2\sum_{j=1}^{m}(p_j+1)\left[\left|u'_{jt}(t,\cdot)\right|^2 - \|u_j(t,\cdot)\|^2\right]$$

$$-2\gamma\sum_{j=1}^{m}(p_j+1)\left\langle u_j(t,\cdot), u'_{jt}(t,\cdot)\right\rangle$$

$$+2(\sum_{\mu=1}^{m} p_\mu + m)\int_{R^n}\prod_{j=1}^{m}|u_j(t,x)|^{p_j+1}\,dx$$

$$= (\sum_{\mu=1}^{m} p_\mu + m + 2)\sum_{j=1}^{m}(p_j+1)\left|u'_{jt}(t,x)\right|^2$$

$$+(\sum_{\mu=1}^{m} p_\mu + m - 2)\sum_{j=1}^{m}(p_j+1)\|u_j(t,\cdot)\|^2$$

$$-2\gamma\sum_{j=1}^{m}(p_j+1)\left\langle u_j(t,\cdot), u'_{jt}(t,\cdot)\right\rangle + (\sum_{\mu=1}^{m} p_\mu + m)\sum_{j=1}^{m}\int_0^t\left|u'_{jt}(s,\cdot)\right|^2 ds$$

$$-2(\sum_{\mu=1}^{m} p_\mu + m)E(0) \geq (4+\varepsilon)\sum_{j=1}^{m}(p_j+1)\left[\left|u'_{jt}(t,\cdot)\right|^2 + \psi(t)\right], \tag{4.8}$$

where

$$\psi(t) = (\sum_{\mu=1}^{m} p_\mu + m - 2 - \varepsilon)\sum_{j=1}^{m}(p_j+1)\left|u'_{jt}(t,x)\right|^2$$

$$+\lambda_1(\sum_{\mu=1}^{m} p_\mu + m - 2)\sum_{j=1}^{m}(p_j+1)\left|u'_{jt}(t,x)\right|^2$$

$$-2\gamma\sum_{j=1}^{m}(p_j+1)\left\langle u_j(t,\cdot), u'_{jt}(t,\cdot)\right\rangle - 2\sum_{\mu=1}^{m}(p_\mu+m)\,E(0). \tag{4.9}$$

Using the Hölder and Young inequality we have

$$\left|2\gamma\sum_{j=1}^{m}(p_j+1)\left\langle u_j(t,\cdot), u'_{jt}(t,\cdot)\right\rangle\right| \leq (\sum_{\mu=1}^{m} p_\mu + m - 2 - \varepsilon)\sum_{j=1}^{m}(p_j+1)\left|u'_{jt}(t,x)\right|^2$$

$$+\frac{\gamma^2}{\sum_{\mu=1}^m p_\mu + m - 2 - \varepsilon}\sum_{j=1}^m (p_j+1)\left|u'_{jt}(t,x)\right|^2. \tag{4.10}$$

From (4.8) - (4.10) for a sufficiently large t_0 should

$$F''(t) \geq (4+\varepsilon)\sum_{\mathrm{j=1}}^{\mathrm{m}} (\mathrm{p_j}+1)\left|u'_{jt}(t,\cdot)\right|^2, t \geq t_0. \tag{4.11}$$

Hence from (4.1) and (4.9) we obtain

$$F''(t)F(t) - (1+\frac{\varepsilon}{4})F'^2 \geq (4+\varepsilon)\sum_{\mathrm{j=1}}^{\mathrm{m}} (\mathrm{p_j}+1)\left|u'_{jt}(t,\cdot)\right|^2 \sum_{\mathrm{j=1}}^{\mathrm{m}} (\mathrm{p_j}+1)\left|u'_{jt}(t,\cdot)\right|^2$$

$$-(1+\frac{\varepsilon}{4})\left[\sum_{j=1}^m (p_j+1)\left\langle u_j(t,\cdot), u'_{jt}(t,\cdot)\right\rangle\right]^2, t \geq t_1.$$

Using Hölder's inequality we have the following inequality

$$F''(t)F(t) - (1+\frac{\varepsilon}{4})F'^2(t) \geq 0,\ t \geq t_1. \tag{4.12}$$

From inequalities (4.11) and (4.12) we obtain the following inequality:

$$\left(F^{-\left(1+\frac{\varepsilon}{4}\right)}(t)\right)'' \leq 0, \quad t \geq t_1.$$

It follows

$$\left(F^{-(1+\frac{\varepsilon}{4})}(t)\right)'' = \frac{-(1+\frac{\varepsilon}{4})F'(t)}{F^{2+\frac{\varepsilon}{2}}(t)} < 0, \quad t \geq t_1. \tag{4.13}$$

Taking (4.7) and (4.13) into account, there exists such $t^* \in (0,t_1)$ that $\lim\limits_{t\to t^*} F^{-1}(t) = 0$, i.e. $\lim\limits_{t\to t^*} F(t) = +\infty$.

