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Asymptotics of the solution of a boundary value problem in a curvilinear trapezoid for a singularly perturbed quasilinear elliptic equation degenerated into a hyperbolic equation

Mahir M. Sabzaliev* · **I.M. Sabzalieva**

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Abstract. *In a curvilinear trapezoid we consider a boundary value problem for a second order singularly perturbed quasilinear elliptic equation degenerated into a hyperbolic equation. The asymptotic expansion of the generalized solution of the problem under consideration is constructed to within any positive degree of the small parameter and the remainder term is estimated.*

Keywords. asymptotics, boundary layer type function, remainder term

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1 Introduction and problem statement

When studying numerous real phenomenon with nonuniform transitions from one physical characteristics to another ones, it is necessary to study singularly perturbed boundary value problems (see e.i. [5], [17]). The study of singularly perturbed boundary value problem originally was carried out from various positions by A.N.Tikhonov [16], L.S.Pontryagin [9], M.I.Vishik and L.A.Lyusternik [18], [19], V.Vazov [21], S.A.Lomov [8], A.M.Il'in [6] and other scientists.

At present there exist various ways for constructing asymptotic expansions of solutions of singularly perturbed boundary value problems. The method developed by M.I.Vishik and L.A.Lyusternik (at present this method is called the Vishik-Lyusternik method [18], [19]) has on undoubted advantage. It is based an two ideas ascending to Prandtle: the idea of regulating stretching and the idea of boundary layer corrections.

The Vishik-Lyusternik method for constructing asymptotics in a small parameter takes the solutions of boundary value problem for linear differential equations also to some classes of nonlinear differential equations. In [20] M.I.Vishik and L.A.Lyusternik illustrated a method for constructing nonlinear differential equation on the following boundary value problem:

$$\varepsilon y'' + \varphi(x,y)\,y' - \psi(x,y) = 0, y(0) = A, y(1) = B.$$

* Corresponding author

M.M. Sabzaliev
Baku Biznes University, Baku, Azerbaijan;
Azerbaijan State Oil and Industry University;
E-mail: sabzalievm@mail.ru

I.M. Sabzalieva
Azerbaijan State Oil and Industry University;
Western Caspian University, Baku, Azerbaijan
E-mail: isabzaliyeva@mail.ru

There are a lot of works devoted to singularly perturbed elliptic differential equation. In [19] M.I.Vishik and L.A.Lyusternik have studied a boundary value problem of the form:

$$\varepsilon L_2 u + L_1 u = n(x,y), \quad u|_{\Gamma} = 0,$$

where $\varepsilon > 0$ is a small parameter, L_2 is a second order general elliptic operator, $L_1 \equiv \frac{\partial}{\partial x} - f(x,y)\,u$, Γ is a boundary of the plane domain. It should be noted that here the solution of the degenerated problem has singularities at the points of intersection of the characteristics of the degenerated equation and the boundary Γ.

In [7], V.Yu. Lunin in the bounded domain Ω with the smooth boundary Γ constructed asymptotics expansion of the solution to the following boundary value problem:

$$-\varepsilon^4 \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^3 - \varepsilon^2 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + F(x,u) = 0,\ u|_{\Gamma} = 0.$$

In the rectangle $D = \{(x,y)|\, 0 \le x \le 1,\ 0 \le y \le 1\}$ the boundary value problem was considered by the author of this paper in [10]

$$-\varepsilon^p \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)^p\right] - \varepsilon \Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + F(x,y,u) = 0,\ u|_{\Gamma} = 0.$$

The Dirichlet problem was studied by I.V.Denisov in the papers [2], [3] in a rectangle for the elliptic equation

$$\varepsilon^2 \Delta u = F(u,x,y,\varepsilon).$$

In the paper [4] the equation

$$\varepsilon \Delta u - v(x)\,u + u^p = 0,\ \ u \in H_0^1(\Omega)$$

was considered in the domain $\Omega \subset R^n$ and the solutions of the final energy of this equation as $\varepsilon \to 0$ were studied.

Boundary value problems in a rectangle, in a semi-infinite strip, in an infinite strip for the following quasilinear elliptic equation degenerated into a parabolic equation have been considered in the papers [11], [12], [13]:

$$-\varepsilon^p \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^p\right] - \varepsilon \Delta u + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} + au - f(x,y) = 0.$$

At present, the construction of the asymptotics of the solution to singularly perturbed boundary value problems is very relevant and in this connection, the studies in this direction are successfully developed. We indicate some of them: [14], [15], [22-24].

The cursory review conducted above shows that the studied boundary value problems for singularly perturbed equations were considered in the domains with smooth boundary, in a rectangular domain, in infinite semi-strips and infinite strips. It is a certain scientific interest to study boundary value problems in domains having the form of a curvilinear trapezoid. I would like to note that the idea of considering such boundary value problem once was given me by prof. M.I.Vishik.

In the present paper we consider a boundary value problem in a curvilinear trapezoid for a singularly perturbed quasilinear elliptic operator degenerated into a hyperbolic equation.

Let $x = \varphi_1(y)$, $x = \varphi_2(y)$ be rather smooth functions determined in $[a,b]$ and satisfy the following conditions:
a) $\varphi_1(y) < \varphi_2(y)$ for $y \in [a,b]$;
b) $\varphi_1(y) < y$ and $\varphi_2(y) > y$ for $y \in [a,b]$;
c) $\varphi_1(a) = a$, $\varphi_2(b) = b$;

d) $\varphi_1'(y) < 1,\ \varphi_2'(y) < 1$ for $y \in [a,b]$.

