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Boundedness criteria of the commutators of G-fractional maximal and G-fractional integral operators on G-Morrey spaces

Elman J. Ibrahimov* · **Saadat Ar. Jafarova** · **Gul A. Aydin**

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Abstract. *In this paper we found boundedness criteria for the commutators of fractional integral and fractional maximal operators generated by the differential Gegenbauer operator on G-Morrey spaces.*

Keywords. Commutator, fractional Gegenbauer integral; maximal Gegenbauer function; generalized shift operator; G-Morrey space.

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1 Introduction

Fractional integral operator I_α of α order has a form

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-a}} dy, \ 0 < \alpha < n.$$

For locally integrable function b, commutator is defined as follows:

$$[b, I_\alpha] f(x) := b(x) I_\alpha f(x) - I_\alpha (bf)(x).$$

This commutator was introduced by Chanillo [2]. Adams [1] studied the boundedness I_α from classical Morrey space $L^{p,\mu}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$. Conditions for the boundedness of $[b, I_\alpha]$ from $L^{p,\mu}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ have been found in [13].

Similar results can be found in [4, 18] and the references cited therein.

Let $1 \le p < \infty$ and $0 \le \mu \le n$. Classical Morrey space is defined as follows:

$$L^{p,\mu}(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) := \|f\|_{L^{p,\mu}} < \infty \right\},$$

* Corresponding author

E.J. Ibrahimov
Institute of Mathematics and Mechanics, Ministry of Science
and Educations of the Republic of Azerbaijan, Baku, Azerbaijan
E-mail: elmanibrahimov@yahoo.com

S.Ar. Jafarova
Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan
E-mail: saadat.cafarova73@gmail.com

G.A. Aydin
Institute of Mathematics and Mechanics, Ministry of Science
and Educations of the Republic of Azerbaijan, Baku, Azerbaijan
E-mail: gdova@mail.ru

there will be

$$\|f\|_{L^{p,\mu}} := \sup_{Q} \left(|Q|^{-\frac{\mu}{n}} \int_{Q} |f(x)^{p} dx \right)^{\frac{1}{p}}, \tag{1.1}$$

supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

It is known that when $1 \leq p < \infty$ we have $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. When $\mu < 0$ or $\mu > n$, then $L^{p,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n)$, where Θ is the set of functions equivalent to zero on $\mathbb{R}_+$.

Classical Morrey space was introduced by Morrey [16]. Morrey spaces are widely used to investigate the local behavior of solutions of second-order quasi-liner elliptic partial differential equations. $L^{p,\mu}$- theory of fractional integral operator and its commutator is based on the following theorems.

Theorem A (Adams [1]) Let $0 < \alpha < n$, $0 \leq \mu < n$ and $1 \leq p < \frac{n-\mu}{\alpha}$.

(i) if $1 < p < \frac{n-\mu}{\alpha}$, then

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\mu}$$

is a necessary and sufficient condition for the boundedness of I_α from $L^{p,\mu}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.

(ii) If $p = 1$, then

$$1 - \frac{1}{q} = \frac{\alpha}{n-\mu}$$

is a necessary and sufficient condition for the boundedness of I_α from $L^{1,\mu}(\mathbb{R}^n)$ to $WL^{q,\mu}(\mathbb{R}^n)$.

Theorem B (Komori and Mizuhara [13]). Let $0 < \alpha < n$ and $1 < p < \frac{n}{\alpha}$, $0 < \mu < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\mu}$. Then, the following conditions are equivalent:

(a) $b \in BMO(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ is bounded from $L^{p,\mu}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.

The following theorem has been proved by Spanne but it was published in the paper of Petre [17].

Theorem C [17] Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $0 < \mu < n - \alpha p$, and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

Then:

(a) if $p > 1$, I_α is bounded from $L^{p,\mu}(\mathbb{R}^n)$ to $L^{q,\theta}(\mathbb{R}^n)$, if and only if $\theta = n\mu/(n - \alpha p)$ (i.e. $\mu/p = \theta/q$).

(b) if $p = 1$, I_α is bounded from $L^{1,\mu}(\mathbb{R}^n)$ to $WL^{q,\theta}(\mathbb{R}^n)$, if and only if $\theta = n\mu/(n-\alpha)$ (i.e. $\theta = \mu q$).

Theorem D (Shirai [18]) Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \mu < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\theta = n\mu/(n - \alpha)$ (i.e. $\mu/p = \theta/q$).

Then, the following conditions are equivalent:

(a) $b \in BMO(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ is bounded from $L^{p,\mu}(\mathbb{R}^n)$ to $L^{q,\theta}(\mathbb{R}^n)$.

2 Definitions, notations and auxiliary results

All of this study is based on the differential Gegenbauer operator

$$G \equiv G_\lambda = \left(x^2 - 1\right) \frac{d^2}{dx^2} + (2\lambda + 1)\, x \frac{d}{dx}, \;\; x \in [1, \infty), \;\; \lambda \in \left(0, \frac{1}{2}\right),$$

which was introduced in [3].

