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The Goursat-Darboux system with two-point boundary condition

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Abstract. *The article studies a system of hyperbolic differential equations of the second-order given with two-point boundary conditions. A Green's function for the boundary problems has been constructed, and the boundary problem has been transformed into an equivalent integral equation. Using the Banach contraction mapping principle, sufficient conditions for the existence and uniqueness of the solution have been found.*

Keywords. nonlocal problem, two-point boundary condition, Goursat-Darboux system, existence and uniqueness.

Mathematics Subject Classification (2010): 35L70, 35R45, 49J20, 49K20

1 Introduction

The assessment of the current state of classical differential equations theory shows that non-local boundary problems hold a special place within the problems of mathematical physics. The emergence of such issues during the investigation of various problems in natural sciences and technology increases the interest in the study of non-local conditional problems. The study of non-local two-point boundary problems occupies an important place in the theory of non-local boundary problems. With such problems, it is usually not possible to directly measure the important characteristics of real processes, but the average value of those quantities is known. In such cases, when mathematically modeling these processes, this information can be represented in the form of a solution with multi-point boundary conditions.

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It should be noted that non-local boundary problems arise in the construction of mathematical models of processes such as turbulence, plasma, heat transfer, demographic, and other processes [17, 19, 25]. The article presented for the first time constructs a Green's function for a system of hyperbolic equations given with two-point boundary conditions and investigates the existence and uniqueness of the solution to the boundary problem.

2 Problem statement

In the paper we consider a nonlocal problem with two point boundary conditions for the Goursat-Darboux system in the domain $Q = [0, T] \times [0, l]$:

$$z_{tx} = f\left(t, x, z(t, x)\right), (t, x) \in Q, \tag{2.1}$$

$$A_1 \; z\left(0, x\right) + A_2 z(T, x) = \varphi(x), x \in [0, l]\,, \tag{2.2}$$

$$B_1 z\left(t, 0\right) + B_2 z\left(t, l\right) = \psi(t), t \in [0, T] \tag{2.3}$$

here $z(t, x) = col\left(z_1(t, x), z_2(t, x), \ldots, z_n(t, x)\right)$ is an unknown n – dimensional vector-function, $f : Q \times R^n \to R^n$ is continuous on $Q \times R^n$, n - dimensional vector-functions $\varphi(x)$ and $\psi(t)$ are continuously-differentiable on $[0, T]$, $[0, l]$ respectively, $A_1, A_2, B_1, B_2 \in R^{n \times n}$ given matrices, $A_i B_j = B_j A_i$ for
$i = 1; 2$, $j = 1; 2$ and $\det(A_1 + A_2) \neq 0$, $\det\left(B_1 + B_2\right) \neq 0$.

It is assumed that the functions $\varphi(x)$ and $\psi(t)$ satisfy the agreement condition

$$B_1 \varphi\left(0\right) + B_2 \varphi\left(l\right) = A_1 \psi\left(0\right) + A_2 \psi(T).$$

Note that problems for hyperbolic type equations have been studied in [1]-[8], [10]-[13], [18], [20], [24], [26], [27]. In these works the conditions of classical, general consistency of problems with nonlocal conditions have been established for second order hyperbolic equations. Analogy similar issues for ordinary differential equations have been studied in [7], [9], [13]-[16] works.

3 Main results

In this paper, for the first time the Green function is constructed for problem (2.1)-(2.3) and this problem is reduced to an equivalent integral equation. Further, using the method of Banach contraction mappings principle, sufficient conditions of classical consistency of the given problem are established.

Theorem 3.1 *A problem* (2.1)-(2.3) *is equivalent to the following integral equation:*

$$z(t, x) = (B_1 + B_2)^{-1} \psi(t) + (A_1 + A_2)^{-1} \varphi(x)$$

$$-(A_1 + A_2)^{-1} (B_1 + B_2)^{-1} B_1 \varphi\left(0\right) - (A_1 + A_2)^{-1} (B_1 + B_2)^{-1} B_2 \varphi\left(l\right)$$

$$+ \int_0^T \int_0^l G\left(t, x, \tau, s\right) f\left(\tau, s, z\right) d\tau ds,$$

where

$$G(t,x,\tau,s) = (A_1+A_2)^{-1}(B_1+B_2)^{-1}\begin{cases} A_1B_1, & 0\le\tau\le t, \quad 0\le s\le x, \\ -A_1B_2, & 0\le\tau\le t, \quad x<s\le l, \\ -A_2B_1, & t<\tau\le T, \quad 0\le s\le x, \\ A_2B_2, & t<\tau\le T, \quad x<s\le l. \end{cases}$$