The functions $\varphi_1(y) = y - (y-a)^3$, $\varphi_2(y) = (b-y)^3 + y$ can be shown as examples of such functions.

We introduce the denotations:

$$\Gamma_1 = \{(x,y)|\, x = \varphi_1(y),\ a \le y \le b\}, \Gamma_2 = \{(x,y)|\, \varphi_1(y) \le x \le \varphi_2(y),\ y = b\},$$

$$\Gamma_3 = \{(x,y)|\, x = \varphi_2(y),\, a \le y \le b\}, \Gamma_4 = \{(x,y)|\, \varphi_1(y) \le x \le \varphi_2(y),\ y = a\}.$$

In $\Omega = \{(x,y)|\, \varphi_1(y) \le x \le \varphi_2(y),\, a \le y \le b\}$ we consider the following boundary value problem:

$$L_\varepsilon u \equiv -\varepsilon^p \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^p\right] - \varepsilon \Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + F(x,y,u) = 0, \quad (1.1)$$

$$u|_\Gamma = 0 \quad (1.2)$$

where $\varepsilon > 0$ is a small parameter, $p = 2k+1$, k is an arbitrary natural number, $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3 \bigcup \Gamma_4$, $F(x,y,u)$ is the given smooth function satisfying the following conditions:

$$\frac{\partial F(x,y,u)}{\partial u} \ge \gamma^2 > 0\ for\ (x,y,u) \in (\Omega \backslash \{(x,y) \in \Omega|\, x = y\}) \times (-\infty, +\infty). \quad (1.3)$$

In this case, the function $F(x,y,u)$ can depend on u as linearly, i.e.

$$F(x,y,u) = d(x,y)\,u - f(x,y),\ d(x,y) \ge \gamma^2 > 0.$$

It is known that for every fixed ε there exists a unique solution to the problem (1.1), (1.2) in the class $\overset{0}{W}_{p+1}(\Omega)$. Obviously, if $F(x,y,0) \equiv 0$ the problem (1.1), (1.2) has only a trivial solution. Therefore, we assume that

$$F(x,y,0) \ne 0\ for\ (x,y) \in \Omega. \quad (1.4)$$

Our goal is to construct the asymptotic expansion in a small parameter of the generalized solution to the boundary value problem (1.1), (1.2) from the space $\overset{0}{W}_{p+1}(\Omega)$. In this connection we conduct iteration processes.

2 Conducting the first iteration process and the solution of the degenerated problem

In the first iteration process, we will look for the approximate solution of the equation (1.1) in the form

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots + \varepsilon^n w_n, \quad (2.1)$$

where the functions $w_i(x,y);\ \ i = 0,1,\ldots,n$ are chosen so that

$$L_\varepsilon w = 0\left(\varepsilon^{n+1}\right). \quad (2.2)$$

Having substituted the expression of w from (2.1) to the equality (2.2), expanding nonlinear terms and regrouping the terms with the same degrees with respect to ε we obtain the following recurrently connected equations whose solutions are the functions $w_i;\ i = 0,1,\ldots,n$:

$$\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y} + F(x,y,w_0) = 0, \quad (2.3)$$

$$\frac{\partial w_i}{\partial x}+\frac{\partial w_i}{\partial y}+\frac{\partial F(x,y,w_0)}{\partial w_0}w_i=f_i;\ \ i=1,2,\ldots,n, \tag{2.4}$$

where $f_i(w_0,w_1,\ldots,w_{i-1})$ are the known functions dependent on $w_0,w_1,\ldots,w_{i-1}$ and their first and second derivatives. We can write formulas for f_i explicitly. However they are very bulky. Here we give formulas only for f_1,f_2 and f_3:

$$f_1=\Delta w_0,\ \ f_2=\Delta w_1-\frac{1}{2!}\cdot\frac{\partial^2 F(x,y,w_0)}{\partial w_0^2}w_1^2,$$

$$f_3=\Delta w_2-\frac{1}{2!}\cdot\frac{\partial^2 F(x,y,w_0)}{\partial w_0^2}2w_1w_2-\frac{1}{3!}\cdot\frac{\partial^3 F(x,y,w_0)}{\partial w_0^3}w_1^3.$$

Equation (2.3) is obtained from equation (1.1) for $\varepsilon=0$. (2.3) is said to be a degenerated equation corresponding to equation (1.1). The equations (2.3), (2.4) will be solved under the following boundary conditions

$$w_i|_{\Gamma_1}=0,\ w_i|_{\Gamma_4}=0; i=0,1,\ldots,n. \tag{2.5}$$

We have the following lemma.

Lemma 2.1. Let $\varphi_1(y)\in C^{2(n+1)}[a,b]$ and the function $F(x,y,u)\in C^{2(n+1)}(\Omega)$ satisfy the conditions (1.3), (1.4) and also the condition

$$\left.\frac{\partial^i f(x,y)}{\partial x^{i_1}\partial y^{i_2}}\right|_{x=y}=0; y\in[a,b],\ i=i_1+i_2;\ \ i=0,1,\ldots,2(n+1), \tag{2.6}$$

in the case of linear dependence of F on u and the conditions

$$F(x,y,u)|_{x=y}=0; y\in[a,b],\ u\in(-\infty,+\infty), \tag{2.7}$$

$$\left.\frac{\partial^i F(x,y,0)}{\partial x^{i_1}\partial y^{i_2}\partial u^{i_3}}\right|_{x=y}=0; y\in[a,b], i=i_1+i_2+i_3;\ \ i=0,1,\ldots,2(n+1), \tag{2.8}$$

in the case of nonlinear dependence of F on u. Then for $i=0$ the problem (2.3), (2.5) has a unique solution, moreover $w_0(x,y)\in C^{2(n+1)}(\Omega)$, and the condition

$$\left.\frac{\partial^i w_0(x,y)}{\partial x^{i_1}\partial y^{i_2}}\right|_{x=y}=0; y\in[a,b], i=i_1+i_2;\ \ i=0,1,\ldots,2(n+1) \tag{2.9}$$

is fulfilled.