Generalized shift operator (GSO) associated with operator G has a form [5]

$$A_{chy}^{\lambda} f(chx) = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int_0^{\pi} f(chxchy - shxshy\cos\varphi)(\sin\varphi)^{2\lambda-1} d\varphi.$$

This operator has all properties of generalized shift operator listed in the works of Levitan ([14], [15]). Denote by $L_p(\mathbb{R}_+, G) \equiv L_{p,\lambda}(\mathbb{R}_+)$, $1 \le p \le \infty$ the space of $\mu_\lambda(x) = sh^{2\lambda}x$ measurable functions on $\mathbb{R}_+ = [0,\infty)$ with the finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left(\int_0^{\infty} |f(chx)|^p \, d\mu_\lambda(x)\right)^{\frac{1}{p}}, \ \ 1 \le p < \infty,$$

$$\|f\|_{L_{\infty,\lambda}(\mathbb{R}_+)} = \|f\|_{L_{\infty}(\mathbb{R}_+)} = ess\sup_{x\in\mathbb{R}_+} |f(chx)|,$$

$d\mu_\lambda(x) = sh^{2\lambda}xdx$.

Let's $\mu_E = |E|_\lambda = \int\limits_E d\mu_\lambda(x)$ from any measurable set $E \subset \mathbb{R}_+$. Denote by $WL_{p,\lambda}(\mathbb{R}_+), 1 \le p < \infty$, the weak space $L_{p,\lambda}(\mathbb{R}_+)$ of locally integrable functions $f(chx), \ x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+)} = \sup_{r>0} r \left|\{x \in \mathbb{R}_+ : |f(chx)| > r\}\right|_\lambda^{\frac{1}{p}}$$
$$= \sup_{r>0} r \left(\int_{\{x\in\mathbb{R}_+ : |f(chx)|>r\}} sh^{2\lambda}xdx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

Further, $A \lesssim B$ will mean that there exits constant C, which may depend on nonessential parameters such that $0 < A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we'll write $A \approx B$ and say that A and B are equivalent.

Let $H_r = (0,r) \subset \mathbb{R}_+$. Below, we'll need the following relation [12, lemma 2.3]

$$|H_r|_\lambda = \int\limits_0^r sh^{2\lambda}xdx \approx \left(sh\frac{r}{2}\right)^{\gamma},$$

where $0 < \lambda < \frac{1}{2}$

$$\gamma = \gamma_\lambda(r) = \begin{cases} 2\lambda+1, & 0 < r < 2, \\ 4\lambda, & 2 \le r < \infty. \end{cases}$$

By analogy with (1.1) in [7] the following definitions are introduced.

Definition 2.1. Let $1 \le p < \infty, 0 < \lambda < \frac{1}{2}$ and $0 \le \nu \le \gamma$. Denote by the Gegenbauer-Morrey (G-Morrey space) $L_{p,\lambda,\nu}(\mathbb{R}_+)$ space associate with the differential Gegenbauer operator G on the set of locally integrable functions $f(chx), \ x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} = \sup_{r>0, x\in\mathbb{R}_+} \left(|H_r|_\lambda^{-\frac{\nu}{\gamma}} \int\limits_{H_r} A_{chy}^{\lambda} |f(chx)|^p \, d\mu_\lambda(x)\right)^{\frac{1}{p}},$$

Therefore, by definition, we have

$$L_{p,\lambda,\nu}(\mathbb{R}_+) = \left(f \in L_{1,\lambda}^{loc}(\mathbb{R}_+) : \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} < \infty\right).$$

Let $1 \leq p \leq \infty$. In [8] it was proved, that $L_{p,\lambda,0}(\mathbb{R}_+) = L_{p,\lambda}(\mathbb{R}_+)$, when $\nu = 0$. If $\nu = \gamma$, then $L_{p,\lambda,\gamma}(\mathbb{R}_+) = L_\infty(\mathbb{R}_+)$, and, if $\nu < 0$ or $\nu > \gamma$, then $L_{p,\lambda,\nu}(\mathbb{R}_+) = \Theta(\mathbb{R}_+)$.

Definition 2.2. [7] Let $1 \leq p < \infty$ and $0 \leq \nu \leq \gamma$. Denote by $WL_{p,\lambda,\nu}(\mathbb{R}_+)$ the weak space $L_{p,\lambda,\nu}(\mathbb{R}_+)$ of locally integrable functions $f(chx),\ x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{WL_{p,\lambda,\nu}(\mathbb{R}_+)} = \sup_{r>0,x,t\in\mathbb{R}_+} r \sup \left(\left(sh\frac{t}{2}\right)^{-\nu} \left|\left\{y \in [0,t) : A^\lambda_{chy} \left|f(chx) > r\right|\right\}\right|\right)^{\frac{1}{p}}$$

$$= \sup_{r>0\,x,t\in\mathbb{R}_+} r \sup \left(\left(sh\frac{t}{2}\right)^{-\nu} \int\limits_{\left\{y\in[0,t):A^\lambda_{chy}|f(chx)|>r\right\}} d\mu_\lambda(x)\right)^{\frac{1}{p}}.$$

The following concept of G-BMO space is given in [9].