From the resulting contradiction we have $T_{\max} < +\infty$.

Remark 4.1 From Theorem 4.1 it follows

$$\lim_{t\to T_{\max}} \sum_{j=1}^m \left[|u'_{jt}(t,x)|^2 + \|u_j(t,\cdot)\|^2\right] = +\infty.$$

Theorem 4.2 *Let the conditions* (1.5)*,* (1.6) *be satisfied and*

$$E(0) > 0, I(u_{10},...,u_{m0}) < 0,$$

$$\sum_{j=1}^m \frac{p_j+1}{2}\|u_{j0}\|^2 > \frac{\sum_{\mu=1}^m p_\mu + m - 2}{\sum_{\mu=1}^m p_\mu + m}E(0).$$

Then the solution to the Cauchy problem (1.3)*,* (1.4) *collapses in a finite period of time.*

Remark 4.2 From theorems 4.1 and 4.2, taking into account conditions (1.5), (1.6), it follows that the stagnant waves corresponding to problem (1.3), (1.4) are not stable.

5 Proof of auxiliary lemmas

Proof of Lemma 2.1. The proof of property (i) follows directly from the following equality:

$$J(\lambda\Phi_1,...,\lambda\Phi_m)=\lambda^2\sum_{j=1}^{m}\frac{p_j+1}{2}\left(|\nabla\Phi_j|^2+|\Phi_j|^2\right)$$

$$-\lambda^{\sum_{\mu=1}^{m}p_\mu+m}\int_{R^n}\prod_{j=1}^{m}|\Phi_j(x)|^{p_j+1}\,dx.$$

(ii) By elementary transformations it can be proven that

$$\frac{d}{d\lambda}J(\lambda\Phi_1,...,\lambda\Phi_m)=\lambda\sum_{j=1}^{m}(p_j+1)\left(|\nabla\Phi_j|^2+|\Phi_j|^2\right)$$

$$-\left[\sum_{\mu=1}^{m}p_\mu+m\right]\lambda^{\sum_{\mu=1}^{m}p_\mu+m-1}\int_{R^n}\prod_{j=1}^{m}|\Phi_j(x)|^{p_j+1}\,dx. \tag{5.1}$$

It follows

$$\lambda^*=\left[\frac{\sum_{j=1}^{m}(p_j+1)\left\|\Phi_j\right\|^2}{\left[\sum_{\mu=1}^{m}p_\mu+m\right]\int_{R^n}\prod_{j=1}^{m}|\Phi_j(x)|^{p_j+1}\,dx}\right]^{\frac{1}{\sum_{\mu=1}^{m}p_\mu+m-2}}.$$

At the point $\lambda=\lambda^*$ the following equality holds:

$$\left.\frac{d}{d\lambda}J(\lambda\Phi_1,...,\lambda\Phi_m)\right|_{\lambda=\lambda^*}=0.$$

(iii) From (5.1) it is clear that when $0<\lambda<\lambda^*$

$$\frac{d}{d\lambda}J(\lambda\Phi_1,...,\lambda\Phi_m)>0,$$

and when $\lambda^*<\lambda<+\infty$

$$\frac{d}{d\lambda}J(\lambda\Phi_1,...,\lambda\Phi_m)<0,$$

those statement (iii) is true.