Proof: The characteristical line of the equation (2.3) passing through the origin of coordinates divides the domain Ω into two parts:

$$\Omega_1=\{(x,y)|\,(x,y)\in\Omega, x<y\}, \Omega_2=\{(x,y)|\,(x,y)\in\Omega, x>y\}.$$

We look for the solution of problem (2.3), (2.5) for $i=0$ in the form:

$$w_0=\begin{cases} w_0^{(1)}, & (x,y)\in\Omega_1,\\ w_0^{(2)}, & (x,y)\in\Omega_2,\end{cases} \tag{2.10}$$

where $w_0^{(1)}$ and $w_0^{(2)}$ are the solutions of the following Cauchy problems:

$$\frac{\partial w_0^{(1)}}{\partial x}+\frac{\partial w_0^{(1)}}{\partial y}+F\left(x,y,w_0^{(1)}\right)=0,\ \ (x,y)\in\Omega_1;\ w_0^{(1)}\Big|_{x=\varphi_1(y)}=0, y\in[a,b], \tag{2.11}$$

$$\frac{\partial w_0^{(2)}}{\partial x} + \frac{\partial w_0^{(2)}}{\partial y} + F\left(x, y, w_0^{(2)}\right) = 0,$$

$$(x, y) \in \Omega_2;\ w_0^{(2)}\Big|_{y=a} = 0, x \in [\varphi_1(a), \varphi_2(a)]. \tag{2.12}$$

When $F(x, y, w_0)$ linearly depends on w_0, i.e. then the explicit representation of the solution of problem (2.3), (2.5) for $i = 0$ is of the form

$$w_0 = \begin{cases} \int\limits_{\psi(y_1)}^{x_1} f(\xi, \xi + y_1) \exp\left[\int\limits_{x_1}^{\xi} d(\tau, \tau + y_1)\, d\tau\right] d\xi, x_1 = x,\ y_1 = y - x,\ y > x, \\ \int\limits_{a}^{y_1} f(x_1 + \xi, \xi) \exp\left[\int\limits_{y_1}^{\xi} d(x_1 + \tau, \tau)\, d\tau\right] d\xi, x_1 = x - y,\ y_1 = y,\ x > y, \end{cases} \tag{2.13}$$

where $x_1 = \psi(y_1)$ is the solution of the equation $x_1 = \varphi_1(x_1 + y_1)$ with respect to x_1. The solvability of the equation $x_1 = \varphi_1(x_1 + y_1)$ with respect to x_1 follows from the first condition in d). Using formula (2.13) we prove that if condition (2.6) is fulfilled, then $w_0(x, y) \in C^{2(n+1)}(\Omega)$ and (2.9) is valid.

In the case of nonlinear dependence of $F(x, y, w_0)$ on w_0 the problems (2.11) and (2.12) are reduced to the following Cauchy problems for ordinary differential equations

$$\frac{dw_0^{(1)}}{dx_1} = -F\left(x_1, x_1 + y_1, w_0^{(1)}\right), w_0^{(1)}\Big|_{x_1=\psi(y_1)} = 0,\ x_1 = x,\ y_1 = y - x, y > x, \tag{2.14}$$

$$\frac{dw_0^{(2)}}{dx_2} = -F\left(x_1 + y_1, w_0^{(2)}\right), w_0^{(2)}\Big|_{y_1=a} = 0,\ \ x_1 = x - y,\ \ y_1 = y, x > y. \tag{2.15}$$

The existence of local solutions to problems (2.14) and (2.15) is obvious. Using condition (1.3) we can obtain a priori estimations for these local solutions. From the obtained a priori estimations it follows the possibility of continuous continuation of local solutions on Ω_1 and Ω_2.

To study differential properties of the solution to problem (2.3), (2.5) for $i = 0$ in the nonlinear case we reduce this problem to the following integral equations

$$w_0(x, y) =$$

$$= \begin{cases} -\int\limits_{\psi(y_1)}^{x_1} F(\tau, \tau + y_1, w_0(\tau, \tau + y_1))\, d\tau,\ \ x_1 = x,\ y_1 = y - x,\ y > x, \\ -\int\limits_{a}^{y_1} F(x_1 + \tau, \tau, w_0(x_1 + \tau, \tau))\, d\tau,\ \ x_1 = x - y,\ y_1 = y,\ x > y. \end{cases} \tag{2.16}$$

Using formula (2.16), we can prove that if conditions (2.7), (2.8) are fulfilled then $w_0(x, y) \in C^{2(n+1)}(\Omega)$ and (2.9) is fulfilled.

Lemma 2.1 is proved.