Definition 2.3. By definition,

$$BMO_G(\mathbb{R}_+) := \left\{f \in L^{loc}_{1,\lambda}(\mathbb{R}_+) : \|f\|_{BMO_G(R_+)} < \infty\right\}$$

where

$$\|f\|_{BMO_G(\mathbb{R}_+)} = \sup_{r>0,x\in\mathbb{R}_+} |H_r|^{-1}_\lambda \int\limits_{H_r} \left|A^\lambda_{chy} f(chx) - f_{H_r}(chx)\right| d\mu_\lambda(y)$$

is a seminorm, and

$$f_{H_r}(chx) = |H_r|^{-1}_\lambda \int\limits_{H_r} A^\lambda_{chy} f(chx) d\mu_\lambda(y).$$

In [5], the fractional maximal function M^α_G and fractional Gegenbauer integral J^α_G, $x \in \mathbb{R}_+$, are defined as follows:

$$M^\alpha_G f(chx) = \sup_{r\in\mathbb{R}_+} |H_r|^{\frac{\alpha}{\gamma}-1} \int\limits_{H_r} A^\lambda_{chy} |f(chx)| d\mu_\lambda(y),$$

$$M^0_G f(chx) \equiv M_G f(chx),$$

$$J^\alpha_G f(chx) = \int\limits_0^\infty \frac{A^\lambda_{chy} f(chx)}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} d\mu_\lambda(y), \quad 0 < \alpha < \gamma.$$

For $b \in L^{loc}_{1,\lambda}(\mathbb{R}_+)$, commutators of these operators are defined in [9] by the following formulas, respectively:

$$M^{b,\alpha}_G f(chx) = \sup_{r\in\mathbb{R}_+} |H_r|^{\frac{\alpha}{\gamma}-1} \int\limits_{H_r} \left|A^\lambda_{chy} f(chx) - b_{H_r}(chx)\right| A^\lambda_{chy} \left|f(chx)\right| d\mu_\lambda(y),$$

$$J^{b,\alpha}_G f(chx) = \int\limits_0^\infty \frac{[A^\lambda_{chy} f(chx) - b_{H_r}(chx)]}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} A^\lambda_{chy} f(chx) d\mu_\lambda(y).$$

Further we will need some auxiliary assertions.

Lemma 2.4. For any $1 < p < \infty$ the following relation [11, lemma 4.2]

$$\sup_{r>0, x\in\mathbb{R}_+} \left(\frac{1}{|H_r|_\lambda} \int\limits_{H_r} \left| A^\lambda_{chy} f(chx) - f_{H_r}(chx) \right|^p d\mu_\lambda(y) \right)^{\frac{1}{p}} \approx \|f\|_{BMO_G(\mathbb{R}_+)} .$$

is true.

Lemma 2.5. [10] Let $f \in BMO_G$. For any interval $H_r \subset \mathbb{R}_+$ and positive integer m, the following inequality

$$|f_{H_r}(chx) - f_{2^{\pm m}H_r}(chx)| \le 2m \|f\|_{BMO_G(\mathbb{R}_+)} .$$

is true.

Lemma 2.6. [7] For any $t \in [0, A] \subset \mathbb{R}_+$, the following $t \le sht \le e^A t$ is true for any $A > 0$.

3 Main results

The following theorems are analogues of the corresponding theorems A, B, C.

Theorem E [10, Adams type]. Let $\gamma_\lambda(r) = 2\lambda + 1$ if $0 < r < 2$ and $\gamma_\lambda(r) = 4\lambda$ if $2 \le r < \infty$, $0 < \alpha < \gamma_\lambda(r)$, $1 < p < \frac{\gamma_\lambda(r)}{\alpha}$, $0 < \nu < \gamma_\lambda(r) - \alpha_p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r)-\nu}$.

Then, $J^{b,\alpha}_G$ is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$, if and only if $b \in BMO_G(\mathbb{R}_+)$.

Theorem F [8, Spanne type.]. Let $0 < \alpha < \gamma_\lambda(r)$, $1 < p < \frac{\gamma_\lambda(r)}{\alpha}$, $0 < \nu < \gamma_\lambda(r) - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r)}$.

Then, J^α_G is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$, if and only if $\frac{\nu}{p} = \frac{\mu}{q}$.

