

## A note on singular integrals along higher dimensional subvarieties

Ahmad Al-Salman\*, Badriya Al-Azri

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**Abstract.** Suppose that  $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  is a homogeneous function of degree zero in the sense (1.5) and satisfying the cancellation property (1.4). Under certain convexity assumptions on the mapping  $\theta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , we prove that the singular integral operator

$$T_{\theta, \Omega} f(x, x_{n+1}) = p.v. \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} f(x - u - v, x_{n+1} - \theta(|u|, |v|)) \frac{\Omega(u', v')}{|u|^n |v|^n} dudv$$

is bounded on  $L^p(\mathbb{R}^n \times \mathbb{R}^n)$ ,  $1 < p < \infty$  provided the kernel function  $\Omega$  is in  $L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ .

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### 1 Introduction and Statement of Results

For  $n \geq 2$ , let  $\mathbb{S}^{n-1}$  be the unit sphere in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . Let  $d\sigma$  be the induced normalized Lebesgue measure on  $\mathbb{S}^{n-1}$ . For  $y \neq 0$ , let  $y' = |y|^{-1}y \in \mathbb{S}^{n-1}$  and  $\Omega \in L^1(\mathbb{S}^{n-1})$  be a homogeneous function of degree zero on  $\mathbb{R}^n$  satisfying

$$\int_{\mathbb{S}^{n-1}} \Omega(y') d\sigma(y') = 0. \quad (1.1)$$

Let  $\phi : [0, \infty) \rightarrow \mathbb{R}$  be a smooth function. Consider the singular integral operator  $S_{\phi, \Omega}$  given by

$$S_{\phi, \Omega} f(x, x_{n+1}) = p.v. \int_{\mathbb{R}^{n+1}} f(x - y, x_{n+1} - \phi(|y|)) \frac{\Omega(y')}{|y|^n} dy. \quad (1.2)$$

\* Corresponding author

Ahmad Al-Salman  
Sultan Qaboos University, College of Science, Department Mathematics, Muscat, Sultanate of Oman  
Department of Mathematics, Yarmouk University, Irbid, Jordan  
E-mail: alsalman@squ.edu.om, alsalman@yu.edu.jo

Badriya Al-Azri  
Sultan Qaboos University, College of Science, Department Mathematics, Muscat, Sultanate of Oman  
E-mail: ab8500703@gmail.com

If  $\phi(t) \equiv 0$ , then the operator  $S_{\varphi, \Omega}$  is the well known classical Calderón-Zygmund singular integral operator  $S_{\Omega}$  given by

$$S_{\Omega}f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{\Omega(y')}{|y|^n} dy. \quad (1.3)$$

In their fundamental papers [9] and [8], Calderón and Zygmund proved that the operator  $S_{\Omega}$  is bounded on  $L^p$  ( $1 < p < \infty$ ) provided that  $\Omega \in L(\log L)(\mathbb{S}^{n-1})$ . Moreover, it was shown in [8] that  $L(\log L)(\mathbb{S}^{n-1})$  is the most desirable size condition in the sense that  $S_{\Omega}$  can fail to boundedness on  $L^p$  if  $\Omega$  is assumed to be in  $L(\log L)^{1-\varepsilon}(\mathbb{S}^{n-1}) \setminus L(\log L)(\mathbb{S}^{n-1})$  for some  $\varepsilon > 0$ . For further results concerning the operator  $S_{\Omega}$ , we cite, among others, [1], [2], [6], [7], [10], [11], [12], [15], [16], [17] and references to their in.

In 1996, Kim, Winger, Wright and Ziesler [13] studied the  $L^p(\mathbb{R}^{n+1})$  boundedness of  $S_{\varphi, \Omega}$  when  $\Omega \in C^{\infty}(\mathbb{S}^{n-1})$  for ( $1 < p < \infty$ ). In [7], Al-Salman and Pan established  $L^p(\mathbb{R}^{n+1})$  ( $1 < p < \infty$ ) the boundedness of  $S_{\varphi, \Omega}$  under the condition  $\Omega \in L(\log L)(\mathbb{S}^{n-1})$ . For more results on this topic, we advise readers to consult [14], [11], [12], among other.