Problems (2.4), (2.5) for $i = 1, 2, \dots, n$, from which the functions $w_1, w_2, \dots, w_n$ will be determined, are linear. The solutions of these problems are written in the explicit from according to formula (2.13). But in it, instead of functions $d(x, y), f(x, y)$ we should take the functions $\frac{\partial F(x,y,w_0)}{\partial w_0}$ and $f_i(x, y)$, respectively. Note that the functions $w_i(x, y);\ i = 1, 2, \dots, n$ also will vanish for $y = x$ together with their derivatives.

Thus, we constructed the function w which is the approximate solution of (1.1) in the sense of (2.2) and satisfies the boundary conditions

$$w|_{\Gamma_1} = 0, w|_{\Gamma_4} = 0. \tag{2.17}$$

The constructed function w does not satisfy, generally speaking, homogeneous boundary conditions on Γ_2 and on Γ_3. To compensate the lost boundary conditions it is necessary to construct boundary layer functions near the boundaries Γ_2 and Γ_3.

3 Constructing boundary layer type functions near the boundaries Γ_2 and Γ_3

To construct a boundary layer function near the boundary Γ_3 at first we have to write a new decomposition of the operator L_ε near this line. We make a change of variables:

$$\varphi_2(y) - x = \varepsilon\tau, \quad y = y_1.$$

Let us consider the auxiliary function $r = \sum\limits_{j=0}^{n+1} \varepsilon^j r_j(\tau, y_1)$, where $r_j(\tau, y_1)$ are some smooth functions determined near the line $x = \varphi_2(y)$. Taking into account this change of variables, having substituted the expression of r in $L_\varepsilon r$, after some transformations we obtain a new decomposition of the operator L_ε in the coordinates (τ, y_1) in the form

$$L_{\varepsilon,1} r \equiv \varepsilon^{-1}\left\{ -\left[\delta_1^2(y_1)\frac{\partial}{\partial\tau}\left(\frac{\partial r_0}{\partial\tau}\right)^{2k+1} + \delta_2^2(y_1)\frac{\partial^2 r_0}{\partial\tau^2} + \delta_3^2(y_1)\frac{\partial r_0}{\partial\tau}\right] + \right.$$
$$+\sum_{j=1}^{n+1}\left[-(2k+1)\delta_1^2(y_1)\frac{\partial}{\partial\tau}\left(\left(\frac{\partial r_0}{\partial\tau}\right)^{2k}\frac{\partial r_j}{\partial\tau}\right) - \delta_2^2(y_1)\frac{\partial^2 r_j}{\partial\tau^2} - \right.$$
$$\left.\left. -\delta_3^2(y_1)\frac{\partial r_j}{\partial\tau} + h_j(r_0, r_1, \ldots, r_{j-1})\right] + 0\left(\varepsilon^{n+2}\right)\right\}. \tag{3.1}$$

Here h_j are the known functions dependent on and $\tau, y_1, r_0, r_1, \ldots, r_{j-1}$ their first and second derivatives. The functions $\delta_1^2(y_1), \delta_2^2(y_1), \delta_3^2(y_1)$ are determined by the following formulae:

$$\delta_1^2(y_1) = 1 + \left[\varphi_2'(y_1)\right]^{2k+2}, \delta_2^2(y_1) = \left[1 + \varphi_2'(y_1)\right]^2, \delta_3^2(y_1) = 1 - \varphi_2'(y_1).$$

We will look for a boundary layer type function near the boundary Γ_3 in the form

$$V = V_0(\tau, y_1) + \varepsilon V_1(\tau, y_1) + \varepsilon^2 V_2(\tau, y_1) + \cdots + \varepsilon^{n+1} V_{n+1}(\tau, y_1), \tag{3.2}$$

as the solution of the equation

$$L_{\varepsilon,1}(w + V) - L_{\varepsilon,1} w = 0\left(\varepsilon^{n+1}\right). \tag{3.3}$$

Expanding each function $w_i(\varphi_2(y_1) - \varepsilon\tau, y_1)$ in Taylor formula at the point $(\varphi_2(y_1), y_1)$, we obtain a new expansion in powers of ε of the function w in the coordinates (τ, y_1).

Having substituted the new expansion of w and expansion (3.2) of the function V in (3.1) we obtain that the functions $V_0, V_1, \ldots, V_{n+1}$ included in the right hand side of (3.2) are the solutions of the following equations, respectively:

$$AV_0 \equiv \delta_1^2(y_1)\frac{\partial}{\partial\tau}\left(\frac{\partial V_0}{\partial\tau}\right)^{2k+1} + \delta_2^2(y_1)\frac{\partial^2 V_0}{\partial\tau^2} + \delta_3^2(y_1)\frac{\partial V_0}{\partial\tau} = 0, \tag{3.4}$$

$$\begin{aligned}&\tfrac{\partial}{\partial\tau}\left\{\left[(2k+1)\delta_1^2(y_1)\left(\tfrac{\partial V_0}{\partial\tau}\right)^{2k} + \delta_2^2(y_1)\right]\tfrac{\partial V_j}{\partial\tau}\right\} + \\ &+\delta_3^2(y_1)\tfrac{\partial V_j}{\partial\tau} = H_j(\tau, y_1),\end{aligned} \tag{3.5}$$

where $H_j; j = 1, 2, \ldots, n+1$ are the known functions dependent on $\tau, y_1, V_0, V_1, \ldots, V_{j-1}$, their first and second derivatives. The boundary conditions for equations (3.4) and (3.5) are obtained from the requirements that the sum $w + V$ satisfies the boundary condition

$$(w+V)|_{\Gamma_3} = 0. \tag{3.6}$$

From (3.6) and taking into account that we look for $V_j;\ j = 0, 1, \ldots, n+1$ as boundary layer type functions we have

$$V_0|_{\tau=0} = \psi_0(y_1), \quad \lim_{\tau\to+\infty} V_0 = 0;\ \ y_1 \in [a,b], \tag{3.7}$$

$$V_j|_{\tau=0} = \psi_j(y_1), \quad \lim_{\tau\to+\infty} V_j = 0;\ j = 1, 2, \ldots, n+1;\ y_1 \in [a,b], \tag{3.8}$$

where $\psi_i(y_1) = -w_i(\varphi_2(y_1), y_1)$ for $i = 0, 1, \ldots, n$; $\psi_{n+1}(y_1) \equiv 0$.