Theorem G [7, Adams type]. Let $\gamma_\lambda(r) = 2\lambda + 1$, if $0 < r < 2$ and $\gamma_\lambda(r) = 4\lambda$, if $2 \le r < \infty$, $0 < \alpha < \gamma_\lambda(r)$, $0 < \nu < \gamma_\lambda(r) - \alpha p$ and $1 \le p < \frac{\gamma_\lambda(r)-\nu}{\alpha}$.

i) if $1 < p < \frac{\gamma_\lambda(r)-\nu}{\alpha}$, then

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r) - \nu}$$

is the necessary and sufficient condition for the boundedness J^α_G from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$.

(ii) If $p = 1 < \frac{\gamma_\lambda(r)-\nu}{\alpha}$, then

$$1 - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r) - \nu}$$

is the necessary and sufficient condition for the boundedness J^α_G from $L_{1,\lambda,\nu}(\mathbb{R}_+)$ to $WL_{q,\lambda,\nu}(\mathbb{R}_+)$.

The proof of the theorem for commutators $J^{b,\alpha}_G$ and $M^{b,\alpha}_G$ which is an analogue of Theorem D [18] is the aim of this paper.

Theorem 3.1 (Main theorem, Spanne type). Let $0 < \alpha < \gamma_\lambda(r)$, $1 < p < \frac{\gamma_\lambda(r)}{\alpha}$, $0 < \nu < \gamma_\lambda(r) - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r)}$ and $\frac{\nu}{p} = \frac{\mu}{q}$.

Then $J^{b,\alpha}_G$ is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$ if and only if $b \in BMO_G(\mathbb{R}_+)$.

Proof. (Sufficiency). Let $0 < \alpha < \gamma$, $1 < p < \frac{\gamma-\nu}{\alpha}$ and $b \in BMO_G(\mathbb{R}_+)$. The proof technique that is implemented here allows us not to consider each case separately when $r \in (0,2)$ or $r \in [2,\infty)$.

Denote
$E_\gamma = \begin{cases} (0,2) & \text{if } \gamma = 2\lambda+1 \\ [2,\infty) & \text{if } \gamma = 4\lambda \end{cases}.$

Let's estimate the commutator $J_G^{b,\alpha}$ above.

$$\left|J_G^{b,\alpha} f(chx)\right| \le \left(\int_0^r + \int_r^\infty\right) \frac{\left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y)$$
$$= J_1(x,r) + J_2(x,r). \tag{3.1}$$

Consider the integral $J_1(x,r)$.

$$J_1(x,r) = \int_0^r \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| A_{chy}^\lambda |f(chx)| \left(sh\frac{y}{2}\right)^{\alpha-\gamma} d\mu_\lambda(y)$$
$$\lesssim \sum_{k=0}^\infty \int_{r/2^{k+1}}^{r/2^k} \frac{\left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| A_{chy}^\lambda |f(chx)|}{(sh\frac{y}{2})^{\gamma-\alpha}} d\mu_\lambda(y)$$

$$\lesssim \sum_{k=0}^\infty \left(sh\frac{r}{2^{k+1}}\right)^\alpha \left(sh\frac{r}{2^{k+1}}\right)^{-\gamma} \int_0^{r/2^k} |A_{chy}^\lambda b(chx) - b_{H_r}(chx)| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \tag{3.2}$$

For $\delta > 0$ and $f \in L_\delta^{loc}(\mathbb{R}_+)$, denote

$$M_{G,\delta} f(chx) = \sup_{r>0} \left(\frac{1}{|H_r|_\lambda} \int_{H_r} |A_{chy}^\lambda f(chx)|^\delta d\mu_\lambda(y)\right)^{\frac{1}{\delta}}$$

Let $\delta < \varepsilon < 1,\ r + r' = rr'$ and $r = \frac{\varepsilon}{\delta} > 1$. By Hölder's inequality, we have

$$(sh\frac{r}{2})^{-\gamma} \int_0^r |A_{chy}^\lambda b(chx) - b_{H_r}(chx)| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y)$$
$$\le [(sh\frac{r}{2})^{-\gamma} \int_0^r |A_{chy}^\lambda b(chx) - b_{H_r}(chx)|^{\delta r'} d\mu_\lambda(y)]^{\frac{1}{\delta r'}}$$
$$\times [(sh\frac{r}{2})^{-\gamma} \int_0^r A_{chy}^\lambda |f(chx)|^{\delta r} d\mu_\lambda(y)]^{\frac{1}{\delta r}}$$

$$\lesssim \|b\|_{BMO_G(\mathbb{R}_+)} M_{G,\varepsilon} f(chx) \lesssim \|b\|_{BMO_G(\mathbb{R}_+)} M_G f(chx), \tag{3.3}$$

Since by the inverse Hölder's inequality [[11], Lemma 4.2], we have $M_{G,\varepsilon} f(chx) \le M_G f(chx)$.

