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Maximal operator in total Morrey-Guliyev spaces defined on Carleson curves

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Abstract. *In this paper we study the maximal operator M_Γ in the total Morrey-Guliyev space $L_{p,\lambda,\mu}(\Gamma)$ defined on Carleson curves Γ. We prove that for $1 < p < \infty$ the maximal operator M_Γ is bounded on the total Morrey-Guliyev space $M_{p,\lambda,\mu}(\Gamma)$, and from the total Morrey-Guliyev space $M_{1,\lambda,\mu}(\Gamma)$ to $WL_{1,\lambda,\mu}(\Gamma)$.*

Keywords. Carleson curves, Morrey spaces, modified Morrey spaces, total Morrey-Guliyev spaces.

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1 Introduction

Morrey spaces, introduced by Morrey [29], play an important role in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for analyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [13], extend the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|f\|_{L_p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \le \lambda$ is equal to the maximum of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, see also [1,2,8,24, 25,27,28,30–33,36–38]. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [3,4,7,12,14–16,22,23,31].

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Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \leq s \leq l \leq \infty\}$ be a rectifiable Jordan curve in the complex plane with arc-length measure $\nu(t) = s$, here $l = \nu\Gamma =$ lengths of Γ.

We denote

$$\Gamma(t,r) = \Gamma \cap B(t,r),\ t \in \Gamma,\ r > 0,$$

where $B(t,r) = \{z \in \mathbb{C} : |z-t| < r\}$.

A rectifiable Jordan curve Γ is called a Carleson curve (regular curve) if the condition

$$\nu\Gamma(t,r) \leq c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r.

It is well known that maximal operator play an important role in harmonic analysis (see [35]). In this paper, in the framework of this analysis in the setting of Carleson curve, we study the boundedness of the maximal operator M_Γ defined on Carleson curves Γ on total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\Gamma)$.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we prove $L_{p,\lambda,\mu}(\Gamma)$-boundedness of the maximal operator M_Γ.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries in the setting on Carleson curves and some auxiliary lemmas

Definition 2.1 *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[r]_1 = \min\{1, r\}$, $r > 0$. We denote by $L_{p,\lambda}(\Gamma)$ the Morrey space [26], by $\widetilde{L}_{p,\lambda}(\Gamma)$ the modified Morrey space [9], and by $L_{p,\lambda,\mu}(\Gamma)$ the total Morrey-Guliyev space [13], associated with the Carleson curves the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{L_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma,\, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))},$$

$$\|f\|_{L_{p,\lambda,\mu}(\Gamma)} = \sup_{t\in\Gamma,\, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f\|_{L_p(\Gamma(t,r))},$$

respectively.

Definition 2.2 *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $L_{p,\lambda}(\Gamma)$ [26] ($\equiv$ weak D_ν-Morrey space), the weak modified Morrey space $\widetilde{L}_{p,\lambda}(\Gamma)$ [9], and the weak total Morrey-Guliyev space $L_{p,\lambda,\mu}(\Gamma)$ [13], associated with the Carleson curves the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{WL_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\Gamma(t,r))},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma,\, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(\Gamma(t,r))},$$

$$\|f\|_{WL_{p,\lambda,\mu}(\Gamma)} = \sup_{t\in\Gamma,\, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\Gamma(t,r))},$$

respectively.

Note that

$$L_{p,0,0}(\Gamma) = \widetilde{L}_{p,0}(\Gamma) = L_{p,0}(\Gamma) = L_p(\Gamma),$$
$$WL_{p,0,0}(\Gamma) = W\widetilde{L}_{p,0}(\Gamma) = WL_{p,0}(\Gamma) = WL_p(\Gamma),$$
$$L_{p,\lambda,\lambda}(\Gamma) = L_{p,\lambda}(\Gamma), \quad L_{p,\lambda,0}(\Gamma) = \widetilde{L}_{p,\lambda}(\Gamma),$$
$$\|f\|_{WL_{p,\lambda,\mu}(\Gamma)} \le \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \text{ and therefore } L_{p,\lambda,\mu}(\Gamma) \subset WL_{p,\lambda,\mu}(\Gamma)$$

and

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{p,\lambda}(\Gamma),\ \mu \le \lambda \text{ and } \|f\|_{L_{p,\lambda}(\Gamma)} \le \|f\|_{L_{p,\lambda,\mu}(\Gamma)}, \tag{2.1}$$
$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{p,\mu}(\Gamma),\ \mu \le \lambda \text{ and } \|f\|_{L_{p,\mu}(\Gamma)} \le \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \tag{2.2}$$
$$\widetilde{L}_{p,\lambda}(\Gamma) \subset_{\succ} L_p(\Gamma) \text{ and } \|f\|_{L_p(\Gamma)} \le \|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)}$$

and if $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(\Gamma) = \widetilde{L}_{p,\lambda}(\Gamma) = WL_{p,\lambda}(\Gamma) = W\widetilde{L}_{p,\lambda}(\Gamma) = \Theta$. Here $\Theta \equiv \Theta(\Gamma)$ is the set of all functions on Γ that are equivalent to 0 on Γ.

