Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **45** (4), 145-154 (2025).

Boundedness of the B–maximal operator in B–total Morrey-Guliyev spaces

Ayhan Serbetci*

Received: 17.11.2024 / Revised: 24.08.2025 / Accepted: 27.09.2025

Abstract. *In this paper we consider a new kind of B–Morrey spaces called total B–Morrey-Guliyev spaces $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. We give basic properties of the spaces $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and study some embeddings into the space $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. We prove that the B–maximal operator M_γ is bounded on the B–total Morrey-Guliyev space $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$.*

Keywords. B–maximal operator, B–shift operator, B–total Morrey-Guliyev space.

Mathematics Subject Classification (2010): Primary 42B20, 42B25, 42B35

1 Introduction

Morrey spaces, introduced by Morrey [19], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for analyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [8], extend the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|f\|_{L_p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \le \lambda$ is equal to the maximum of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [1,2,7,9–12,18,20, 21].

* Corresponding author

A. Serbetci
Department of Mathematics, Ankara University, Ankara, Turkey
E-mail: serbetci@ankara.edu.tr

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^n x_i^2$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$, $n \geq 2$, $\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n; \, x_1 > 0, \dots, x_k > 0\}$, $1 \leq k \leq n$, $E(x, r) = \{y \in \mathbb{R}^n_{k,+} \, ; \, |x - y| < r\}$, $E_r = E(0, r)$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$.

For measurable $E \subset \mathbb{R}^n_{k,+}$ suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E_r|_\gamma = \omega(n, k, \gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i + 1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Denote by T^y the generalized shift operator (B–shift operator) acting according to the law

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left((x', y')_\beta, x'' - y''\right) \, d\nu(\beta),$$

where $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \leq i \leq k$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \dots, (x_k, y_k)_{\beta_k})$, $d\nu\,(\beta) = \prod_{i=1}^k \sin^{\gamma_i - 1} \beta_i \, d\beta_1 \dots d\beta_k$, $1 \leq k \leq n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k} \omega(2k, k, \gamma).$$

We remark that the generalized shift operator T^y is closely connected with the Bessel differential operator B (for example, $n = k = 1$ see [16], $n > 1$, $k = 1$ see [15] and $n, k > 1$ see [17] for details).

Let $f : \mathbb{R}^n_{k,+} \to \mathbb{R}$ be a locally integrable function, the B–maximal function $M_\gamma f$ associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^k \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \dots, \gamma_k > 0$$

is given by

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int\limits_{E_r} T^y |f(x)| (y')^\gamma dy.$$

The B–maximal operator M_γ introduced and studied by Guliyev in [3]. In [3] the strong $\left(L_{p,\gamma}, L_{p,\gamma}\right)$, $1 < p \leq \infty$ and weak $\left(L_{1,\gamma}, L_{1,\gamma}\right)$–boundedness of B–maximal operator M_γ were proved (see also [4,5]). The boundedness of the B–maximal operator M_γ on the B–Morrey spaces $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ is proved in [6]. In this paper we study the bundedness of the B–maximal operator M_γ on the total B–Morrey-Guliyev spaces $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$.

The paper is organized as follows. In Section 2 we present some basic properties of the B–total Morrey-Guliyev spaces $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and embeddings into these spaces. In Section 3 we prove the boundedness of the B–maximal operator M_γ on B–total Morrey-Guliyev space $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$. We obtain that the operator M_γ is bounded on $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ and from $L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ to weak $WL_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$.

2 Some embeddings into the B–total Morrey-Guliyev spaces

Definition 2.1 *Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm*

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} |f(x)|.$$

Let $1 \le p < \infty$. We denote by $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ the weak $L_{p,\gamma}$ space defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$ with the finite norms

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left| \{ x \in \mathbb{R}^n_{k,+} \ : \ |f(x)| > r \} \right|_\gamma^{1/p}.$$

The translation operator T^y generates the corresponding B–convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)[T^y g(x)](y')^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \le \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \le p,q \le r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Definition 2.2 *Let $0 < p < \infty$, $\lambda, \mu \in \mathbb{R}$, $[t]_1 = \min\{1,t\}$, $t > 0$. We denote by $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ the total Morrey space ($\equiv$ total B–Morrey-Guliyev space), associated with the Laplace-Bessel differential operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norm*