Proof. We will look for any solution of equation (2.1) in the from

$$z(t,x) = a(t) + b(x) + \int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds, \tag{3.1}$$

here $a(t)$ and $b(x)$ are unknown continuous functions and are determined in the segments $[0,T]\,,[0,l]$ respectively. Let the function determined by equality (3.1) satisfy conditions (2.2) and (2.3). Then

$$A_1\left[a(0)+b(x)\right] + A_2\left[a(T)+b(x)+\int_0^T\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds\right]$$

$$= A_1a(0)+A_2a(T)+(A_1+A_2)\,b(x)+A_2\int_0^T\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds = \varphi(x), x\in[0,l]\,. \tag{3.2}$$

$$B_1\left[a(t)+b(0)\right] + B_2\left[a(t)+b(l)+\int_0^t\int_0^l f(\tau,s,z(\tau,s))\,d\tau ds\right]$$

$$= (B_1+B_2)\,a(t)+B_1b(0)+B_2b(l)+B_2\int_0^t\int_0^l f(\tau,s,z(\tau,s))\,d\tau ds = \psi(t), t\in[0,T]\,. \tag{3.3}$$

Without loss of generality, we assume that the relationship

$$A_1a(0) + A_2a(T) = 0$$

is valid.

From equality (3.2) we obtain the following relationship:

$$b(x) = (A_1+A_2)^{-1}\varphi(x) - (A_1+A_2)^{-1}A_2\times\int_0^T\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds, x\in[0,l]\,. \tag{3.4}$$

$$b(0) = (A_1+A_2)^{-1}\varphi(0)\,,$$

$$b(l) = (A_1+A_2)^{-1}\varphi(l) - (A_1+A_2)^{-1}A_2\int_0^T\int_0^l f(\tau,s,z(\tau,s))\,d\tau ds.$$

We'll take into account equality (3.3) in the function $b(0)\,,b(l)$ determined by equality (3.4).

Then

$$(B_1+B_2)\,a(t) + (A_1+A_2)^{-1}B_1\varphi(0) + (A_1+A_2)^{-1}B_2\varphi(l)$$

$$-(A_1+A_2)^{-1}A_2B_2\int_0^T\int_0^l f(\tau,s,z(\tau,s))\,d\tau ds$$

$$+B_2 \int_0^t \int_0^l f(\tau, s, z(\tau, s))\, d\tau ds = \psi(t).$$

From here

$$a(t) = (B_1 + B_2)^{-1}\psi(t) - (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}$$
$$\times B_1\varphi(0) - (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}B_2\varphi(l) + (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}A_2B_2$$
$$\times \int_0^T \int_0^l f(\tau, s, z(\tau, s))\, d\tau ds - (B_1 + B_2)^{-1}B_2 \int_0^t \int_0^l f(\tau, s, z(\tau, s))\, d\tau ds. \quad (3.5)$$

We'll take into account expressions (3.4) and (3.5) obtained for the functions $a(t)$ and $b(x)$ in the (3.1).
Then

$$z(t,x) = (B_1 + B_2)^{-1}\psi(t) + (A_1 + A_2)^{-1}\varphi(x) - (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}B_1\varphi(0)$$
$$-(A_1 + A_2)^{-1}(B_1 + B_2)^{-1}$$
$$\times B_2\varphi(l) - (A_1 + A_2)^{-1}A_2 \int_0^T \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds + (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}A_2B_2$$
$$\times \int_0^T \int_0^l f(\tau, s, z(\tau, s))\, d\tau ds - (B_1 + B_2)^{-1}B_2$$
$$\times \int_0^t \int_0^l f(\tau, s, z(\tau, s))\, d\tau ds + \int_0^t \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds,\ (t,x) \in Q. \quad (3.6)$$