Let  $\theta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a smooth mapping. Suppose that  $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  is satisfied

$$\int_{\mathbb{S}^{n-1}} \Omega(u', \cdot) d\sigma(u') = \int_{\mathbb{S}^{n-1}} \Omega(\cdot, v') d\sigma(v') = 0 \quad (1.4)$$

and

$$\Omega(tu, sv) = \Omega(u, v) \quad (1.5)$$

for any  $t, s > 0$ . Consider the singular integral operator

$$T_{\theta, \Omega}f(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} f(x-u-v, x_{n+1} - \theta(|u|, |v|)) \frac{\Omega(u', v')}{|u|^n |v|^n} dudv. \quad (1.6)$$

In order to state our results in this paper, we cite the following remarks:

(i) When  $\theta = 0$ , then the corresponding operator  $T_{\theta, \Omega}$  reduces to the operator

$$T_{\Omega}f(x) = \text{p.v.} \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x-u-v) \frac{\Omega(u', v')}{|u|^n |v|^n} dudv; \quad (1.7)$$

which was introduced in [3]. In [3], Al-Salman proved that the operator  $T_{\Omega}$  is bounded on  $L^p(\mathbb{R}^n)$ ,  $1 < p < \infty$ , provided that  $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ , i.e.,

$$\int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} |\Omega(u, v)| (\log 2 + |\Omega(u, v)|)^2 d\sigma(u) d\sigma(v) < \infty. \quad (1.8)$$

It is worth pointing that,

$$L(\log^+ L)^s(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \subset L(\log^+ L)^r(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \quad \text{whenever } r < s$$

and

$$L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \subsetneq L(\log^+ L)^r(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \subsetneq L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$$

whenever  $q > 1$  and  $r \geq 1$ . In addition, it was pointed out in [3] that the condition  $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  is nearly optimal. Namely, there exists an  $\Omega$  in  $L(\log L)^{2-\varepsilon}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  for some  $\varepsilon > 0$  such that  $T_{\gamma, \Omega}$  is not bounded on  $L^p(\mathbb{R}^n)$ .

(ii) When  $\theta$  is separable in the sense that  $\theta(s, t) = \varphi_1(s) + \varphi_2(t)$  and  $\Omega(x, y) = \Omega_1(x) \Omega_2(y)$  for some  $\Omega_1, \Omega_2 \in L^1(\mathbb{S}^{n-1})$  where  $\varphi_1$  and  $\varphi_2$  are suitable real valued functions, then

the special operator  $T_{\varphi_1, \varphi_2, \Omega} = T_{\theta, \Omega}$  is a composition of two singular integral operators. Namely,

$$T_{\varphi_1, \varphi_2, \Omega}(f)(x, x_{n+1}) = S_{\varphi_1, \Omega_1} \circ S_{\varphi_2, \Omega_2}(f)(x, x_{n+1}) \quad (1.9)$$

where  $S_{\varphi_1, \Omega_1}$  is the operator given by (1.2) with  $\varphi$  replaced by  $\varphi_1$  and  $\Omega$  replaced by  $\Omega_1$ . Similarly, the operator  $S_{\varphi_2, \Omega_2}$ .

(iii) By Theorem 1.2 in [7], it follows that if  $\Omega_1, \Omega_2 \in L(\log L)(\mathbb{S}^{n-1})$  and  $\varphi_1$  and  $\varphi_2$  are  $\mathcal{C}^2$ , convex, and increasing functions satisfying  $\varphi_1(0) = \varphi_2(0) = 0$ , then the operators  $S_{\varphi_1, \Omega_1}$  and  $S_{\varphi_2, \Omega_2}$  are bounded on  $L^p(\mathbb{R}^{n+1})$  for all  $p \in (1, \infty)$ . Hence, by this and (1.9), we deduce that the operator  $T_{\varphi_1, \varphi_2, \Omega}$  in (1.9) is bounded on  $L^p$  for all  $1 < p < \infty$ .

(iv) For general  $\Omega$  and  $\theta$ , it is not visible if the operator  $T_{\theta, \Omega}$  can be written as a composition of two operators of convolution type.

In light of above remarks, it is natural to investigate the  $L^p$  boundedness of the operator  $T_{\theta, \Omega}$  for various functions  $\theta$  under the condition  $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ . Our results are the following:

**Theorem 1.1.** *Suppose that  $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  satisfies (1.4)-(1.5). If  $\theta(s, t) = \varphi_1(s) + \varphi_2(t)$  where  $\varphi_1$  and  $\varphi_2$  are  $\mathcal{C}^2$ , convex, and increasing functions satisfying  $\varphi_1(0) = \varphi_2(0) = 0$ , then  $T_{\theta, \Omega}$  is bounded on  $L^p(\mathbb{R}^{n+1})$  for  $1 < p < \infty$ .*

**Theorem 1.2.** *Suppose that  $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  satisfies (1.4)-(1.5). If  $\theta(s, t) = \varphi_1(s) + \varphi_2(t)$  where (i)  $\varphi_1$  is  $\mathcal{C}^2$ , convex, and increasing function with  $\varphi_1(0) = 0$  and (ii)  $\varphi_2$  is a polynomial mapping with  $\varphi_2(0) = 0$ , then  $T_{\theta, \Omega}$  is bounded on  $L^p(\mathbb{R}^{n+1})$  for  $1 < p < \infty$ . The  $L^p$  may depend on the degree of the polynomial  $\varphi_2$  but it is independent of its coefficients.*