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}} = \sup_{t>0,\, x \in \mathbb{R}^n_{k,+}} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \left(\int_{E_t} T^y |f(x)|^p (y')^\gamma dy \right)^{1/p}.$$

We denote by $WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ the weak total B–Morrey-Guliyev space the set of all classes of locally integrable functions f with the finite norm

$$\|f\|_{WL_{p,(\lambda,\mu),\gamma}} = \sup_{t>0,\, x \in \mathbb{R}^n_{k,+}} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \sup_{r>0} r \left(\int_{\{y \in E_t : T^y |f(x)|^p > r\}} (y')^\gamma dy \right)^{1/p}.$$

We note that

$$L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \text{ and } \|f\|_{WL_{p,\lambda,\gamma}} \le \|f\|_{L_{p,\lambda,\gamma}}.$$

Let us note that if $\lambda = \mu$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ is the B–Morrey space, $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ is the weak B–Morrey space, see [4,6] and if $\mu = 0$, then $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,(\lambda,0),\gamma}(\mathbb{R}^n_{k,+})$ is the modified B–Morrey space, $W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,(\lambda,0),\gamma}(\mathbb{R}^n_{k,+})$ is the weak modified B–Morrey space, see [14].

We note that

$$L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+}),$$

and if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$, where $\Theta \equiv \Theta(\mathbb{R}^n_{k,+})$ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$.

Note that

$$L_{p,(0,0),\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,0}(\mathbb{R}^n_{k,+}) = L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+}),$$
$$WL_{p,(0,0),\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,0}(\mathbb{R}^n_{k,+}) = WL_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+}),$$
$$L_{p,(\lambda,\lambda),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}), \quad L_{p,(\lambda,0),\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}),$$
$$\|f\|_{WL_{p,(\lambda,\mu),\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}} \text{ and therefore } L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$$

and

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}),\ \mu \le \lambda \text{ and } \|f\|_{L_{p,\lambda,\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}, \tag{2.1}$$

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+}),\ \mu \le \lambda \text{ and } \|f\|_{L_{p,\mu,\gamma}} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}, \tag{2.2}$$

and if $\lambda < 0$ or $\lambda > Q$, then

$$L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta.$$

Lemma 2.1 *If $0 < p < \infty$ and $0 \le \mu \le \lambda \le Q$, then*

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\}.$$

Proof. Let $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and $0 \le \mu \le \lambda \le Q$. Then from (2.1) and (2.2) we have that $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$ and $\max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} \le \|f\|_{L_{p,(\lambda,\mu),\gamma}}$.

Now let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$. Then

$$\|f\|_{L_{p,(\lambda,\mu),\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \left(\int_{E_t} T^y|f(x)|^p (y')^\gamma dy\right)^{1/p}$$
$$= \max\left\{ \sup_{x\in\mathbb{R}^n_{k,+},0<t\le 1} \left(t^{-\lambda}\int_{E_t} T^y|f(x)|^p (y')^\gamma dy\right)^{1/p},\right.$$
$$\left.\sup_{x\in\mathbb{R}^n_{k,+},t>1} \left(t^{-\mu}\int_{E_t} T^y|f(x)|^p (y')^\gamma dy\right)^{1/p}\right\} \le \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\}.$$

Therefore, $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and the embedding $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ is valid.

Thus $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$ and $\max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} = \|f\|_{L_{p,(\lambda,\mu),\gamma}}$.