(iv) From the purposes of the functionals $J$and $I$and (5.1) we have

$$I(\lambda\Phi_1,...,\lambda\Phi_m)=\frac{\lambda}{\sum_{\mu=1}^{m}p_\mu+m}\frac{d}{d\lambda}J(\lambda\Phi_1,...,\lambda\Phi_m).$$

Let us define the following set

$$N=\left\{(\varphi_1,...,\varphi_m):(\varphi_1,...,\varphi_m)\in H^1\times...\times H^1\backslash\{(0,...,0)\}\,,I(\varphi_1,...,\varphi_m)=0\right\}.$$

We should $(\varphi_1,...,\varphi_m)\in N$, then

$$J(\varphi_1,...,\varphi_m)=\left(1-\frac{2}{\sum_{\mu=1}^{m}p_\mu+m}\right)\sum_{j=1}^{m}\frac{p_j+1}{2}\left\|\varphi_j\right\|^2>0, \tag{5.2}$$

those J bounded below in N. Consider the following variation problem

$$d = \inf_{(\varphi_1,...,\varphi_m)\in N} J(\varphi_1, ..., \varphi_m).$$

Proof of Lemma 2.2. If $(u_1, ..., u_m) \in N$, then from (5.2) we have

$$J(u_1, ..., u_m) = \frac{\sum_{\mu=1}^{m} p_\mu + m - 2}{\sum_{\mu=1}^{m} p_\mu + m} \sum_{j=1}^{m} \frac{p_j + 1}{2} \|u_j\|^2 > 0.$$

Let $(u_{1r}, ..., u_{mr})$ the sequence ensuring minimization, i.e.

$$\lim_{r\to\infty} J(u_{1r}, ..., u_{mr}) = \inf_{(u_1,...,u_m)\in N} J(u_1, ..., u_m) = d.$$

Allow us denote $u_{j\lambda} = ru_j, j = 1, ..., m$ and Schwartz symmetrization with respect to the variable x, $y_{jr} = \mu_r u_{jr}$ through $v_{jr} = (u^*_{jr})_{\mu_r}$ [17–19]. Here μ_r it is chosen so that $(v_{1r}, ..., v_{mr}) \in N$.

Taking into account (5.2), we have

$$J(v_{1r}, ..., v_{mr}) = \left(1 - \frac{2\delta}{\sum_{\mu=1}^{m} p_\mu + m}\right) \sum_{j=1}^{m} \frac{p_j + 1}{2} \|v_{jr}\|^2. \tag{5.3}$$

On the other side ([17,18])

$$\begin{aligned} \int\limits_{R^n} |\nabla v_{jr}|^2\, dx &= \int\limits_{R^n} \left|\nabla (u^*_{jr})_{\mu_r}\right|^2 dx \\ &= \int\limits_{R^n} |(\nabla (u_{jr})_{\mu_r})^*|^2 dx \le \int\limits_{R^n} |\nabla (u_{jr})_{\mu_r}|^2 dx. \end{aligned} \tag{5.4}$$

From (5.3), (5.4) follows

$$J(v_{1r}, ..., v_{mr}) \le J((u_{1r})_{\mu_r}, ..., (u_{mr})_{\mu_r}). \tag{5.5}$$

On the other hand, based on the choice μ_r we get

$$J((u_{1r})_{\mu_r}, ..., (u_{mr})_{\mu_r}) \le J(u_{1r}, ..., u_{mr}). \tag{5.6}$$

Thus,

$$\lim_{r\to\infty} J(v_{1r}, ..., v_{mr}) = d.$$

From here

$$\|\nabla v_{jr}\| \le c. \tag{5.7}$$

Therefore, there is such an element $(v_{1\infty}, ..., v_{m\infty}) \in H^1 \times ... \times H^1$, what for its subsequence at $r \to +\infty$

$$v_{jr} \to v_{j\infty} \text{ in } \mathrm{H}^1 \text{ weakly, } j = 1, ..., m. \tag{5.8}$$

Then for $p \le \frac{2}{n-2}$ from the compactness of the embedding $\mathrm{H}^1 \subset L_p(R^n)$([20]) at $r \to +\infty$

$$v_{jr} \to v_{j\infty} \text{ in } L_p(R^n), j = 1, ..., m. \tag{5.9}$$

Let's prove that $(v_{1\infty}, ..., v_{m\infty}) \ne (0, ..., 0)$. Let's assume the opposite, i.e.

$$(v_{1\infty}, ..., v_{m\infty}) = (0, ..., 0). \tag{5.10}$$

Then from (2.20), (5.9) and (5.10) at $r \to +\infty$ we have

$$G(v_{1r}, ..., v_{mr}) \to 0.$$

On the other hand, since $I(v_{1r}, ..., v_{mr}) = 0$, from (5.9) at $r \to +\infty$ we have

$$v_{jr} \to 0 \ \text{ in } \ \mathrm{H}^1 \ \text{ strong}, \ \ j = 1, ..., m. \tag{5.11}$$