**Theorem 1.3.** *Suppose that  $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  satisfies (1.4)-(1.5). If  $\theta(s, t) = \varphi_1(s) + \varphi_2(t)$  where  $\varphi_1$  and  $\varphi_2$  are polynomial mappings with  $\varphi_1(0) = \varphi_2(0) = 0$ , then  $T_{\theta, \Omega}$  is bounded on  $L^p(\mathbb{R}^{n+1})$  for  $1 < p < \infty$ . The  $L^p$  may depend on the degrees of the polynomials  $\varphi_1$  and  $\varphi_2$  but it is independent of their coefficients.*

Throughout this paper, the letter  $C$  will stand for a constant that may vary at each occurrence but it is independent of the essential variables.

## 2 Preliminary Lemmas

For  $j \in \mathbb{Z}$  and  $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  for some  $q > 1$  and satisfying (1.4)-(1.5), let  $\alpha(q, \Omega) = \log_2(e + \|\Omega\|_q)$  and let  $I_{j, q}$  be the interval  $[2^{\alpha(q, \Omega)j}, 2^{\alpha(q, \Omega)(j+1)})$  in  $\mathbb{R}$ . Let  $\theta(t, s) = \varphi_1(s) + \varphi_2(t)$  where  $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$  are suitable real valued functions. Define the sequence  $\{\mathbb{M}_{q, \theta, j, k} : j, k \in \mathbb{Z}\}$  of multipliers on  $\mathbb{R}^{n+1}$  by

$$\mathbb{M}_{q, \theta, j, k}(\xi, \eta) = \int \int_{(|u|, |v|) \in I_{j, q} \times I_{k, q}} e^{-i((u+v) \cdot \xi + \theta(|u|, |v|)\eta)} \frac{\Omega(u', v')}{|u|^n |v|^n} du dv \quad (2.1)$$

where  $\xi \in \mathbb{R}^n$  and  $\eta \in \mathbb{R}$ .

For  $j \in \mathbb{Z}$  and  $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  for some  $q > 1$ , we let  $e_{j, q} : \mathbb{R} \rightarrow \mathbb{C}$  be the mapping defined by

$$e_{j, q}(t) = e^{-i2^{\alpha(q, \Omega)j}t}. \quad (2.2)$$

Now, we have the following lemma:

**Lemma 2.1.** Let  $\mathbb{M}_{q,\varphi_1,\varphi_2,j,k}$  be as in (2.1). Let

$$C_{q,\Omega} = (\alpha(q,\Omega))^2 \|\Omega\|_1^{2-\frac{2}{\alpha(q,\Omega)}}$$

and let  $\theta, \varphi_1$ , and  $\varphi_2(t)$  be as above. Then

$$|\mathbb{M}_{q,\theta,j,k}(\xi, \eta)| \leq \left| 2^{\alpha(q,\Omega)j} \xi \right|^{-\frac{1}{4q'\alpha(q,\Omega)}} \left| 2^{\alpha(q,\Omega)k} \xi \right|^{-\frac{1}{4q'\alpha(q,\Omega)}} C C_{q,\Omega}, \quad (2.3)$$

$$|\mathbb{M}_{q,\theta,j,k}(\xi, \eta)| \leq \left| 2^{\alpha(q,\Omega)j} \xi \right|^{-\frac{1}{4\alpha(q,\Omega)}} \left| 2^{\alpha(q,\Omega)(k+1)} \xi \right|^{\frac{1}{\alpha(q,\Omega)}} C C_{q,\Omega}, \quad (2.4)$$

$$|\mathbb{M}_{q,\theta,j,k}(\xi, \eta)| \leq \left| 2^{\alpha(q,\Omega)(j+1)} \xi \right|^{\frac{1}{\alpha(q,\Omega)}} \left| 2^{\alpha(q,\Omega)k} \xi \right|^{-\frac{1}{4\alpha(q,\Omega)}} C C_{q,\Omega}, \quad (2.5)$$

$$|\mathbb{M}_{q,\theta,j,k}(\xi, \eta)| \leq (\alpha(q,\Omega))^2 \|\Omega\|_1 \left| 2^{\alpha(q,\Omega)(j+1)} \xi \right| \left| 2^{\alpha(q,\Omega)(k+1)} \xi \right| C \quad (2.6)$$

for all  $j, k \in \mathbb{Z}$ ,  $\xi \in \mathbb{R}^n$ ,  $\eta \in \mathbb{R}$ , and mappings  $\varphi_1$  and  $\varphi_2$ . The constant  $C$  is independent of the essential variables.