Corollary 2.1 *If $0 < p < \infty$, $0 \le \lambda \le Q$, then*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\gamma}}\right\}.$$

Lemma 2.2 *If* $0 < p < \infty$ *and* $0 \le \mu \le \lambda \le Q$*, then*

$$WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap WL_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{WL_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})} = \max\left\{\|f\|_{WL_{p,\lambda,\gamma}}, \|f\|_{WL_{p,\mu,\gamma}}\right\}.$$

Lemma 2.3 *If* $0 < p < \infty$*,* $0 \le \lambda_2 \le \lambda_1 \le Q$ *and* $0 \le \mu_1 \le \mu_2 \le Q$*, then*

$$L_{p,(\lambda_1,\mu_1),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,(\lambda_2,\mu_2),\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{L_{p,(\lambda_2,\mu_2),\gamma}} \le \|f\|_{L_{p,(\lambda_1,\mu_1),\gamma}}.$$

Proof. Let $f \in L_{p,(\lambda_1,\mu_1),\gamma}$, $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le Q$, $0 \le \mu_1 \le \mu_2 \le Q$. Then

$$\begin{aligned}\|f\|_{L_{p,(\lambda_2,\mu_2),\gamma}} = \max\Big\{ & \sup_{x\in\mathbb{R}^n_{k,+},\, 0<t\le 1} \left(t^{\lambda_1-\lambda_2} t^{-\lambda_1} \int_{E_t} T^y |f(x)|^p \,(y')^\gamma dy\right)^{1/p}, \\ & \sup_{x\in\mathbb{R}^n_{k,+},\, t\ge 1} \left(t^{\mu_1-\mu_2} t^{-\mu_1} \int_{E_t} T^y |f(x)|^p \,(y')^\gamma dy\right)^{1/p}\Big\} \\ & \le \|f\|_{L_{p,(\lambda_1,\mu_1),\gamma}}.\end{aligned}$$

Lemma 2.4 *If* $0 < p < \infty$*,* $0 \le \lambda \le Q$ *and* $0 \le \mu \le Q$*, then*

$$L_{p,(Q,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_\infty(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,(\lambda,Q),\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{L_{p,(\lambda,Q),\gamma}} \le \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty} \le \|f\|_{L_{p,(Q,\mu),\gamma}}.$$

Proof. Let $f \in L_\infty(\mathbb{R}^n_{k,+})$. Then for all $x \in \mathbb{R}^n_{k,+}$ and $0 < t \le 1$

$$\left(t^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p} \le \omega(n,k,\gamma)^{1/p} \,\|f\|_{L_\infty},\ 0 \le \lambda \le Q$$

and for all $x \in \mathbb{R}^n_{k,+}$ and $t > 1$

$$\left(t^{-Q} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p} \le \omega(n,k,\gamma)^{1/p} \,\|f\|_{L_\infty}.$$

Therefore $f \in L_{p,(\lambda,Q),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{p,(\lambda,Q),\gamma}} \le \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty}.$$

Let $f \in L_{p,(Q,\mu),\gamma}(\mathbb{R}^n_{k,+})$. By the Lebesgue's Theorem we have

$$\lim_{t\to 0} |E_t|_\gamma^{-1} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy = |f(x)|^p \quad a.e.\ x \in \mathbb{R}^n_{k,+}.$$

Then for a.e. $x \in \mathbb{R}^n_{k,+}$

$$|f(x)| = \left(\lim_{t\to 0} |E_t|_\gamma^{-1} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy \right)^{1/p}$$
$$\leq \omega(n,k,\gamma)^{-1/p} \times \sup_{x\in\mathbb{R}^n_{k,+},\, 0<t\leq 1} \left(t^{-Q} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy \right)^{1/p}$$
$$\leq \omega(n,k,\gamma)^{-1/p} \, \|f\|_{L_{p,(Q,\mu),\gamma}}.$$

Therefore $f \in L_\infty(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_\infty} \leq \omega(n,k,\gamma)^{-1/p} \, \|f\|_{L_{p,(Q,\mu),\gamma}}.$$

Corollary 2.2 *If* $0 < p < \infty$*, then*

$$L_{p,Q,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\succ L_\infty(\mathbb{R}^n_{k,+}) \subset\!\succ \widetilde{L}_{p,Q,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{\widetilde{L}_{p,Q,\gamma}} \leq \omega(n,k,\gamma)^{1/p} \, \|f\|_{L_\infty} \leq \|f\|_{L_{p,Q,\gamma}}.$$

Lemma 2.5 *If* $0 \leq \lambda < Q$, $0 \leq \mu < Q$, $0 \leq \alpha < Q-\lambda$ *and* $0 \leq \beta < Q-\mu$*, then for* $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q-\mu}{\beta}$

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\succ L_{1,(Q-\alpha,Q-\beta),\gamma}(\mathbb{R}^n_{k,+})$$

and for $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ *the following inequality*

$$\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} \leq \omega(n,k,\gamma)^{1/p'} \, \|f\|_{L_{p,(\lambda,\mu),\gamma}}$$

is valid, where $1/p + 1/p' = 1$.