Using (3.3) in (3.2), we get the following

$$J_1(x,r) \lesssim \|b\|_{BMO_G(\mathbb{R}_+)} M_G f(chx) \sum_{k=0}^{\infty} \left(sh\frac{r}{2^{k+1}} \right)^{\alpha}$$
$$\lesssim (sh\frac{r}{2})^{\alpha} \|b\|_{BMO_G(\mathbb{R}_+)} M_G f(chx) \sum_{k=0}^{\infty} 2^{-k\alpha}$$
$$\lesssim (sh\frac{r}{2})^{\alpha} \|b\|_{BMO_G(\mathbb{R}_+)} M_G f(chx).$$

By Hölder's inequality, we have

$$J_1(x,r) \approx (sh\frac{r}{2})^{\alpha} \sup_{r>0} (sh\frac{r}{2})^{-\gamma} \int_0^r A^{\lambda}_{chy} |f(chx)| d\mu_{\lambda}(y)$$
$$\lesssim (sh\frac{r}{2})^{\alpha} \sup_{r>0} (sh\frac{r}{2})^{-\gamma} \left(\int_0^r A^{\lambda}_{chy} |f(chx)|^p d\mu_{\lambda}(y) \right)^{\frac{1}{p}} \left(\int_0^r d\mu_{\lambda}(y) \right)^{\frac{1}{p'}}$$
$$\lesssim (sh\frac{r}{2})^{\alpha} \sup_{r>0} (sh\frac{r}{2})^{\frac{\nu}{p} + \frac{\gamma}{p'} - \gamma} \|f\|_{L_{p,\lambda,\nu}}(\mathbb{R}_+)$$
$$\lesssim (sh\frac{r}{2})^{\alpha} \sup_{r>0} (sh\frac{r}{2})^{\frac{\nu-\gamma}{p}} \|f\|_{L_{p,\lambda,\nu}}(\mathbb{R}_+), \ r \in E_{\gamma} \tag{3.4}$$

Consider the integral $J_2(x,r)$. According to Hölder's inequality, we have

$$J_2(x,r) \lesssim \left(\int_r^{\infty} A^{\lambda}_{chy} |f(chx)|^p \left(sh\frac{y}{2} \right)^{-\beta} d\mu_{\lambda}(y) \right)^{\frac{1}{p}}$$
$$\times \left(\int_r^{\infty} \frac{\left| A^{\lambda}_{chy} b(chx) - b_{H_r}(chx) \right|^{p'}}{\left(sh\frac{y}{2} \right)^{(\gamma-\alpha-\beta/p)p'}} d\mu_{\lambda}(y) \right)^{\frac{1}{p'}}$$
$$= J_{2.1}(x,r) \cdot J_{2.2}(x,r), \ \ r \in E_{\gamma}. \tag{3.5}$$

Let $\nu < \beta < \gamma - \alpha p$. Taking into account the inequality $shat \geq asht$ where $a \geq 1$, and Lemma 2.6, we get the following:

$$J_{2.1}(x,r) \leq \left(\sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} A^{\lambda}_{chy} |f(chx)|^p \left(sh\frac{y}{2} \right)^{-\beta} d\mu_{\lambda}(y) \right)^{\frac{1}{p}}$$
$$\lesssim \left(\sum_{j=0}^{\infty} \frac{\left(sh2^{j+1}\frac{r}{2} \right)^{\nu-\beta}}{\left(sh2^{j+1}\frac{r}{2} \right)^{\nu}} \int_0^{2^{j+1} r} A^{\lambda}_{chy} |f(chx)|^p d\mu_{\lambda}(y) \right)^{\frac{1}{p}}$$

$$\lesssim \left(\sum_{j=0}^{\infty} \frac{\left(2^{j+1} sh\frac{r}{2}\right)^{\nu-\beta}}{\left(sh 2^{j+1}\frac{r}{2}\right)^{\nu}} \int_0^{2^{j+1}r} A^{\lambda}_{chy} |f(chx)|^p d\mu_\lambda(y) \right)^{\frac{1}{p}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{\nu-\beta}{p}} \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} \left(\sum_{j=0}^{\infty} 2^{(j+1)(\nu-\beta)} \right)^{\frac{1}{p}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{\nu-\beta}{p}} \|f\|_{L_{p,\lambda,\nu}}, \ r \in E_\gamma \tag{3.6}$$