**Proof.** We start by the proof of (2.3). We write  $\mathbb{M}_{q,\theta,j,k}(\xi, \eta)$  as

$$\begin{aligned} & \left| \mathbb{M}_{q,\theta,j,k}(\xi, \eta) \right|^2 \\ & \leq \left( \int_1^{2^{\alpha(q,\Omega)}} \int_1^{2^{\alpha(q,\Omega)}} \left| \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \Omega(u', v') e_{j,q}(r\xi \cdot u') e_{k,q}(s\xi \cdot v') d\sigma(u') d\sigma(v') \right| \frac{drds}{rs} \right)^2 \\ & \leq \left( \int_1^{2^{\alpha(q,\Omega)}} \int_1^{2^{\alpha(q,\Omega)}} \left| \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \Omega(u', v') e_{j,q}(r\xi \cdot u') e_{k,q}(s\xi \cdot v') d\sigma(u') d\sigma(v') \right| \frac{drds}{rs} \right)^2 \\ & \leq \|\Omega\|_q^4 \left( \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \left| \int_1^{2^{\alpha(q,\Omega)}} \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot u') e_{k,q}(s\xi \cdot v') \frac{drds}{rs} \right|^{q'} d\sigma(u', v', z', w') \right)^{\frac{1}{q'}} \end{aligned} \quad (2.7)$$

where

$$d\sigma(u', v', z', w') = d\sigma(u') d\sigma(v') d\sigma(z') d\sigma(w').$$

Now,

$$\begin{aligned} & \left| \int_1^{2^{\alpha(q,\Omega)}} \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot (u' - w')) e_{k,q}(s\xi \cdot (v' - z')) \frac{drds}{rs} \right|^{q'} \\ & = \left| \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot (u' - w')) \frac{dr}{r} \right|^{q'} \left| \int_1^{2^{\alpha(q,\Omega)}} e_{k,q}(s\xi \cdot (v' - z')) \frac{ds}{s} \right|^{q'}. \end{aligned}$$

Thus, by integration by parts, we have

$$\begin{aligned} & \left| \int_1^{2^{\|\Omega\|_q}} \int_1^{2^{\|\Omega\|_q}} e_{j,q}(r\xi \cdot (u' - w')) e_{k,q}(s\xi \cdot (v' - z')) \frac{drds}{rs} \right|^{q'} \\ & \leq \left| 2^{\alpha(q,\Omega)j} \xi \cdot (u' - w') \right|^{-q'} \left| 2^{\alpha(q,\Omega)k} s\xi \cdot (v' - z') \right|^{-q'}. \end{aligned} \quad (2.8)$$

On the other hand, we have

$$\begin{aligned} & \left| \int_1^{2^{\alpha(q,\Omega)}} \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot (u' - w')) e_{k,q}(s\xi \cdot (v' - z')) \frac{drds}{rs} \right|^{q'} \\ & \leq \alpha(q, \Omega)^{2q'}. \end{aligned} \quad (2.9)$$

Thus, by (2.8) and (2.9), we have

$$\begin{aligned} & \left| \int_1^{2^{\alpha(q,\Omega)}} \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot (u' - w')) e_{k,q}(s\xi \cdot (v' - z')) \frac{drds}{rs} \right|^{q'} \\ & \leq \alpha(q, \Omega)^{2q'(1-\frac{1}{2q'})} \left| 2^{\alpha(q,\Omega)j} \xi \cdot (u' - w') \right|^{-\frac{1}{2}} \left| 2^{\alpha(q,\Omega)k} s\xi \cdot (v' - z') \right|^{-\frac{1}{2}}. \end{aligned} \quad (2.10)$$

By (2.10), (2.7), and the fact that

$$\sup_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\xi \cdot (u' - v')|^{-\delta} d\sigma(u') d\sigma(v') = C_\delta < \infty \quad (2.11)$$

for  $0 < \delta < 1$ , we have

$$|\mathbb{M}_{q,\varphi_1,\varphi_2,j,k}(\xi, \eta)| \leq \left| 2^{\alpha(q,\Omega)j} \xi \right|^{-\frac{1}{4q'}} \left| 2^{\alpha(q,\Omega)k} \xi \right|^{-\frac{1}{4q'}} CC_{q,\Omega} \quad (2.12)$$

where  $C$  is independent of the essential variables. By (2.12) and the observation that

$$|\mathbb{M}_{q,\varphi_1,\varphi_2,j,k}(\xi, \eta)| \leq \alpha(q, \Omega)^2 \|\Omega\|_1^2. \quad (2.13)$$

By (2.12) and (2.13), we get

$$\begin{aligned} & |\mathbb{M}_{q,\varphi_1,\varphi_2,j,k}(\xi, \eta)| \\ & \leq C_{q,\Omega} \left( \|\Omega\|_q^2 (\alpha(q, \Omega))^{2(1-\frac{1}{2q'})} C \right)^{\frac{1}{\alpha(q,\Omega)}} \left| 2^{\alpha(q,\Omega)(j+k)} |\xi|^2 \right|^{-\frac{1}{4q'\alpha(q,\Omega)}}. \end{aligned} \quad (2.14)$$

By (2.14) and the observation

$$C_{q,\Omega} \left( \|\Omega\|_q^2 (\alpha(q, \Omega))^{2(1-\frac{1}{2q'})} C \right)^{\frac{1}{\alpha(q,\Omega)}} \leq (\alpha(q, \Omega))^2 \|\Omega\|_1^{2-\frac{2}{\alpha(q,\Omega)}} CC_{q,\Omega}, \quad (2.15)$$

we obtain (2.3).