Proof. Let $0 < \alpha < Q$, $0 \leq \lambda < Q$, $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q-\mu}{\beta}$. By the Hölder's inequality we have

$$\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} [t]_1^{\alpha-Q} \, [1/t]_1^{Q-\beta} \int_{E_t} T^y |f(x)| (y')^\gamma dy$$
$$\leq \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} [t]_1^{\alpha-Q} \, [1/t]_1^{Q-\beta} \left(\int_{E_t} (T^y |f(x)|)^p \, (y')^\gamma dy \right)^{1/p} |E_t|_\gamma^{1/p'}$$
$$\leq \omega(n,k,\gamma)^{1/p'} \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1 \, t^{-1} \right)^{-Q/p'} [t]_1^{\alpha - \frac{Q-\lambda}{p}} \, [1/t]_1^{Q-\beta-\frac{\mu}{p}}$$
$$\times \left([t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy \right)^{1/p}$$
$$\leq \omega(n,k,\gamma)^{1/p'} \, \|f\|_{L_{p,\lambda,\mu}} \sup_{t>0} \left([t]_1 \, t^{-1} \right)^{\frac{Q-\mu}{p}-\beta} \, [t]_1^{\alpha-\frac{Q-\lambda}{p}}.$$

Note that

$$\sup_{t>0} \left([t]_1 \, t^{-1} \right)^{\frac{Q-\mu}{p}-\beta} [t]_1^{\alpha-\frac{Q-\lambda}{p}} = \max\left\{ \sup_{0<t\leq 1} t^{\alpha-\frac{Q-\lambda}{p}}, \sup_{t>1} t^{\beta-\frac{Q-\mu}{p}} \right\} < \infty$$
$$\Longleftrightarrow \frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q-\mu}{\beta}.$$

Therefore $f \in L_{1,(Q-\alpha,Q-\beta),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} \leq \omega(n,k,\gamma)^{1/p'} \, \|f\|_{L_{p,(\lambda,\mu),\gamma}}.$$

From Lemma 2.5 we get the following

Corollary 2.3 *If* $0 \leq \mu \leq \lambda < Q$, $0 \leq \alpha < Q-\lambda$, *then for* $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q-\mu}{\alpha}$

$$L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+}) \subset\!\succ L_{1,Q-\alpha,\gamma}(\mathbb{R}^n_{k,+})$$

and for $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ *the following inequality*

$$\|f\|_{L_{1,Q-\alpha,\gamma}} \leq \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,(\lambda,\mu),\gamma}}$$

is valid.

Corollary 2.4 *If* $0 \leq \lambda < Q$ *and* $0 \leq \alpha < Q-\lambda$, *then for* $p = \frac{Q-\lambda}{\alpha}$

$$L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset L_{1,Q-\alpha,\gamma}(\mathbb{R}^n_{k,+}) \quad \text{and} \quad \|f\|_{L_{1,Q-\alpha,\gamma}} \leq \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,\lambda,\gamma}}.$$

Corollary 2.5 *If* $0 \leq \lambda < Q$ *and* $0 \leq \alpha < Q-\lambda$, *then for* $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q}{\alpha}$

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset L_{1,Q-\alpha,\gamma}(\mathbb{R}^n_{k,+}) \quad \text{and} \quad \|f\|_{L_{1,Q-\alpha,\gamma}} \leq \omega(n,k,\gamma)^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Remark 2.1 Note that in the case of the B–Morrey space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, Corollaries 2.2 and 2.4 were proved in [6, Lemmas 5 and 6] and in the case of the modified B–Morrey space $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, Corollaries 2.1 and 2.5 were proved in [14, Lemmas 5 and 6].