In the same way, taking into account the Lemma 2.4, we get the following for $J_{2.2}(x,r)$:

$$J_{2.2}(x,r) \lesssim \left(\sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1}r} \frac{\left|A^{\lambda}_{chy} b(chx) - b_{H_r}(chx)\right|^{p'}}{\left(sh\frac{y}{2}\right)^{(\gamma-\alpha-\beta/p)p'}} d\mu_\lambda(y) \right)^{\frac{1}{p'}}$$

$$\lesssim \left(\sum_{j=0}^{\infty} \left(sh 2^j \frac{r}{2}\right)^{(\beta/p+\alpha-\gamma)p'} \int_0^{2^{j+1}r} \left|A^{\lambda}_{chy} b(chx) - b_{H_r}(chx)\right|^{p'} d\mu_\lambda(y) \right)^{\frac{1}{p'}}$$

$$\lesssim \left(\sum_{j=0}^{\infty} \frac{\left(sh 2^j \frac{r}{2}\right)^{\gamma-(\gamma-\alpha-\beta/p)p'}}{\left(sh 2^j \frac{r}{2}\right)^{\gamma}} \int_0^{2^{j+1}r} \left|A^{\lambda}_{chy} b(chx) - b_{H_r}(chx)\right|^{p'} d\mu_\lambda(y) \right)^{\frac{1}{p'}}.$$

Taking into account the Minkowski inequality and the Lemma 2.5, we have

$$\left(\int_0^{2^{j+1}r} \left|A^{\lambda}_{chy} b(chx) - b_{H_r}(chx)\right|^{p'} d\mu_\lambda(y) \right)^{\frac{1}{p'}}$$

$$\le \left(\int_0^{2^{j+1}r} \left|A^{\lambda}_{chy} b(chx) - b_{2^{j+1}H_r}(chx)\right|^{p'} d\mu_\lambda(y) \right)^{\frac{1}{p'}}$$

$$+ \left(\int_0^{2^{j+1}r} \left|b_{H_r}(chx) - b_{2^{j+1}H_r}(chx)\right|^{p'} d\mu_\lambda(y) \right)^{\frac{1}{p'}}.$$

Then

$$J_{2.2}(x,r) \lesssim \left(sh\frac{r}{2}\right)^{\gamma/p'+\beta/p+\alpha-\gamma} \|b\|_{BMO_G(\mathbb{R}_+)} \left(\sum_{j=0}^{\infty} 2^{j(\gamma-(\gamma-\alpha-\beta/p)p'} \right)^{\frac{1}{p'}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha+(\beta-\gamma)/p} \|b\|_{BMO_G(\mathbb{R}_+)}, \ \ r \in E_\gamma. \tag{3.7}$$

Using (3.6) and (3.7) in (3.5), we get

$$J(x,r) \lesssim \left(sh\frac{r}{2}\right)^{\alpha+(\nu-\gamma)/p} \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} \ \|b\|_{BMO_G(\mathbb{R}_+)}, \ \ r \in E_\gamma \tag{3.8}$$

From (3.4), (3.8) and (3.1), we have

$$\left|J^{b,\alpha}_G f(chx)\right| \lesssim \left(sh\frac{r}{2}\right)^{\alpha} \sup_{r>0} (sh\frac{r}{2})^{(\nu-\gamma)/p} \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)}, \ \ r \in E_\gamma.$$

From here it follows that

$$\left\|J_G^{b,\alpha} f\right\|_{L_{q,\lambda,\mu}(\mathbb{R}_+)} = \sup_{r>0, x\in\mathbb{R}_+} \left(\left(sh\frac{r}{2}\right)^{-\mu} \int_0^r \left|J_G^{b,\alpha} f(chx\right|^q d\mu_\lambda(y)\right)^{\frac{1}{q}}$$
$$\lesssim \sup_{r>0} \left(sh\frac{r}{2}\right)^{\alpha} (sh\frac{r}{2})^{(\nu-\gamma)/p-\mu/q} \left(\int_0^r d\mu_\lambda(y)\right)^{\frac{1}{q}} \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)}$$
$$\lesssim \sup_{r>0} \left(sh\frac{r}{2}\right)^{\alpha} (sh\frac{r}{2})^{(\nu-\gamma)/p-\mu/q+\frac{\gamma}{q}} \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)}$$
$$\lesssim \sup_{r>0} \left(sh\frac{r}{2}\right)^{\alpha} (sh\frac{r}{2})^{-\gamma(1/p-1/q)} \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)}$$
$$= \sup_{r>0} (sh\frac{r}{2})^{\alpha} (sh\frac{r}{2})^{-\alpha} \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} \lesssim \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)}.$$

Necessity. Let $1 < p < \gamma/\alpha,\ f \in L_{p,\lambda,\nu}(\mathbb{R}_+)$ and $J_G^{b,\alpha}$ be bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$, that is

$$\left\|J_G^{b,\alpha} f\right\|_{L_{q,\lambda,\mu}} \lesssim \|f\|_{L_{p,\lambda,\nu}}. \tag{3.9}$$