Because $(v_{1r}, ..., v_{mr}) \in N$, from Hölder's inequality and embedding $\mathrm{H}^1 \subset L_p(R^n)$ ([21]) we get

$$\sum_{\mu=1}^{m} \frac{p_j+1}{\sum\limits_{\mu=1}^{m} p_\mu + m} \|v_{jr}\|^2 =$$

$$= \int\limits_{R^n} \prod_{j=1}^{m} |v_j(x)|^{p_j+1}\, dx \le \|v_{1r}\|^{p_1+1}_{L_{\sum\limits_{\mu=1}^{m} p_\mu+m}(R^n)} \times ... \times \|v_{mr}\|^{p_m+1}_{L_{\sum\limits_{\mu=1}^{m} p_\mu+m}(R^n)}.$$

Begining at multiplicative inequalities of the Gagliardo - Nirenberg type it follows

$$\|v_{jr}\|^{p_j+1}_{L_{\sum_{\mu=1}^m p_\mu+m}(R^n)} \le \|\nabla v_{jr}\|^{(p_j+1)\theta} \|v_{jr}\|^{(p_j+1)(1-\theta)} \text{ ([21])}, \tag{5.12}$$

where

$$\theta = n\left(\frac{1}{2} - \frac{1}{\sum_{\mu=1}^m p_\mu + m}\right), \quad j = 1, ..., m.$$

From (5.7) and (5.12) we have

$$\|v_{jr}\|^{p_j+1}_{L_{\sum_{\mu=1}^m p_\mu+m}(R^n)} \le c \|\nabla v_{jr}\|^{(p_j+1)\theta}, \quad j = 1, ..., m.$$

As a result we get

$$\sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^m p_\mu + m} \|v_{jr}\|^2 \le c^{\sum_{\mu=1}^m p_\mu+m} \left(\sum_{j=1}^{m} \|v_{jr}\|^2\right)^{\frac{3n}{2}\left(\sum_{\mu=1}^m p_\mu+m-2\right)}.$$

Here

$$\sum_{j=1}^{m} \|v_{jr}\|^2 \ge c_1 > 0.$$

And this contradicts our assumption. Thus, $d > 0$.

Proof of Lemma 2.3. From inequality (2.20), $\mathrm{H}^1 \subset L_{\sum_{\mu=1}^m p_\mu+m}(R^n)$ and Young's inequalities we have

$$G \le C^{\sum_{\mu=1}^m p_\mu+m} |u_1|^{p_1+1}_{L_{\sum_{\mu=1}^m p_\mu+m}(R^n)} \times ... \times |u_m|^{p_m+1}_{L_{\sum_{\mu=1}^m p_\mu+m}(R^n)} \le$$

$$\le C^{\sum_{\mu=1}^m p_\mu+m} \left[\sum_{j=1}^m \frac{p_j+1}{\sum_{\mu=1}^m p_\mu+m} \|u_j\|^2\right]^{\frac{\sum_{\mu=1}^m p_\mu+m-2}{2}+1}.$$

If

$$\sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^m p_\mu + m} \|u_j\|^2 < r(\delta)$$

we get

$$G \leq \delta \sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m} \|u_j\|^2 .$$

From the definition $I_\delta(u_1,...,u_m)$ it follows $I_\delta(u_1,...,u_m) > 0$.

Proof of Lemma 2.4. If $(u_1,...,u_m) \in H^1 \times ... \times H^1$, $\|u_j\| \neq 0, \quad j = 1,...,m$ and $I_\delta(u_1,...,u_m) < 0$, then the following inequality is true:

$$\delta \sum_{\mu=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m} \|u_j\|^2 <$$

$$< \int_{R^n} \prod_{j=1}^{m} |u_j(x)|^{p_j+1}\, dx \leq C^{\sum_{\mu=1}^{m} p_\mu + m} \left[\sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m} \|u_j\|^2 \right]^{\frac{\sum_{\mu=1}^{m} p_\mu + m - 2}{2} + 1} .$$

This implies the required inequality.