The following lemma is valid.

Lemma 3.1. If $\varphi_2(y_1) \in C^{2(n+1)}[a,b]$, then for every fixed $y_1 \in [a,b]$ the problem (3.4), (3.7) has a unique solution that is infinitely differentiable with respect to τ and has continuous derivatives up to the $(2n+2)-$th order inclusively with respect to y_1 the function $V_0(\tau, y_1)$ and all its derivatives exponentially tend to zero as $\tau \to +\infty$.

Proof. At first we prove the uniqueness of the solution to the problem (3.4), (3.7). Let $V_0^{(1)}(\tau, y_1)$ and $V_0^{(2)}(\tau, y_1)$ be twice continuously differentiable solutions to problem (3.4), (3.7). Obviously, the function $H = V_0^{(1)} - V_0^{(2)}$ satisfies the boundary conditions $H|_{\tau=0} = 0$, $\lim\limits_{\tau\to+\infty} H = 0$.

Subtracting the equation $AV_0^{(2)} = 0$ from the equation $AV_0^{(1)} = 0$, we have:

$$\delta_1^2(y_1)\frac{\partial}{\partial\tau}\left[\left(\frac{\partial V_0^{(1)}}{\partial\tau}\right)^{2k+1} - \left(\frac{\partial V_0^{(2)}}{\partial\tau}\right)^{2k+1}\right] + \delta_2^2(y_1)\frac{\partial^2 H}{\partial\tau^2} + \delta_3^2(y_1)\frac{\partial H}{\partial\tau} = 0.$$

Multiplying the both hand sides of the last equality by $-H = -\left(V_0^{(1)} - V_0^{(2)}\right)$ and integrating by parts allowing for boundary conditions for H and also using the inequality $\left(a^{2k+1} - b^{2k+1}\right)(a-b) \geq 2^{-(2k+2)}(a-b)^{2k+2}$ we obtain

$$2^{-(2k+2)}\delta_1^2(y_1)\int_0^{+\infty}\left(\frac{\partial H}{\partial\tau}\right)^{2k+2} d\tau + \delta_2^2(y_1)\int_0^{+\infty}\left(\frac{\partial H}{\partial\tau}\right)^2 d\tau \leq 0.$$

From the last inequality and from the first boundary condition for H it follows that $V_0^{(1)} \equiv V_0^{(2)}$.

From the uniqueness of the solution to the problem it follows that in the case $y_1 = a$ and $y_1 = b$ the solution of the problem (3.4), (3.7) is predetermined by the identical zero.

We can prove that for every $y_1 \in (a,b)$ the solution to the problem (3.4), (3.7) in the parametric form is as follows:

$$\tau = \frac{2k+1}{2k}\cdot\frac{\delta_1^2(y_1)}{\delta_3^2(y_1)}\cdot\left(t_0^{2k} - t^{2k}\right) + \frac{\delta_2^2(y_1)}{\delta_3^2(y_1)}\ln\left|\frac{t_0}{t}\right|, \tag{3.9}$$

$$V_0 = -\cdot\frac{\delta_1^2(y_1)}{\delta_3^2(y_1)}\cdot t^{2k+1} - \frac{\delta_2^2(y_1)}{\delta_3^2(y_1)}\cdot t. \tag{3.10}$$

Here t is a parameter, $t_0(y_1)$ is a real root of the algebraic equation

$$t_0^{2k+1} + p(y_1)\,t_0 + q(y_1) = 0,$$

where

$$p(y_1) = \frac{\delta_2^2(y_1)}{\delta_1^2(y_1)} > 0, q(y_1) = \frac{\delta_3^2(y_1)}{\delta_1^2(y_1)}\psi_0(y_1), \quad y_1 \in (a,b).$$

Using the explicit form (3.9), (3.10) of the solution of problem (3.4), (3.7) it is easy to prove that $V_0(\tau, y_1)$ and all its derivatives exponentially decay as $\tau \to +\infty$.

Lemma 3.1 is proved.