Next, we verify (2.4). Notice that by the cancellation property (1.4), we have

$$\int_{(|u|,|v|) \in I_{j,q} \times I_{k,q}} \int e^{-i((u \cdot \xi + \theta(|u|,|v|)\eta))} \frac{\Omega(u', v')}{|u|^n |v|^n} du dv = 0.$$

Thus,

$$\begin{aligned} & |\mathbb{M}_{q,\theta,j,k}(\xi, \eta)| \\ &= \left| \mathbb{M}_{q,\theta,j,k}(\xi, \eta) - \int_{(|u|,|v|) \in I_{j,q} \times I_{k,q}} \int e^{-i((u \cdot \xi + \theta(|u|,|v|)\eta))} \frac{\Omega(u', v')}{|u|^n |v|^n} du dv \right| \\ &\leq \int_1^{2^{\alpha(q,\Omega)}} \int_1^{2^{\alpha(q,\Omega)}} \int_{\mathbb{S}^{n-1}} |(e_{k,q}(s\xi \cdot v') - 1)| \left| \int_{\mathbb{S}^{n-1}} \Omega(u', v') e_{j,q}(r\xi \cdot u') d\sigma(u') \right| d\sigma(v') \frac{dr ds}{rs} \\ &\leq C_{q,\Omega} \left| 2^{\alpha(q,\Omega)(k+1)} \xi \right| \int_1^{2^{\alpha(q,\Omega)}} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} \Omega(u', v') e_{j,q}(r\xi \cdot u') d\sigma(u') \right| d\sigma(v') \frac{dr}{r}. \quad (2.16) \end{aligned}$$

Now,

$$\begin{aligned} & \int_1^{2^{\alpha(q,\Omega)}} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} \Omega(u', v') e_{j,q}(r\xi \cdot u') d\sigma(u') \right| d\sigma(v') \frac{dr}{r} \\ &\leq \int_1^{2^{\alpha(q,\Omega)}} \left( \left| \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \Omega(u', v') e_{j,q}(r\xi \cdot u') d\sigma(u') \right|^2 d\sigma(v') \right)^{\frac{1}{2}} \frac{dr}{r} \\ &\leq (\alpha(q, \Omega))^{\frac{1}{2}} \left( \int_1^{2^{\alpha(q,\Omega)}} \left| \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \Omega(u', v') e_{j,q}(r\xi \cdot u') d\sigma(u') \right|^2 d\sigma(v') \frac{dr}{r} \right)^{\frac{1}{2}} \\ &\leq (|\mathbb{S}^{n-1}| \alpha(q, \Omega))^{\frac{1}{2}} \left( \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |I_{j,q,\Omega}(u', w')| d\sigma(u', v', z', w') \right)^{\frac{1}{2}} \quad (2.17) \end{aligned}$$

where

$$I_{j,q,\Omega}(u', w') = \Omega(u', v') \Omega(w', v') \left| \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot (u' - w')) \frac{dr}{r} \right|.$$

By integration by parts, we have

$$\left| \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot (u' - w')) \frac{dr}{r} \right| \leq \alpha(q, \Omega) \left| 2^{\alpha(q,\Omega)j} \xi \cdot (u' - w') \right|^{-1}$$

which when combined with the estimate

$$\left| \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot (u' - w')) \frac{dr}{r} \right| \leq \alpha(q, \Omega) \quad (2.18)$$

implies that

$$\left| \int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi \cdot (u' - w')) \frac{dr}{r} \right| \leq \alpha(q, \Omega) \left| 2^{\alpha(q,\Omega)j} \xi \cdot (u' - w') \right|^{-\frac{1}{2q'}}. \quad (2.19)$$

By (2.16), (2.17), (2.19), (2.11), and Hölder's inequality, we have

$$|\mathbb{M}_{q,\theta,j,k}(\xi, \eta)| \leq (\alpha(q, \Omega))^2 |\mathbb{S}^{n-1}|^{\frac{1}{2}} \|\Omega\|_q \left| 2^{\alpha(q,\Omega)(k+1)} \xi \right| \left| 2^{\alpha(q,\Omega)j} \xi \right|^{-\frac{1}{4}}. \quad (2.20)$$