3 $L_{p,(\lambda,\mu),\gamma}$-boundedness of the B–maximal operator

In this section we study the $L_{p,(\lambda,\mu),\gamma}$-boundedness of the B–maximal operator (see [3])

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int_{E_r} T^y |f(x)| (y')^\gamma dy.$$

The following theorem proves the (B–Morrey spaces) $L_{p,\lambda,\gamma}$-boundedness of the B–maximal operator.

Theorem 3.1 *[6, Theorem 1]*
1. If $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $0 \leq \lambda < Q$, *then* $M_\gamma f \in WL_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \leq C_{1,\lambda,\gamma} \|f\|_{L_{1,\lambda,\gamma}},$$

where $C_{1,\lambda,\gamma}$ *depends only on* λ,γ,k *and* n.
2. If $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$,$0 \leq \lambda < Q$, *then* $M_\gamma f \in L_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *and*

$$\|M_\gamma f\|_{L_{p,\lambda,\gamma}} \leq C_{p,\lambda,\gamma} \|f\|_{L_{p,\lambda,\gamma}},$$

where $C_{p,\lambda,\gamma}$ *depends only on* p,λ,γ,k *and* n.

The following theorem proves the (modified B–Morrey spaces) $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$-boundedness of the B–maximal operator.

Theorem 3.2 *[14, Theorem 1]*
1. If $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \lambda < Q$, then $M_\gamma f \in W\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f\|_{W\widetilde{L}_{1,\lambda,\gamma}} \le C_{1,\lambda,\gamma}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}},$$

where $C_{1,\lambda,\gamma}$ is independent of f.
2.If $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$, $0 \le \lambda < Q$, then $M_\gamma f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le C_{p,\lambda,\gamma}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

where $C_{p,\lambda,\gamma}$ depends only on p,λ,γ,k and n.

The following theorem is the main result of this paper. We prove the (B–total Morrey-Guliyev spaces) $L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$-boundedness of the B–maximal operator

Theorem 3.3 *1. If $f \in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \mu \le \lambda < Q$, then $M_\gamma f \in WL_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \le C_{1,\lambda,\gamma}\,\|f\|_{L_{1,(\lambda,\mu),\gamma}},$$

where $C_{1,\lambda,\gamma}$ depends only on λ, μ, γ, k and n.
2. If $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$, $0 \le \mu \le \lambda < Q$, then $M_\gamma f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}} \le C_{p,\lambda,\gamma}\|f\|_{L_{p,(\lambda,\mu),\gamma}},$$

where $C_{p,\lambda,\gamma}$ depends only on $p, \lambda, \mu, \gamma, k$ and n.

Proof. Let $f \in L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$ and $0 \le \mu \le \lambda < Q$. Then from Lemma 2.2 and Corollary 3.3 we obtain

$$\begin{aligned}\|M_\gamma f\|_{WL_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})} &= \max\left\{\|M_\gamma f\|_{WL_{1,\lambda,\gamma}}, \|M_\gamma f\|_{WL_{1,\mu,\gamma}}\right\} \\ &\lesssim \max\left\{\|f\|_{L_{1,\lambda,\gamma}}, \|f\|_{L_{1,\mu,\gamma}}\right\} = \|f\|_{L_{1,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})}.\end{aligned}$$

Let now $f \in L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ and $0 \le \mu \le \lambda < Q$. Then from Lemma 2.1 and Corollary 3.3 we obtain

$$\begin{aligned}\|M_\gamma f\|_{L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})} &= \max\left\{\|M_\gamma f\|_{L_{p,\lambda,\gamma}}, \|M_\gamma f\|_{L_{p,\mu,\gamma}}\right\} \\ &\lesssim \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} = \|f\|_{L_{p,(\lambda,\mu),\gamma}(\mathbb{R}^n_{k,+})}.\end{aligned}$$

Remark 3.1 Note that in the case of $\lambda = \mu$ and $\lambda = 0$ from Theorem 3.3 we get Theorem 3.1 and Theorem 3.2, respectively.

Acknowledgements. The author thanks the referees for his/her careful reading the paper and useful comments.