We pass to the construction of functions $V_1, V_2, \ldots, V_{n+1}$ that are the solutions of equations (3.5) satisfying boundary conditions (3.8) for $j = 1, 2, \ldots, n+1$, respectively. For example, the function $V_1(\tau, y)$ is determined by the formula

$$V_1(\tau, y_1) = \left\{\int_0^\tau \tilde{H}(\xi_1, y_1)\exp\left[\delta_3^2(y_1)\int_0^{\xi_1}\frac{d\xi}{A(\xi, y_1)}\right]d\xi_1 + \psi_1(y_1)\right\}\times$$

$$\times\exp\left[-\delta_3^2(y_1)\int_0^\tau\frac{d\xi}{A(\xi, y_1)}\right], \tag{3.11}$$

where

$$\tilde{H}_1(\tau, y_1) = -\frac{1}{A(\xi, y_1)}\cdot\int_\tau^{+\infty} H_1(\xi, y_1)\,d\xi, \tag{3.12}$$

$$A(\tau, y_1) = (2k+1)\,\delta_1^2(y_1)\left(\frac{\partial V_0}{\partial\tau}\right)^{2k} + \delta_2^2(y_1). \tag{3.13}$$

Using (3.11)-(3.13) we introduce the following estimations for $V_1(\tau, y_1)$:

$$|V_1(\tau, y_1)| \le \left(C_1 + C_2\tau + C_3\tau^2\right)\exp(-\tau) \;\; for\; \delta_3^2(y_1) = \delta_2^2(y_1), \tag{3.14}$$

$$|V_1(\tau, y_1)| \le (C_4 + C_5\tau)\exp\left[-\frac{\delta_3^2(y_1)}{\delta_2^2(y_1)}\tau\right] + C_6\exp\left[-\frac{\delta_2^2(y_1)}{\delta_3^2(y_1)}\tau\right]$$

$$for\; \delta_3^2(y_1) \ne \delta_2^2(y_1), \tag{3.15}$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are positive constants.

We now estimate $\frac{\partial V_1(\tau, y_1)}{\partial\tau}$. From (3.11) we obtain the following formula

$$\frac{\partial V_1}{\partial\tau} = -\frac{1}{A(\tau, y_1)}\cdot\left[\int_\tau^{+\infty} H_1(\xi, y_1)\,d\xi + \delta_3^2(y_1)\,V_1\right]. \tag{3.16}$$

Following estimation (3.14) for $V_1(\tau, y_1)$ in the case $\delta_3^2(y_1) = \delta_2^2(y_1)$ from (3.16) we obtain:

$$\left|\frac{\partial V_1}{\partial\tau}\right| \le \frac{1}{\delta_2^2(y_1)}\left[C_7 + C_8\tau + \delta_3^2(y_1)\left(C_9 + C_{10}\tau + C_{11}\tau^2\right)\right]\exp(-\tau),$$

where $C_7, C_8, C_9, C_{10}, C_{11}$ are positive constants.

The estimation $\frac{\partial V_1(\tau, y_1)}{\partial\tau}$ for $\delta_3^2(y_1) \ne \delta_2^2(y_1)$ will have the same form as (3.15).

We can obtain estimations for the higher order derivatives $V_1(\tau, y_1)$ with respect to τ by differentiating successively the both hand sides of (3.16) with respect to τ and taking into account each time the estimations of the previous derivatives. This estimation also will be of the form (3.14) or (3.15) depending on the fact that $\delta_3^2(y_1) = \delta_2^2(y_1)$ or $\delta_3^2(y_1) \ne \delta_2^2(y_1)$.

The estimations for mixed derivatives are obtained in the same way. In principle, the construction of functions $V_2, V_3, \ldots, V_{n+1}$ is not different from the construction of the function V_1 . All the functions $V_j; j = 0, 1, \ldots, n+1$ tend to zero as $\tau \to +\infty$.

We multiply all the functions $V_j;\ \ 0,1,\ldots,n+1$ by the smoothing multiplier and for the obtained new functions we leave previous denotations. Note that due to the smoothing functions V does not violate the fulfillment of the first condition from (2.5) i.e. the sum $w+V$ in addition to condition (3.6) satisfies the condition

$$(w+V)|_{\Gamma_1}=0 \tag{3.17}$$

as well.

But the function V can violate the fulfillment of the second condition from (2.17) for the sum $w+V$. For the condition

$$(w+V)|_{\Gamma_4}=0 \tag{3.18}$$

to be fulfilled, it is necessary to ensure that all functions for $y=a$ vanish, e.i.

$$V_j|_{y=a}=0;\ \ j=0,1,\ldots,n+1. \tag{3.19}$$

Obviously, the condition (3.19) for $j=0$ is fulfilled. Assume that the function $F\left(x,y,u\right)$ satisfies the condition

$$\frac{\partial^k f\left(\varphi_2\left(a\right),a\right)}{\partial x^{k_1}\partial y^{k_2}}=0; k=k_1+k_2;\ \ k=0,1,\ldots,2n+1, \tag{3.20}$$

in the case of linear dependence of F or u and the condition

$$\frac{\partial^k F\left(\varphi_2\left(a\right),a,0\right)}{\partial x^{k_1}\partial y^{k_2}\partial u^{k_3}}=0; k=k_1+k_2+k_3;\ \ k=0,1,\ldots,2n+1, \tag{3.21}$$

in the case of nonlinear dependence of F on u. Then condition (3.19) will be fulfilled also for $j=1,2,\ldots,n+1$.

Thus, the constructed sum $w+V$ satisfies boundary conditions (3.6), (3.17), (3.18). But this sum does not satisfy, generally speaking, the homogeneous boundary condition on Γ_2. Therefore, it is necessary to construct the boundary layer function

$$\eta=\eta_0+\varepsilon\eta_1+\varepsilon^2\eta_2+\cdots+\varepsilon^{n+1}\eta_{n+1}, \tag{3.22}$$

near the boundary Γ_2 that must provide the fulfillment of the boundary condition

$$(w+V+\eta)|_{\Gamma_2}=0. \tag{3.23}$$

In addition, equations from which the functions $\eta_j; j=0,1,\ldots,n+1$ will be defined are obtained from the equality

$$L_{\varepsilon,2}\left(w+V+\eta\right)-L_{\varepsilon,2}\left(w+V\right)=0\left(\varepsilon^{n+1}\right) \tag{3.24}$$

where $L_{\varepsilon,2}$ is another decomposition of the operator L_ε near the boundary Γ_2.