From the equality (3.6), we obtain that

$$z(t,x) = (B_1 + B_2)^{-1}\psi(t) + (A_1 + A_2)^{-1}\varphi(x)$$

$$-(A_1 + A_2)^{-1}(B_1 + B_2)^{-1}B_1\varphi(0) - (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}B_2\varphi(l)$$

$$+ \int_0^t \int_0^x \left[E - (B_1 + B_2)^{-1}B_2 + (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}A_2B_2 - (A_1 + A_2)^{-1}A_2\right]$$
$$\times f(\tau, s, z(\tau, s))\, d\tau ds$$

$$+ \int_0^t \int_x^l \left[(A_1 + A_2)^{-1}(B_1 + B_2)^{-1}A_2B_2 - (B_1 + B_2)^{-1}B_2\right]$$
$$\times f(\tau, s, z(\tau, s))\, d\tau ds$$

$$+ \int_t^T \int_0^x \left[-(A_1 + A_2)^{-1}A_2 + (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}A_2B_2\right]$$
$$\times f(\tau, s, z(\tau, s))\, d\tau ds$$

$$+ \int_t^T \int_x^l (A_1 + A_2)^{-1}(B_1 + B_2)^{-1}A_2B_2\ f(\tau, s, z(\tau, s))\, d\tau ds,\ (t,x) \in Q. \quad (3.7)$$

Taking into account that the equalities,

$$E-(B_1+B_2)^{-1}B_2+(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_2$$

$$-(A_1+A_2)^{-1}A_2=(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_1B_1,$$

$$(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_2-(B_1+B_2)^{-1}B_2$$

$$=-(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_1B_2,$$

$$-(A_1+A_2)^{-1}A_2+(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_2$$

$$=-(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_1$$

are valid, we can write the equality (3.1) in the following from

$$z(t,x)=(B_1+B_2)^{-1}\psi(t)+(A_1+A_2)^{-1}\varphi(x)$$

$$+(A_1+A_2)^{-1}(B_1+B_2)^{-1}B_1\varphi(0)-(A_1+A_2)^{-1}(B_1+B_2)^{-1}B_2\varphi(l)$$

$$+\int_0^t\int_0^x(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_1B_1f(\tau,s,z(\tau,s))\,d\tau ds$$

$$-\int_0^t\int_x^l(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_1B_2f(\tau,s,z(\tau,s))\,d\tau ds$$

$$-\int_t^T\int_0^x(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_1f(\tau,s,z(\tau,s))\,d\tau ds$$

$$+\int_t^T\int_x^l(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_2\ f(\tau,s,z(\tau,s))\,d\tau ds. \tag{3.8}$$

In this equality, introducing the matrix-function $G(t,x,\tau,s)$, we prove first part of the theorem. Now show that the function determined by the equality (3.8) is the solution of the boundary value problem (2.2), (2.3). For that from the equality (3.8) we derive a derivative with respect to t and x.

$$z_{tx}(t,x)=\frac{\partial^2}{\partial t\partial x}[(B_1+B_2)^{-1}\psi(t)+(A_1+A_2)^{-1}\varphi(x)$$

$$-(A_1+A_2)^{-1}(B_1+B_2)^{-1}B_1\varphi(0)-(A_1+A_2)^{-1}(B_1+B_2)^{-1}B_2\varphi(l)]$$

$$+\frac{\partial^2}{\partial t\partial x}\left[\int_0^t\int_0^x(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_1B_1f(\tau,s,z(\tau,s))\,d\tau ds\right]$$

$$-\frac{\partial^2}{\partial t\partial x}\left[\int_0^t\int_x^l(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_1B_2f(\tau,s,z(\tau,s))\,d\tau ds\right]$$

$$-\frac{\partial^2}{\partial t\partial x}\int_t^T\int_0^x (A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_1 f\left(\tau,s,z(\tau,s)\right)d\tau ds$$

$$+\frac{\partial^2}{\partial t\partial x}\left[\int_t^T\int_x^l (A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_2\ f\left(\tau,s,z(\tau,s)\right)d\tau ds\right]$$

$$=(A_1+A_2)^{-1}(B_1+B_2)^{-1}\left[A_1B_1+A_1B_2+A_2B_1+A_2B_2\right]$$

$$\times f(t,x,z(t,x)=(A_1+A_2)^{-1}(B_1+B_2)^{-1}\left(A_1+A_2\right)\left(B_1+B_2\right)$$

$$\times f(t,x,z\left(t,x\right))=f(t,x,z\left(t,x\right)).$$

Now, show that the function determined by the equality (3.8) satisfies the conditions (2.2), (2.3) as well.