The necessity of this theorem is proved in the same way as the necessity of theorem F. In order to do this, it is sufficient to replace the fractional integral J_G^{α} with the commutator $J_G^{b,\alpha}$. Therefore, we only provide a schematic proof of the necessity. In order to do this, we use the stretch operator f_t which was introduced in [7]. Let f be a positive and increasing function. The stretch operator f_t has a form

$$f\left(ch\left(th\frac{t}{2}\right)x\right) \le f_t(chx) \le f\left(ch\left(ch\frac{t}{2}\right)x\right),\ \ 0 < t < 2,$$
$$f\left(ch\left(th\frac{t}{2}\right)x\right) \le f_t(chx) \le f\left(ch\left(sh\frac{t}{2}\right)x\right),\ \ 2 \le t < \infty. \tag{3.10}$$

According to (3.10), it is proved that [see [7], for (3.37)]

$$\|f_t\|_{L_{p,\lambda,\nu}} \approx \sup_{x,r\in\mathbb{R}_+} \left(|H_r|_\lambda^{-\frac{\nu}{\gamma}} \int\limits_{H_r} A_{chy}^\lambda |f_t(chx)|^p d\mu_\lambda(y)\right)^{\frac{1}{p}}$$
$$\approx \left(sh\frac{t}{2}\right)^{\alpha+(\nu-\gamma)/p} \|f\|_{L_{p,\lambda,\nu}},\ t \in E_\gamma, \tag{3.11}$$

and also [see [13], for (3.48)]

$$\left\|J_G^{b,\alpha} f\right\|_{L_{q,\lambda,\mu}} \approx \left(sh\frac{t}{2}\right)^{(\gamma-\mu)/q} \left\|J_G^{b,\alpha} f_t\right\|_{L_{q,\lambda,\mu}},\ t \in E_\gamma. \tag{3.12}$$

Then, according to (3.9), from (3.11) and (3.12), it follows that

$$\left\|J_G^{b,\alpha} f\right\|_{L_{q,\lambda,\mu}} \approx \left(sh\frac{t}{2}\right)^{(\gamma-\mu)/q} \left\|J_G^{b,\alpha} f_t\right\|_{L_{q,\lambda,\mu}}$$
$$\lesssim \left(sh\frac{t}{2}\right)^{\nu/p-\mu/q} \|f\|_{L_{p,\lambda,\nu}},\ t \in E_\gamma.$$

Now, if $\frac{\nu}{p} - \frac{\mu}{q} > 0$ then $t \to 0$, $\left\| J_G^{b,\alpha} f \right\|_{L_{q,\lambda,\mu}} = 0$, for any $f \in L_{p,\lambda,\nu}(\mathbb{R}_+)$, and, if $\frac{\nu}{p} - \frac{\mu}{q} < 0$, then $t \to \infty$, $\left\| J_G^{b,\alpha} f \right\|_{L_{q,\lambda,\mu}} = 0$, for any $f \in L_{p,\lambda,\nu}(\mathbb{R}_+)$.

Therefore, $\frac{\nu}{p} = \frac{\mu}{q}$

We still need to prove $b \in BMO_G$.

Let χ_{H_r} - be a characteristic function for interval H_r. Using the property of symmetry of GSO, $A_{chx}^{\lambda} f(chy) = A_{chy}^{\lambda} f(chx)$, and inequality (3.9), we get

$$\frac{1}{|H_r|_\lambda} \int\limits_{H_r} \left| A_{chy}^{\lambda} b(chx) - b_{H_r}(chx) \right| d\mu_\lambda(y)$$

$$= \frac{1}{|H_r|_\lambda} \int\limits_{H_r} \frac{\left| A_{chy}^{\lambda} b(chx) - b_{H_r}(chx) \right| \left(sh\frac{y}{2}\right)^{\gamma-\alpha}}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} d\mu_\lambda(y)$$

$$\lesssim \frac{|H_r|^{1-\frac{\alpha}{\gamma}}}{|H_r|_\lambda} \int\limits_0^\infty \frac{\left| A_{chy}^{\lambda} b(chx) - b_{H_r}(chx) \right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} \chi_{H_r}(chy) d\mu_\lambda(y)$$

$$\lesssim \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} \int\limits_0^\infty \frac{\left| A_{chy}^{\lambda} b(chx) - b_{H_r}(chx) \right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} \chi_{H_r}(chy) d\mu_\lambda(y) \int\limits_0^\infty A_{chy}^{\lambda} \chi_{H_r} d\mu_\lambda(x)$$

$$= \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} \int\limits_0^\infty \left(\int\limits_0^\infty \frac{\left| A_{chx}^{\lambda} b(chy) - b_{H_r}(chx) \right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} A_{chx}^{\lambda} \chi_{H_r}(chy) d\mu_\lambda(x) \right) \chi_{H_r}(chy) d\mu_\lambda(y)$$

$$= \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} \int\limits_0^\infty J_G^{b,\alpha}(\chi_{H_r}(chy)) d\mu_\lambda(y)$$