Here, exchange of variables near the boundary Γ_2 is conducted by the formulas $x=x$, $b-y=\varepsilon\xi$. Expanding every function $w_i\left(x;b-\varepsilon\xi\right); i=0,1,\ldots,n$ and $V_j\left(\tau;b-\varepsilon\xi\right);$ $j=0,1,\ldots,n+1$ in Taylor formula, at the points (x,b) and (τ,b) from (3.24) we obtain the following equations:

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\eta_0}{\partial\xi}\right)^{2k+1}+\frac{\partial^2\eta_0}{\partial\xi^2}+\frac{\partial\eta_0}{\partial\xi}=0, \tag{3.25}$$

$$\frac{\partial}{\partial\xi}\left[\psi\left(x,\xi\right)\frac{\partial\eta_j}{\partial\xi}\right]+\frac{\partial\eta_j}{\partial\xi}=G_j;\ \ j=1,2,\ldots,n+1. \tag{3.26}$$

Here by $\psi(x,\xi)$ we denote the function

$$\psi(x,\xi) = (2k+1)\left(\frac{\partial \eta_0}{\partial \xi}\right)^{2k} + 1, \tag{3.27}$$

while G_j are the known functions.

From (3.23) and from the fact that we look for $\eta_j;\ j = 0, 1, \ldots, n+1$ as boundary layer type functions, we obtain the following boundary conditions for the equations (3.25), (3.26)

$$\left.\eta_j\right|_{\xi=0} = g_j(x), \quad \lim_{\xi\to+\infty} \eta_j = 0;\ \ j = 0, 1, \ldots, n, \tag{3.28}$$

$$\left.\eta_{n+1}\right|_{\xi=0} = g_{n+1}(x), \quad \lim_{\xi\to+\infty} \eta_{n+1} = 0, \tag{3.29}$$

where $g_j(x) = -\left.(w_j + v_j)\right|_{y=b};j = 0, 1, \ldots, n;\ \ g_{n+1}(x) = -\left.V_{n+1}\right|_{y=b}$.

The construction of functions η_j will be little different from the procedure for finding the functions $V_j; j = 0, 1, \ldots, n+1$. Therefore, we will not dwell on the construction of η_j in detail.

We multiply all the functions $\eta_0, \eta_1, \ldots, \eta_{n+1}$by the smoothing functions and for the obtained new functions leave previous denotations. Due to the smoothing multipliers, the functions $\eta_j; j = 0, 1, \ldots, n+1$ vanish in Γ_4. Therefore, it follows from (3.18) that the sum $w + V + \eta$ in addition to the condition (3.23) satisfies the condition

$$\left.(w + V + \eta)\right|_{\Gamma_4} = 0 \tag{3.30}$$

as well.

Using vanishing of functions $w_i(x,y);\ i = 0, 1, \ldots, n$ and their derivatives for $x = \varphi_2(b) = b,\ y = b$, we obtain $\left.\eta_j\right|_{x=\varphi_2(y)} = 0;\ \ j = 0, 1, \ldots, n+1$. Hence and from (3.6) it follows that the sum $w + V + \eta$ satisfies the boundary condition

$$\left.(w + V + \eta)\right|_{\Gamma_3} = 0 \tag{3.31}$$

as well.

Assume that $F(x, y, u)$ satisfies the condition

$$\frac{\partial^k f(\varphi_1(b), b)}{\partial x^{k_1} \partial y^{k_2}} = 0; k = k_1 + k_2; k = 0, 1, \ldots, 2n+1, \tag{3.32}$$

when the function F depends on u linearly, and the condition

$$\frac{\partial^k F(\varphi_1(b), b, 0)}{\partial x^{k_1} \partial y^{k_2} \partial u^{k_3}} = 0; k = k_1 + k_2 + k_3; k = 0, 1, \ldots, 2n+1, \tag{3.33}$$

when F depends on u nonlinearly. Then all the functions η_j will vanish for $x = \varphi_1(y)$: $\left.\eta_j\right|_{x=\varphi_1(y)} = 0;\ j = 0, 1, \ldots, n+1$.

Hence and from (3.17) it follows that the sum $w+V+\eta$ in addition to boundary conditions (3.23), (3.30), (3.31) will satisfy the boundary condition

$$\left.(w + V + \eta)\right|_{\Gamma_1} = 0. \tag{3.34}$$

Thus, we constructed the sum $\tilde{u} = w+V+\eta$ that following (3.23), (3.30), (3.31), (3.34) satisfies the boundary condition

$$\left.\tilde{u}\right|_{\Gamma} = 0. \tag{3.35}$$

Summing (2.2), (3.3), (3.24), we have that $\tilde{u}$ satisfies the equation

$$L_\varepsilon \tilde{u} = \varepsilon^{n+1} \Phi(\varepsilon, x, y), \tag{3.36}$$

where $\|\Phi(\varepsilon,x,y)\|_{L_2(\Omega)} \le C$, for any $\varepsilon \in [0,\varepsilon_0)$ and $c>0$ is independent of ε.