$$A_1\Big[(B_1+B_2)^{-1}\psi\left(0\right)+(A_1+A_2)^{-1}\varphi(x)$$

$$-\left(A_1+A_2\right)^{-1}(B_1+B_2)^{-1}B_1\varphi\left(0\right)-\left(A_1+A_2\right)^{-1}(B_1+B_2)^{-1}B_2\varphi\left(l\right)$$

$$-(A_1+A_2)^{-1}A_2\int_0^T\int_0^x f\left(\tau,s,z(\tau,s)\right)d\tau ds$$

$$+(A_1+A_2)^{-1}(B_1+B_2)^{-1}A_2B_2\int_0^T\int_0^l f\left(\tau,s,z(\tau,s)\right)d\tau ds\Big]$$

$$A_2\Big[(B_1+B_2)^{-1}\psi(t)+(A_1+A_2)^{-1}\varphi(x)$$

$$-\left(A_1+A_2\right)^{-1}(B_1+B_2)^{-1}B_1\varphi\left(0\right)-\left(A_1+A_2\right)^{-1}(B_1+B_2)^{-1}B_2\varphi\left(l\right)$$

$$-(A_1+A_2)^{-1}A_2\int_0^T\int_0^x f\left(\tau,s,z(\tau,s)\right)d\tau ds$$

$$+\left(A_1+A_2\right)^{-1}(B_1+B_2)^{-1}A_2B_2\int_0^T\int_0^l f\left(\tau,s,z(\tau,s)\right)d\tau ds$$

$$-(B_1+B_2)^{-1}B_2\int_0^T\int_0^l f\left(\tau,s,z(\tau,s)\right)d\tau ds+\int_0^T\int_0^x f\left(\tau,s,z(\tau,s)\right)d\tau ds\Big]$$

$$=\left(A_1+A_2\right)^{-1}\left(A_1+A_2\right)\varphi(x)-\left(A_1+A_2\right)^{-1}\left(A_1+A_2\right)\left(B_1+B_2\right)^{-1}$$

$$\times\left[B_1\varphi\left(0\right)+B_2\varphi\left(l\right)\right]+\left(B_1+B_2\right)^{-1}\left[A_1\psi\left(0\right)+A_2\psi(t)\right]$$

$$-(A_1+A_2)^{-1}\left(A_1+A_2\right)A_2\int_0^T\int_0^x f\left(\tau,s,z(\tau,s)\right)d\tau ds$$

$$+ (B_1 + B_2)^{-1} A_2 B_2 \int_0^T \int_0^l f(\tau, s, z(\tau, s))\, d\tau ds$$

$$- (B_1 + B_2)^{-1} A_2 B_2 \int_0^T \int_0^l f(\tau, s, z(\tau, s))\, d\tau ds$$

$$+ A_2 \int_0^T \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds = \varphi(x).$$

We can show that the condition

$$B_1 z(t, 0) + B_2 z(t, l) = \psi(t), \quad t \in [0, T]$$

is also satisfied similarly.

This completed the proof of Teorem 3.1.

4 Existence and uniqueness

To prove the uniqueness of the solution of the stated problem, we determined the operator $P : C(Q; ; R^n) \to C(Q; R^n)$ as

$$(Pz)(tx) = (B_1 + B_2)^{-1} \psi(t) + (A_1 + A_2)^{-1} \varphi(x)$$

$$-(A_1 + A_2)^{-1} (B_1 + B_2)^{-1} B_1 \varphi(0) - (A_1 + A_2)^{-1} (B_1 + B_2)^{-1} B_2 \varphi(l)$$

$$+ \int_0^T \int_0^l G(t, x, \tau, s)\, f(\tau, s, z)\, d\tau ds.$$

It is known that problem (2.1)-(2.3) is equivalent to the problem on a fixed point $z = Pz$. So, problem (2.1)-(2.3) has a solution if and only if the operator P has a fixed point.

Theorem 4.1 *Assume that the following conditions:*

$$|f(t, x, z_2) - f(t, x, z_1)| \leq M |z_2 - z_1| \tag{4.1}$$

are satisfied for each $(t, x) \in Q$ *and for all* $z_1, z_2 \in R^n$*, the constant* $M \geq 0$ *and*

$$L = lTSM < 1, \tag{4.2}$$

where

$$S = \max_{Q \times Q} \| G(t, x, \tau, s) \| .$$

Then boundary value problem (2.1)-(2.3) *has a unique solution on* Q*.*

Proof. Denoting

$$N = \max_{Q} \Big| (B_1 + B_2)^{-1} \psi(t) + (A_1 + A_2)^{-1} \varphi(x)$$

$$- (A_1 + A_2)^{-1} (B_1 + B_2)^{-1} B_1 \varphi(0) - (A_1 + A_2)^{-1} (B_1 + B_2)^{-1} B_2 \varphi(l) \Big|,$$