Proof of Lemma 2.5. If $\|u_j\| \neq 0, \quad j = 1,...,m$, then from $I_\delta(u_1,...,u_m) = 0$ we get

$$\delta \sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m - 1} \|u_j\|^2 =$$

$$= \int_{R^n} \prod_{j=1}^{m} |u_j(x)|^{p_j+1}\, dx \leq C^{\sum_{\mu=1}^{m} p_\mu + m} \left[\sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m} \|u_j\|^2 \right]^{\frac{\sum_{\mu=1}^{m} p_\mu + m - 2}{2} + 1} .$$

Thus,

$$\sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m - 1} \|u_j\|^2 \geq r(\delta) = \left(\frac{\delta}{C^{\sum_{\mu=1}^{m} p_\mu + m}} \right)^{\frac{2}{\sum_{\mu=1}^{m} p_\mu + m - 2}} .$$

Proof of Lemma 2.6. From Lemma 2.5 for each $(u_1,...,u_m) \in N$ we have

$$\sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m - 1} \|u_j\|^2 \geq r(\delta).$$

Therefore

$$J(u_1,...,u_m) = \left(\frac{\sum_{\mu=1}^{m} p_\mu + m}{2} - \delta \right) \sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m} \|u_j\|^2 \geq a(\delta) r(\delta).$$

Here $0 < \delta < \frac{\sum_{\mu=1}^{m} p_\mu + m}{2}$, $\quad a(\delta) = \frac{\sum_{\mu=1}^{m} p_\mu + m}{2} - \delta$, $d(\delta) \geq a(\delta) r(\delta)$.

Let's say $(\bar{u}_1,...,\bar{u}_m) \in N$ minimum element, i.e. $d = J(\bar{u}_1,...,\bar{u}_m)$.

For anyone $\delta > 0$ let's choose something like $\lambda = \lambda(\delta)$, this

$$\delta \sum_{j=1}^{m} \frac{p_j+1}{\sum_{\mu=1}^{m} p_\mu + m} \|\lambda \bar{u}_j\|^2 = \int_{R^n} \prod_{j=1}^{m} |\lambda \bar{u}_j(x)|^{p_j+1}\, dx. \tag{5.13}$$

Hence,

$$\lambda(\delta) = \left[\frac{\delta \sum_{j=1}^{m} (p_j + 1) \|\bar{u}_j\|^2}{\left(\sum_{\mu=1}^{m} p_\mu + m\right) \int_{R^n} \prod_{j=1}^{m} |\bar{u}_j(x)|^{p_j+1} dx} \right]^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m - 2}} = \delta^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m - 2}}. \tag{5.14}$$

Because $(\lambda(\delta)\bar{u}_1, ..., \lambda(\delta)\bar{u}_m) \in N_\delta$, from the definition $d\,(\delta)$ the following inequality is true

$$d(\delta) \leq J(\lambda(\delta)\bar{u}_1, ..., \lambda(\delta)\bar{u}_m) =$$

$$= \delta^{\frac{2}{\sum_{\mu=1}^{m} p_\mu + m - 2}} \sum_{j=1}^{m} \frac{p_j + 1}{2} \|\bar{u}_j\|^2 - \delta^{1 + \frac{2}{\sum_{\mu=1}^{m} p_\mu + m - 2}} \int_{R^n} \prod_{j=1}^{m} |\bar{u}_j(x)|^{p_j+1} dx. \tag{5.15}$$

On the other side,

$$(\bar{u}_1, ..., \bar{u}_m) \in N. \tag{5.16}$$

Therefore

$$\int_{R^n} \prod_{j=1}^{m} |\bar{u}_j(x)|^{p_j+1} dx = \sum_{j=1}^{m} \frac{p_j + 1}{\sum_{\mu=1}^{m} p_\mu + m} \|\bar{u}_j\|^2. \tag{5.17}$$

From (5.13) and (5.17) we have

$$d(\delta) \leq \delta^{\frac{2}{\sum_{\mu=1}^{m} p_\mu + m - 2}} \left(1 - \frac{2\delta}{\sum_{\mu=1}^{m} p_\mu + m}\right) \sum_{j=1}^{m} \frac{p_j + 1}{2} \|\bar{u}_j\|^2. \tag{5.18}$$

Therefore, if $(\bar{u}_1, ..., \bar{u}_m)$ the minimum element, then

$$d = J(\bar{u}_1, ..., \bar{u}_m) = \frac{\sum_{\mu=1}^{m} p_\mu + m - 2}{\sum_{\mu=1}^{m} p_\mu + m} \sum_{j=1}^{m} \frac{p_j + 1}{2} \|\bar{u}_j\|^2,$$

those

$$\sum_{j=1}^{m} \frac{p_j + 1}{2} \|\bar{u}_j\|^2 = \frac{\sum_{\mu=1}^{m} p_\mu + m}{\sum_{\mu=1}^{m} p_\mu + m - 2} d. \tag{5.19}$$