By (2.13) and (2.20), we obtain (2.4). By symmetry, we can obtain (2.5). To see (2.6), we make use of the observation

$$\int_{(|u|,|v|) \in I_{j,q} \times I_{k,q}} e^{-i\theta(|u|,|v|)\eta} \frac{\Omega(u', v')}{|u|^n |v|^n} du dv = 0.$$

In fact,

$$\begin{aligned} & |\mathbb{M}_{q,\theta,j,k}(\xi, \eta)| \\ &= \left| \mathbb{M}_{q,\theta,j,k}(\xi, \eta) - \int_{(|u|,|v|) \in I_{j,q} \times I_{k,q}} e^{-i\theta(|u|,|v|)\eta} \frac{\Omega(u', v')}{|u|^n |v|^n} du dv \right| \\ &\leq \int_1^{2^{\alpha(q,\Omega)}} \int_1^{2^{\alpha(q,\Omega)}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\Omega(u', v')| |e_{k,q}(s\xi \cdot v') - 1| |e_{j,q}(r\xi \cdot u') - 1| d\sigma(u') d\sigma(v') \frac{dr ds}{rs} \\ &\leq (\alpha(q, \Omega))^2 \|\Omega\|_1 \left| e^{-i2^{\alpha(q,\Omega)(j+1)} \xi} \right| \left| e^{-i2^{\alpha(q,\Omega)(k+1)} \xi} \right|. \end{aligned}$$

This completes the proof of Lemma 2.1.

Now, let  $\bar{\mu}_{\Omega}^{(\varphi_1, \varphi_2)}$  be the maximal function given by

$$\bar{\mu}_{\Omega,q}^{(\varphi_1, \varphi_2)}(f)(x, x_{n+1}) = \sup_{j,k} \left| \int_{|u| \in I_{j,q}} \int_{|v| \in I_{k,q}} |f(x - u - v, x_{n+1} - \theta(|u|, |v|))| \frac{|\Omega(u', v')|}{|u|^n |v|^n} \right|. \quad (2.21)$$

**Lemma 2.2.** *Suppose that  $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$  for some  $q > 1$ . Let  $\varphi_1$  and  $\varphi_2$  be as in Theorem 1.1 or Theorem 1.2 or Theorem 1.3. Then*

$$\|\bar{\mu}_{\Omega,q}^{(\varphi_1, \varphi_2)}(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C (\alpha(q, \Omega))^2 \|\Omega\|_{L^1} \|f\|_{L^p(\mathbb{R}^{n+1})} \quad (2.22)$$

for all  $1 < p < \infty$  and  $C$  is a constant independent of of the essential variables.

**Proof.** Notice that

$$\overline{\mu}_{\Omega, q}^{(\varphi_1, \varphi_2)}(f)(x, x_{n+1}) \leq (\alpha(q, \Omega))^2 \|\Omega\|_1 \overline{\mathcal{M}}_{\Omega, q}^{(\varphi_1, \varphi_2)}(f)(x, x_{n+1}) \quad (2.23)$$

where

$$\overline{\mathcal{M}}^{(\theta)}(f)(x, x_{n+1}) = \sup_{j, k \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |f(x - ru' - sv', x_{n+1} - \theta(r, s))| \frac{dr ds}{rs}. \quad (2.24)$$

It can be shown that

$$\overline{\mathcal{M}}^{(\theta)}(f)(x, x_{n+1})(f)(x, x_{n+1}) \leq \overline{\mathcal{M}}^{\varphi_1} \circ \overline{\mathcal{M}}^{\varphi_2}(f)(x, x_{n+1}) \quad (2.25)$$

where

$$\overline{\mathcal{M}}^{\varphi_1}(f)(x, x_{n+1}) = \sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |f(x - ru', x_{n+1} - \varphi_1(r))| \frac{dr}{r} \quad (2.26)$$

and

$$\overline{\mathcal{M}}^{\varphi_2}(f)(x, x_{n+1}) = \sup_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} |f(x - sv', x_{n+1} - \varphi_2(s))| \frac{ds}{s}. \quad (2.27)$$

Under the given assumptions on  $\varphi_1$  and  $\varphi_2$ , it follows by Theorem 2.4 in [4] that

$$\|\overline{\mathcal{M}}^{\varphi_1}(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C \|f\|_{L^p(\mathbb{R}^{n+1})} \quad (2.28)$$

and

$$\|\overline{\mathcal{M}}^{\varphi_2}\|_{L^p(\mathbb{R}^{n+1})} \leq C \|f\|_{L^p(\mathbb{R}^{n+1})} \quad (2.29)$$

for all  $1 < p < \infty$  with constant  $C$  independent of  $u'$  and  $v'$ . Hence, by (2.23), (2.28), and (2.29), we obtain (2.22). This completes the proof of Lemma 2.2.