$$\max_{(t,x)\in Q} |f(t,x,0)| = M_f$$

and choose $r \geq \frac{N + M_f TS}{1-L}$. We'll prove that $PB_r \subset B_r$ that, where

$$B_r = \{x \in C(Q, R^n) : \|z\| \leq r\}.$$

For $z \in B_r$ we have

$$\|Pz(t,x)\| \leq N$$

$$+ \int_0^T \int_0^l |G(t,x,\tau,s)| \left(|f(\tau,s,z(\tau,s)) - f(\tau,s,0)| + |f(\tau,s,0)| \right) d\tau ds$$

$$\leq N + S \int_0^T \int_0^l (M|z| + M_f) \, dt dx \leq N + SM_r Tl + M_f TlS$$

$$\leq \frac{N + M_f TS}{1 - L} \leq r.$$

Further, by (4.1), for any $z_1, z_2 \in B_r$

$$|Pz_2 - Pz_1| \leq \int_0^T \int_0^l |G(t,x,\tau,s)| \left(|f(\tau,s,z_2(\tau,s) - f(\tau,s,z_1(\tau,s)| \right)$$

$$\leq S \int_o^T \int_0^l M |z_2(t,x) - z_1(t,x)| \, dt dx \leq MSTl \max_{Q\times Q} |z_2(t,x) - z_1(t,x)|$$

$$\leq MSTl \, \|z_2 - z_1\|$$

is valid, or

$$\|Pz_2 - Pz_1\| \leq L \|z_2 - z_1\|.$$

It is clear that by condition (4.2) P is contraction operator. Thus, boundary value problem (2.1)-(2.3) has a unique solution.

5 Example

We give an example illustrating the main result obtained in the paper. Let's consider the following system of differential equations with an two-point boundary condition:

$$\begin{cases} y_{1tx} = cos(0,1y_2), \\ y_{2tx} = \frac{|y_1|}{(9+e^{tx})(1+|y_1|)} \end{cases}, (t,x) \in [0,1] \times [0,1], \tag{5.1}$$

$$\begin{cases} z_1(0,x) + z_2(1,x) = x, \quad z_2(0,x) = x^2, \\ z_1(t,0) + z_2(t,1) = t, \quad z_2(t,1) = t^2. \end{cases}$$

If we indicate

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we can write the boundary conditions as follows

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1(0,x) \\ z_2(0,x) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(1,x) \\ z_2(1,x) \end{pmatrix} = \begin{pmatrix} x \\ x^2 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(t,0) \\ z_2(t,0) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(t,1) \\ z_2(t,1) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}. \end{cases} \tag{5.2}$$

Obviously, the agreement condition is satisfied. Condition (4.2) is satisfied due to conditions (4.1) and $G_{max} < 2$, $M = 0,1$. Consequently,

$$L = G_{max}MTl = 2 \cdot 0,1 = 0,2 < 1.$$

So, by Theorem 4.1, boundary value problem (5.1), (5.2) has a unique solution on $[0,1] \times [0,1]$.

6 Conclusion

In this paper, the existence and uniqueness of solutions for nonlinear hyperbolic differential equations with two-point boundary conditions is established. Note that the method introduced in the paper can be successfully used in more complicated problems for hyperbolic differential equation. For example, we can consider the following problem:

$$z_{tx} = f(t,x,z(t,x)), \qquad (t,x) \in Q,$$

with point and integral boundary condition

$$Az(0,x) + \int_0^T n(t)z(t,x)dt = \varphi(x), \quad x \in [0,l],$$

$$Bz(t,0) + \int_0^l m(x)z(t,x)dx = \psi(t), \quad t \in [0,T].$$

here $n(t)$,$m(x) \in R^{n\times n}$ are the given matrices; $\varphi(x),\ x \in [0,l]$, $\psi(t),\ t \in [0,T]$ are the given functions, and $det \int_0^T n(t)dt \neq 0, \quad det \int_0^l m(x)dx \neq 0.$