References

1. Abasova, G.A. Omarova, M.N.: *Commutator of anisotropic maximal function with BMO functions on anisotropic total Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **43**(1), 3-15 (2023).
2. Celik, S., Akbulut, A., Omarova, M.N.: *Characterizations of anisotropic Lipschitz functions via the commutators of anisotropic maximal function in total anisotropic Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci **45**(1) Mathematics, 25-37 (2025).
3. Guliyev, V.S.: *Sobolev theorems for the Riesz B-potentials,* (Russian) Doklady Academy Nauk Russia **358**(4), 450-451 (1998).
4. Guliyev, V.S.: *Sobolev theorems for anisotropic Riesz–Bessel potentials on Morrey-Bessel spaces,* (Russian) Doklady Academy Nauk Russia **367**(2), 155-156 (1999).
5. Guliyev, V.S.: *On maximal function and fractional integral, associated with the Bessel differential operator,* Math. Ineq. Appl. **6**(2), 317-330 (2003).
6. Guliyev, V.S., Hasanov, J.J.: *Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces*, J. Math. Anal. Appl. **347**, 113-122 (2008).
7. Guliyev, V.S., Hasanov J.J., Zeren Y.: *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. **5**(4), 491-506 (2011).
8. Guliyev, V.S.: *Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal. **16**(4), 1509-1524 (2022).
9. Guliyev, V.S., Isayev, F.A., Serbetci, A.: *Boundedness of the anisotropic fractional maximal operator in total anisotropic Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **44**(1), 41-50 (2024).
10. Guliyev, V.S.: *Characterizations of Lipschitz functions via the commutators of maximal function in total Morrey spaces*, Math. Meth. Appl. Sci. **47**(11), 8669-8682 (2024).
11. Guliyev, V.S.: *Characterizations for the fractional maximal operator and its commutators on total Morrey spaces*, Positivity **28**, Article number: 51 (2024).
12. Guliyev, V.S.: *Characterizations of commutators of the maximal function in total Morrey spaces on stratified Lie groups*, Anal. Math. Phys. **15**, Article number: 42 (2025).
13. Guliyev, V.S., Akbulut, A., Isayev, F.A., Serbetci, A.: *Commutator of maximal function with BMO functions on total mixed Morrey spaces*, Journal of Contemporary Applied Mathematics **16**(1), 119-138 (2026).
14. Hasanov, J.J., Badalov, X.A., Fleydanli, A.E.: *Necessary and sufficient conditions for the boundedness of B-Riesz potential in modified B-Morrey spaces*, Journal of Contemporary Applied Mathematics **4**(2), 18-41 (2014).
15. Kipriyanov, I.A.: *Fourier-Bessel transformations and imbedding theorems*, Trudy Math. Inst. Steklov 89, 130-213 (1967).
16. Levitan, B.M.: *Expansion in Fourier series and integrals with Bessel functions*, (Russian) Uspehi Matem. Nauk (N.S.) **6**(2)(42), 102-143 (1951).
17. Lyakhov, L.N.: *Multipliers of the mixed Fourier-Bessel transform*, (Russian) Proc. Steklov Inst. Math. **214**(3), 234-249 (1997).
18. Mammadov, Y.Y., Guliyev, V.S., Muslimova, F.A.: *Commutator of the maximal function in total Morrey spaces for the Dunkl operator on the real line*, Azerb. J. Math. **15**(2), 85-104 (2025).
19. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
20. Omarova, M.N.: *Commutators of parabolic fractional maximal operators on parabolic total Morrey spaces*, Math. Meth. Appl. Sci. **48**(11), 11037-11044 (2025).
21. Omarova, M.N.: *Commutators of anisotropic maximal operators with BMO functions on anisotropic total Morrey spaces*, Azerb. J. Math. **15**(2), 150-162 (2025).

22. Stempak, K.: *Almost everywhere summability of Laguerre series*, Studia Math. **100**(2), 129-147 (1991).
23. Trimeche, K.: *Inversion of the Lions transmutation operators using generalized wavelets*, Applied and Computational Harmonic Analysis **4**, 97-112 (1997).