### 3 Proof of Main result

**Proof of Theorem 1.1.** We start by choosing a sequence  $\{A_m : m \in \mathbb{N}\}$  of functions on  $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$  and a sequence  $\{\lambda_m : m \in \mathbb{N}\} \subset \mathbb{R}$  such that

$$\int_{\mathbb{S}^{n-1}} A_m(u', \cdot) d\sigma(u') = \int_{\mathbb{S}^{n-1}} A_m(\cdot, v') d\sigma(v') = 0, \quad (3.1)$$

$$A_m(ru', sv') = A_m(u', v'), r, s > 0, \quad (3.2)$$

$$\|A_m\|_1 \leq 4, \quad \|A_m\|_2 \leq 2^{2m+2}, \quad (3.3)$$

$$\Omega(x, y) = \sum_{m=1}^{\infty} \lambda_m A_m(x, y), \quad (3.4)$$

$$\sum_{m=1}^{\infty} (m+2)^2 \lambda_m \leq \|\Omega\|_{L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})}. \quad (3.5)$$

By (3.4), it follows that

$$T_{\theta, \Omega}(f)(x, x_{n+1}) = \sum_{m=1}^{\infty} \lambda_m T_{\theta, A_m}(f)(x, x_{n+1}), \quad (3.6)$$

where  $T_{\theta, A_m}$  is given by (1.6) with  $\Omega$  is replaced by  $A_m$ . Let  $\{\sigma_{m,j,k} : j, k \in \mathbb{Z}\}$  be the sequence of measures such that

$$\hat{\sigma}_{m,j,k}(\xi, \eta) = \mathbb{M}_{2,\theta,j,k}(\xi, \eta) \quad (3.7)$$

where  $\mathbb{M}_{2,\theta,j,k}$  is given by (2.1) with  $q = 2$  and  $\alpha(q, A_m) = \log_2(e + \|A_m\|_2)$ . Thus, the operator  $T_{\theta, A_m}$  is decomposed as follows:

$$T_{\theta, A_m}(f)(x, x_{n+1}) = \sum_{j,k \in \mathbb{Z}} \sigma_{m,j,k} * f(x, x_{n+1}). \quad (3.8)$$

Thus, by (3.6), (3.8), and Minkowski's inequality, it suffices to prove that

$$\|T_{\theta, A_m}\|_{L^p(\mathbb{R}^{n+1})} \leq (m+2)^2 C \|f\|_{L^p(\mathbb{R}^{n+1})} \quad (3.9)$$

where  $C$  is a constant independent of  $m$ . Now, by an argument similar to that used in [7], we let  $\{\varpi_j^{(1)}\}_{-\infty}^{\infty}$  and  $\{\varpi_k^{(2)}\}_{-\infty}^{\infty}$  be smooth partitions of unity on  $(0, \infty)$  in the sense that

$$\text{supp}(\varpi_j^{(1)}) \subseteq \left\{ t : 2^{-2\alpha(2, A_m)(j+1)} < t < 2^{-2\alpha(2, A_m)(j-1)} \right\}, \quad (3.10)$$

$$\text{supp}(\varpi_k^{(2)}) \subseteq \left\{ t : 2^{-2\alpha(2, A_m)(k+1)} < t < 2^{-2\alpha(2, A_m)(k-1)} \right\}, \quad (3.11)$$

$$0 \leq \varpi_j^{(1)}, \varpi_k^{(2)} \leq 1; \quad (3.12)$$

$$\sum_{j \in \mathbb{Z}} \varpi_j^{(1)}(t) = \sum_{k \in \mathbb{Z}} \varpi_k^{(2)}(t) = 1; \quad (3.13)$$

$$\left| \frac{d^r \varpi_j^{(1)}}{dt^r}(t) \right|, \left| \frac{d^r \varpi_k^{(2)}}{dt^r}(u) \right| \leq \frac{C_r}{t^r} \quad (3.14)$$

where  $C_r$  is independent of  $m$ . Defined the multiplier operator  $\mathcal{F}_{j,k}^{(m)}$  in  $\mathbb{R}^{n+1}$  by

$$\widehat{(\mathcal{F}_{j,k}^{(m)} f)}(\xi, \eta) = \varpi_j^{(1)}(|2^{\alpha(2, A_m)j}\xi|) \varpi_k^{(2)}(|2^{\alpha(2, A_m)k}\xi|) \hat{f}(\xi, \eta).$$

Thus, for  $r, s \in \mathbb{Z}$ , we have

$$(\mathcal{F}_{j+r, k+s}^{(m)} \widehat{\mathcal{F}_{j+r, k+s}^{(m)} f})(\xi, \eta) = \hat{f}(\xi, \eta) \left( \varpi_{j+r}^{(1)}(|2^{\alpha(2, A_m)j}\xi|) \right)^2 \left( \varpi_{k+s}^{(2)}(|2^{\alpha(2, A_m)k}\xi|) \right)^2.$$