References

1. Assanova, A.T.: *Nonlocal problem with integral conditions for a system of hyperbolic equations in characteristic rectangle*, Russian Math. (Iz. VUZ), **61**(5), 7–20 (2017).
2. Assanova, A.T.: *A generalized integral problem for a system of hyperbolic equations and its applications*, Hacet. J. Math. Stat. **52**, 1513–1532 (2023).
3. Assanova, A.T., Dzhumabaev D.S.: *Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations*, J. Math. Anal. Appl. **402**, 167-178 (2013).
4. Assanova, A.T.: *On the theory of nonlocal problems with integral conditions for systems of equations of hyperbolic type*, Ukr. Math. J. **70**, 1514-1525 (2019).
5. Assanova, A.T.: *On a nonlocal problem with integral conditions for the system of hyperbolic equations*, Diff. Equat. **54**, 201–214 (2018).
6. Bouziani, A.: *Solution forte d'un probleme mixte avec conditions non-locales pour une classe liquations hyperboliques*, Bulletin de la Classe des sciences **8**(1), 53–70 (1997).
7. Byszewski, L.: *Existence and uniqueness of solution of nonlocal problems for hyperbolic equation* $u_{xt} = F(x, t, u, u_x)$, J. Appl. Math. Stoch. Anal. **3**, 163–168 (1990).
8. Beilin, S.: *Existence of solution for one-dimensional wave equations with nonlocal conditions*, Electron. J. Different. Equat. **76**, 1–8 (2001).
9. Gasimov, Y.S., Jafari, H., Mardanov, M.J., Sardarova, R.A., Sharifov, Y.A.: *Existence and uniqueness of the solutions of the nonlinear impulse differential equations with nonlocal boundary conditions*, Quaest. Math. **45**, 1399–1412.
10. Golubeva, N.D., Pulkina, L.S.: *A nonlocal problem with integral conditions*, Math. Notes. **59**, 326–328 1996.
11. Gilev, A.V.: *A nonlocal problem for a hyperbolic equation with a dominant mixed derivative*, Vestnik SamU. Estestvenno-Nauchnaya Ser. **26**(4), 25–35 (2020).
12. Kozhanov, A.I., Pulkina L.S.: *On the solvability of boundary-value problems with nonlocal integral boundary condition for multidimensional hyperbolic equations*, Diff. Equat. **42**, 1233–1246 (2006).
13. Mardanov, M.J., Sharifov, Y.A.: *An optimal control problem for the systems with integral boundary conditions*, Bulletin of the Karaganda University, Math. Series **109** (1), 110-123.
14. Mardanov, M.J., Sharifov, Y.A., Zeynally, F.M.: *Existence and uniqueness of solutions for nonlinear impulsive differential equations with nonlocal boundary conditions* , Vestn. Tomsk. Gos. Univ. Mat. Mekh. **60**, 61–72 (2019).
15. Mardanov, M.J., Mahmudov, N.I., Sharifov, Y.A.: *Existence and uniqueness results for q-fractional difference equations with p-Laplacian operators*, Adv. Differ. Equ. 1–13 (2015).
16. Mekhtiyev, M.F., Djabrailov, S.I., Sharifov, Y.A.: *Necessary optimality conditions of second order in classical sense in optimal control problems of three-point conditions*, J. Automat. Inf. Scien. **42** (3), 47–57 (2010).
17. Nakhushev, A.M.: Problems with replacement for partial differential equations. *Nuaka, Moscow* (2006).
18. Oussaeif, T.E., Bouziani, A.: *Solvability of nonlinear goursat type problem for hyperbolic equation with integral condition*, Khayyam J. Math. **4**, 198–213 (2018).
19. Ptashnyck, B.I.: Ill-posed boundary value problems for partial differential equations. *Naukova, Dumka Kiev, Ukraine* (1984).

20. Paneah, B., Paneah, P.: *Nonlocal problems in the theory of hyperbolic differential equations*, Trans. Moscow Math. Soc. 135–170 (2009).
21. Pulkina, L.S.: Problem with nonclassical conditions for hyperbolic equations. *Samara: Izdatel'stwo Samarskiii Universiet* (2012).
22. Pulkina, L.S.: *A nonlocal problem with integral conditions for a hyperbolic equation*, Differ. Equ. **40**(7), 947–953 (2004).
23. Pulkina, L.S.: *A non-local problem with integral conditions for hyperbolic equations*, Electron. J. Differential Equations **1999**, 1–6 (1999).
24. Pulkina, L.S.: *A nonlocal problem for a loaded hyperbolic equation*, Tr. Steklov Mat. Inst. **236**, 298–303 (2002).
25. Samarskii, A.A.: *Some problems of the theory of differential equations*, Differential Equations **16**, 1221–1228 (1980).
26. Utkina, E.A.: *On the uniqueness of the solution of a semi-integral problem for a fourth-order equation*, Vestn. Samar. Gos. Univ. Ser. Estestvennonauchn. **4**, 98–102 (2010).
27. Zhestkov, S.V.: *The Goursat problem with integral boundary conditions*, Ukr. Mat. Zh. **42**, 132–135 (1990).