Having denoted $u-\tilde{u}=z$, we obtain the following asymptotic expansion in a small parameter of the problem (1.1), (1.2)

$$u=\sum_{i=1}^{n}\varepsilon^i w_i+\sum_{j=0}^{n+1}\varepsilon^j V_j+\sum_{j=0}^{n+1}\varepsilon^j \eta_j+z, \tag{3.37}$$

where z is a remainder term. Now we have to estimate the remainder term.

4 Estimation of the remainder term and formulation of the result

It follows from (2.1) and (3.35) that the remainder term z satisfies the boundary condition

$$z|_\Gamma=0. \tag{4.1}$$

Subtracting (3.36) from (1.1), we multiply the both hand sides of the equality by $z=u-\tilde{u}$ and integrate the obtained expression in the domain Ω

$$-\varepsilon^p\iint_\Omega \frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}\right)^p-\left(\frac{\partial \tilde{u}}{\partial x}\right)^p\right](u-\tilde{u})\,dxdy-$$
$$-\varepsilon^p\iint_\Omega \frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^p-\left(\frac{\partial \tilde{u}}{\partial y}\right)^p\right](u-\tilde{u})\,dxdy-$$
$$-\varepsilon\iint_\Omega \frac{\partial^2 z}{\partial x^2}zdxdy-\varepsilon\iint_\Omega \frac{\partial^2 z}{\partial y^2}zdxdy+\iint_\Omega \frac{\partial z}{\partial x}zdxdy+\iint_\Omega \frac{\partial z}{\partial y}zdxdy+$$
$$+\iint_\Omega [F(x,y,u)-F(x,y,\tilde{u})](u-\tilde{u})\,dxdy=\varepsilon^{n+1}\iint_\Omega \Phi(\varepsilon,x,y)\,zdxdy. \tag{4.2}$$

Transforming some terms in the left hand side of the equality (4.2) and applying the Green formula allowing for boundary condition (4.1); using the Lagrange formula for the difference $F(x,y,u)-F(x,y,\tilde{u})$ and following the condition (1.3) we have

$$\varepsilon^p\iint_\Omega\left[\left(\tfrac{\partial u}{\partial x}\right)^p-\left(\tfrac{\partial \tilde{u}}{\partial x}\right)^p\right]\left(\tfrac{\partial u}{\partial x}-\tfrac{\partial \tilde{u}}{\partial x}\right)dxdy+$$
$$+\varepsilon^p\iint_\Omega\left[\left(\tfrac{\partial u}{\partial y}\right)^p-\left(\tfrac{\partial \tilde{u}}{\partial y}\right)^p\right]\left(\tfrac{\partial u}{\partial y}-\tfrac{\partial \tilde{u}}{\partial y}\right)dxdy+$$
$$+\varepsilon\iint_\Omega\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]dxdy+\gamma^2\iint_\Omega z^2dxdy\le$$
$$\le\varepsilon^{n+1}\iint_\Omega \Phi(\varepsilon,x,y)\,zdxdy. \tag{4.3}$$

Using in the left hand side of (4.3) the inequality $(a^p-b^p)(a-b)\ge\frac{1}{2^{p+1}}(a-b)^{p+1}$, and in the right hand side $ab\le\delta^2a^2+\frac{1}{4\delta^2}b^2$, we have

$$\frac{\varepsilon^p}{2^{p+1}}\iint_\Omega\left[\left(\frac{\partial z}{\partial x}\right)^{p+1}+\left(\frac{\partial z}{\partial y}\right)^{p+1}\right]dxdy+\varepsilon\iint_\Omega\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]dxdy+$$

$$+\left(\gamma^2-\delta^2\right)\iint\limits_{\Omega} z^2 dxdy \le \varepsilon^{2(n+1)}\cdot\frac{1}{4\delta^2}\iint\limits_{\Omega}\left[\Phi\left(\varepsilon,x,y\right)\right]^2 dxdy.$$

Choosing δ^2 so small that $\gamma^2-\delta^2=C_1>0$, we obtain the following estimation for z :

$$\frac{\varepsilon^p}{2^{p+1}}\iint\limits_{\Omega}\left[\left(\frac{\partial z}{\partial x}\right)^{p+1}+\left(\frac{\partial z}{\partial y}\right)^{p+1}\right]dxdy+\varepsilon\iint\limits_{\Omega}\left[\left(\frac{\partial z}{\partial x}\right)^{2}+\left(\frac{\partial z}{\partial y}\right)^{2}\right]dxdy+$$

$$+C_1\iint\limits_{\Omega} z^2 dxdy \le C_2\varepsilon^{2(n+1)}, \tag{4.4}$$

where $C_1>0, C_2>0$ are the constants independent of ε. The results obtained in the paper can be generalized in the form of the following theorem.

Theorem 4.1 *Assume that* $F(x,y,u)\in C^{2(n+1)}(\Omega\times(-\infty,+\infty))$ *and the conditions* (1.3)*,* (1.4) (2.7)*,* (3.20)*,* (3.32) *are fulfilled in the case of linear dependence of* F *on* u *and the conditions* (2.8)*,* (3.21)*,* (3.33) *in the case of nonlinear dependence of* F *on* u*. Then, for the generalized solution of problem* (1.1)*,* (1.2) *the asymptotic representation* (3.37) *to valid, where the functions* w_i *are determined by the first iteration process,* V_j,η_j *are boundary layer type functions near the boundaries* Γ_3,Γ_2 *and are determined by the appropriate iteration processes, z is a remainder term and estimation* (4.4) *is valid for it.*

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