Therefore, by (3.13), we get

$$\sum_{r,s \in \mathbb{Z}} (\mathcal{F}_{j+r, k+s}^{(m)} \mathcal{F}_{j+r, k+s}^{(m)} f)(x, x_{n+1}) = f(x, x_{n+1}). \quad (3.15)$$

By (3.15), we decompose the operator  $T_{\theta, A_m}$  as follows

$$T_{\theta, A_m} f(x, x_{n+1}) = \sum_{j,k \in \mathbb{Z}} \sigma_{m,j,k} * f(x, x_{n+1}) = \sum_{j,k \in \mathbb{Z}} \sigma_{m,j,k} * \left( \sum_{r,s \in \mathbb{Z}} \mathcal{F}_{j+r, k+s}^{(m)} \mathcal{F}_{j+r, k+s}^{(m)} f \right).$$

Now, let

$$\mathcal{N}_{r,s}^{(m)}(f) = \sum_{j,k \in \mathbb{Z}} \mathcal{F}_{j+r,k+s}^{(m)} \left( \sigma_{m,j,k} * \mathcal{F}_{j+r,k+s}^{(m)} f \right). \quad (3.16)$$

Hence, by (3.16),  $T_{\theta, A_m}$  reduces to

$$T_{\theta, A_m} f(x, x_{n+1}) = \sum_{r,s \in \mathbb{Z}} \mathcal{N}_{r,s}^{(m)}(f)(x, x_{n+1}). \quad (3.17)$$

Now, for  $p > 2$ , we have

$$\begin{aligned} \|\mathcal{N}_{r,s}^{(m)} f\|_p &\leq C_p \left\| \left( \sum_{j,k \in \mathbb{Z}} \left| \sigma_{m,j,k} * \mathcal{F}_{j+r,k+s}^{(m)} f \right|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\leq C_p (\alpha(2, A_m))^2 \|A_m\|_1 \left\| \left( \sum_{j,k \in \mathbb{Z}} \left| \mathcal{F}_{j+r,k+s}^{(m)} f \right|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\leq C_p (\alpha(2, A_m))^2 \|A_m\|_1 \|f\|_p \end{aligned} \quad (3.18)$$

for all  $r, s \in \mathbb{Z}$  and  $C_p$  is a constant independent of  $m$ . Here, the first and the last inequalities follow by Littlewood-Paley theory while the second inequality follows by (2.22)

Next, we consider the  $L^2$ - norm. Notice that by Plancherel's theorem and the properties of the portions  $\{\varpi_j^1\}_{-\infty}^{\infty}$  and  $\{\varpi_k^2\}_{-\infty}^{\infty}$ , we have

$$\|\mathcal{N}_{r,s}^{(m)} f\|_{L^2}^2 \leq \sum_{j,k \in \mathbb{Z}} \int_{A_{j+r,k+s}} |f(\xi, \eta)|^2 |\mathbb{M}_{2,\theta,j,k}(\xi, \eta)|^2 d\xi d\eta, \quad (3.19)$$

where

$$A_{j+r,k+s} = A_{j+r}^{(m)} \cap A_{k+s}^{(m)}$$

$$A_{j+r}^{(m)} = \left\{ \xi : 2^{-2\alpha(2, A_m)(j+r+1)} < |2^{\alpha(2, A_m)j} \xi| < 2^{-2\alpha(2, A_m)(j+r+1)} \right\},$$

and

$$A_{k+s}^{(m)} = \left\{ \xi : 2^{-2\alpha(2, A_m)(k+s+1)} < |2^{\alpha(2, A_m)k} \xi| < 2^{-2\alpha(2, A_m)(k+s+1)} \right\}.$$

By Lemma 2.1, (2.13), and the fact that  $\|A_m\|_1 \leq 4$ , we get

$$\|\mathcal{N}_{r,s}^{(m)} f\|_{L^2} \leq (\alpha(2, A_m))^2 C 2^{-\tilde{\alpha}_l |r|} 2^{-\tilde{\beta}_l |s|} \|f\|_{L^2} \quad (3.20)$$

for some real  $\tilde{\alpha}_l > 0$  and  $\tilde{\beta}_l > 0$ . Hence, by interpolation between (3.18) and (3.20) along with the fact that  $\alpha(2, A_m) \leq 2(m+2)$ , we get (3.9) for  $p \geq 2$ . The case for  $1 < p < 2$  follows by duality. This completes the proof of Theorem 1.1.

**Proof of Theorems 1.2 and 1.3.** The proofs follows by similar argument as that for the proof of Theorem 1.1 with minor modifications. We omit the details.

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