$$\lesssim \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} \left(\int\limits_{H_r} d\mu_\lambda(y) \right)^{\frac{1}{q'}} \left(\int\limits_{H_r} \left(J_G^{b,\alpha}(\chi_{H_r}(chy) \right)^q d\mu_\lambda(y) \right)^{\frac{1}{q}}$$

$$\lesssim \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} |H_r|_\lambda^{\frac{1}{q'}} |H_r|_\lambda^{\frac{\mu}{\gamma q}} \left\| J_G^{b,\alpha}(\chi_{H_r}) \right\|_{L_{q,\lambda,\mu}} \lesssim |H_r|_\lambda^{-\frac{\alpha}{\gamma}-\frac{1}{q}} |H_r|_\lambda^{\frac{\mu}{\gamma q}} \|\chi_{H_r}\|_{L_{p,\lambda,\nu}}$$

$$\lesssim |H_r|_\lambda^{-\frac{\alpha}{\gamma}-\frac{1}{q}} |H_r|_\lambda^{(1-\frac{\nu}{\gamma})/p} |H_r|_\lambda^{\frac{\mu}{\gamma q}} = 1, \; r \in E_\gamma.$$

4 Commutator of fractional maximal operator

In this section, we find the necessary and sufficient conditions for the boundedness of $M_G^{b,\alpha}$ from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$.

Theorem 4.1. Let $0 < \alpha < \gamma_\lambda(r), 1 < p < \frac{\gamma_\lambda(r)}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma_\lambda(r)}$ and $\frac{\nu}{p} = \frac{\mu}{q}$. Then the commutator $M_G^{b,\alpha}$ is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$ if and only if $b \in BMO_G(\mathbb{R}_+)$.

Proof. Let $b \in BMO_G(\mathbb{R}_+)$. For fixed $x \in \mathbb{R}_+$ we have

$$\begin{aligned} J_G^{b,\alpha}(|f|)(chx) &= \int_0^\infty \frac{\left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\ &\geq \int_0^r \frac{\left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \qquad (4.1) \\ &\geq \left(sh\frac{r}{2}\right)^{\alpha-\gamma} \int_0^r \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\ &\gtrsim |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_r} \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y). \end{aligned}$$

By taking supremum with respect to $r > 0$ on both sides (4.1), we get

$$M_G^{b,\alpha} f(chx) \lesssim J_G^{b,\alpha}(|f|)(chx), \quad \forall x \in \mathbb{R}_+.$$

Then, for $b \in BMO_G(\mathbb{R}_+)$, by Theorem 3.1, we have

$$\left\| M_G^{b,\alpha} f \right\|_{L_{q,\lambda,\mu}} \lesssim \|f\|_{L_{p,\lambda,\nu}}.$$

Now, let $M_G^{b,\alpha}$ be bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$, then taking into account the symmetry of the GSO, we get

$$\begin{aligned} &\frac{1}{|H_r|_\lambda} \int_{H_r} \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| d\mu_\lambda(y) \\ &= \frac{1}{|H_r|_\lambda^2} \int_{H_r} \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| d\mu_\lambda(y) \int_{H_r} A_{chx}^\lambda \chi_{H_r}(chy) d\mu_\lambda(x) \\ &= \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} \int_{H_r} \left(\frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_{H_r} \left|A_{chx}^\lambda b(chy) - b_{H_r}(chy)\right| A_{chx}^\lambda \chi_{H_r}(chy) d\mu_\lambda(x) \right) d\mu_\lambda(y) \\ &\leq \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} \int_{H_r} M_G^{b,\alpha}\left(\chi_{H_r}(chy)\right) d\mu_\lambda(y) \\ &\leq \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} \left(\int_{H_r} d\mu_\lambda(y) \right)^{\frac{1}{q'}} \left(\int_{H_r} \left(M_G^{b,\alpha}\left(\chi_{H_r}(chy)\right)\right)^q d\mu_\lambda(y) \right)^{\frac{1}{q}} \\ &\leq \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} |H_r|_\lambda^{\frac{1}{q'}} \left\| M_G^{b,\alpha} \chi_{H_r} \right\|_{L_{q,\lambda,\mu}} |H_r|_\lambda^{\frac{\mu}{\gamma q}} \\ &\lesssim \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} |H_r|_\lambda^{\frac{1}{q'}} \|\chi_{H_r}\|_{L_{p,\lambda,\nu}} |H_r|_\lambda^{\frac{\mu}{\gamma q}} \\ &\lesssim \frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}} |H_r|_\lambda^{\frac{1}{q'}} |H_r|_\lambda^{\left(1-\frac{\nu}{\gamma}\right)\frac{1}{p}} |H_r|_\lambda^{\frac{\mu}{\gamma q}} = 1, \quad r \in E_r. \end{aligned}$$

Thus, $b \in BMO_G(\mathbb{R}_+)$.

Remark 4.2 Similar results can be found in the work [12].

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