From (5.17) and (5.19) we obtain

$$d(\delta) \leq \frac{\sum_{\mu=1}^{m} p_\mu + m - 2\delta}{\sum_{\mu=1}^{m} p_\mu} \delta^{\frac{2}{\sum_{\mu=1}^{m} p_\mu + m - 2}} d. \tag{5.20}$$

Let's $(\bar{v}_1, ..., \bar{v}_m) \in N_\delta$ an element that provides a minimum of functionality $J(u_1, ..., u_m)$, i.e.

$$J(\bar{v}_1, ..., \bar{v}_m) = \min_{(\bar{v}_1, ..., \bar{v}_m) \in N_\delta} J(v_1, ..., v_m) = d(\delta).$$

We should choose the parameter $\mu = \mu\,(\delta)$ so that $(\mu v_1, ..., \mu v_m) \in N$, those

$$I(\mu\bar{v}_1, ..., \mu\bar{v}_m) = 0. \tag{5.21}$$

Then

$$\mu = \mu(\delta) = \left[\frac{\sum_{j=1}^{m} (p_j + 1) \|\bar{v}_j\|^2}{\left(\sum_{\mu=1}^{m} p_\mu + m\right) \int_{R^n} \prod_{j=1}^{m} |\bar{v}_j(x)|^{p_j+1} dx} \right]^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m - 2}} = \left(\frac{1}{\delta}\right)^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m - 2}}.$$

Considering the definition d we have

$$d \leq J(\mu\bar{v}_1, ..., \mu\bar{v}_m)$$

$$= \left(\frac{1}{\delta}\right)^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m-2}} \sum_{j=1}^{m} \frac{p_j+1}{2} \|\bar{v}_j\|^2 - \left(\frac{1}{\delta}\right)^{\frac{\sum_{\mu=1}^{m} p_\mu + m}{\sum_{\mu=1}^{m} p_\mu + m-2}} \int_{R^n} \prod_{j=1}^{m} |\bar{v}_j|^{p_j+1}\, dx$$

$$= \left(\frac{1}{\delta}\right)^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m-2}} \frac{\sum_{\mu=1}^{m} p_\mu + m - 2}{\sum_{\mu=1}^{m} p_\mu + m} \sum_{j=1}^{m} \frac{p_j+1}{2} \|\bar{v}_j\|^2 . \tag{5.22}$$

On the other hand, from (5.21) and (5.22) we obtain

$$J(\bar{v}_1, ..., \bar{v}_m) = \left(1 - \frac{2\delta}{\sum_{\mu=1}^{m} p_\mu + m}\right) \sum_{j=1}^{m} \frac{p_j+1}{2} \|\bar{v}_j\|^2 .$$

Therefore

$$\sum_{j=1}^{m} \frac{p_j+1}{2} \|\bar{v}_j\|^2 = \frac{\sum_{\mu=1}^{m} p_\mu + m}{\sum_{\mu=1}^{m} p_\mu + m - 2\delta} J(\bar{v}_1, ..., \bar{v}_m) = \frac{\sum_{\mu=1}^{m} p_\mu + m}{\sum_{\mu=1}^{m} p_\mu + m - 2\delta} d(\delta). \tag{5.23}$$

From (5.21) and (5.22) it follows

$$d \leq \left(\frac{1}{\delta}\right)^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m-2}} \frac{\sum_{\mu=1}^{m} p_\mu + m - 2}{\sum_{\mu=1}^{m} p_\mu + m - 2\delta} d(\delta),$$

those

$$d(\delta) \geq \frac{\sum_{\mu=1}^{m} p_\mu + m - 2\delta}{\sum_{\mu=1}^{m} p_\mu + m - 2} \delta^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m-2}} d. \tag{5.24}$$

If we compare (5.20) and (5.23) we get

$$d(\delta) = \frac{\sum_{\mu=1}^{m} p_\mu + m - 2\delta}{\sum_{\mu=1}^{m} p_\mu + m - 2} \delta^{\frac{1}{\sum_{\mu=1}^{m} p_\mu + m-2}} d.$$

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