Maximal operators and potential operators in various spaces defined on Carleson curves have been widely studied by many authors (see, for example [5,6,17–21,34]). In Morrey spaces defined on quasimetric measure spaces, in particular Morrey spaces $L_{p,\lambda}(\Gamma)$ defined on Carleson curves N. Samko [34] studied the boundedness of the maximal operator M_Γ defined by

$$M_\Gamma f(t) = \sup_{r>0} (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau)$$

and proved the following:

Theorem A. [34] *Let Γ be a Carleson curve, $1 \le p < \infty$ and $0 \le \lambda < 1$.*

1) If $1 < p < \infty$, then M_Γ is bounded from $L_{p,\lambda}(\Gamma)$ to $L_{p,\lambda}(\Gamma)$.

2) If $p = 1$, then M_Γ is bounded from $L_{1,\lambda}(\Gamma)$ to $WL_{1,\lambda}(\Gamma)$.

Theorem B. [26] *Let Γ be a Carleson curve, $1 \le p < \infty$ and $0 \le \lambda < 1$.*

1) If $1 < p < \infty$, then M_Γ is bounded from $\widetilde{L}_{p,\lambda}(\Gamma)$ to $\widetilde{L}_{p,\lambda}(\Gamma)$.

2) If $p = 1$, then M_Γ is bounded from $\widetilde{L}_{1,\lambda}(\Gamma)$ to $W\widetilde{L}_{1,\lambda}(\Gamma)$.

To prove our theorems we need the following lemmas.

Lemma 2.1 *Let Γ be a Carleson curve, $1 \le p < \infty$, $0 \le \mu \le \lambda \le 1$. Then*

$$L_{p,\lambda,\mu}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\Gamma)} = \max\left\{\|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_{p,\mu}(\Gamma)}\right\}.$$

Proof. Let $f \in L_{p,\lambda,\mu}(\Gamma)$. Then from (2.1) and (2.2) we have that $f \in L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma)$ and $\max\left\{\|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_{p,\mu}(\Gamma)}\right\} \le \|f\|_{L_{p,\lambda,\mu}(\Gamma)}$.

Let now $f \in L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma)$. Then

$$\|f\|_{L_{p,\lambda,\mu}(\Gamma)} = \sup_{t\in\Gamma, r>0} \left([r]_1^{-\lambda}\, [1/r]_1^{\lambda} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau)\right)^{1/p}$$
$$= \max\left\{\sup_{t\in\Gamma, 0<r\le 1} \left(r^{-\lambda} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau)\right)^{1/p}, \sup_{t\in\Gamma, r>1} \left(r^{-\mu} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau)\right)^{1/p}\right\}$$
$$\le \max\left\{\|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_{p,\mu}(\Gamma)}\right\}.$$

Therefore, $f \in L_{p,\lambda,\mu}(\Gamma)$ and the embedding $L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma) \subset\!\!\succ L_{p,\lambda,\mu}(\Gamma)$ is valid.

Thus $L_{p,\lambda,\mu}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma)$ and $\max\left\{\|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_{p,\mu}(\Gamma)}\right\} = \|f\|_{L_{p,\lambda,\mu}(\Gamma)}$.

Corollary 2.1 *Let Γ be a Carleson curve, $1 \le p < \infty$, $0 \le \lambda \le 1$. Then*

$$\widetilde{L}_{p,\lambda}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_p(\Gamma)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)} = \max\left\{\|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_p(\Gamma)}\right\}.$$

Remark 2.1 If $1 \le p < \infty$, and $\mu < 0$ or $\lambda > 1$, then

$$L_{p,\lambda,\mu}(\Gamma) = WL_{p,\lambda,\mu}(\Gamma) = \Theta(\Gamma).$$

The following statement can be proved analogously.

Lemma 2.2 *Let Γ be a Carleson curve, $1 \le p < \infty$, $0 \le \mu \le \lambda \le 1$. Then*

$$W\widetilde{L}_{p,\lambda}(\Gamma) = WL_{p,\lambda}(\Gamma) \cap WL_{p,\mu}(\Gamma)$$

and

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\Gamma)} = \max\left\{\|f\|_{WL_{p,\lambda}(\Gamma)}, \|f\|_{WL_{p,\mu}(\Gamma)}\right\}.$$

Lemma 2.3 *Let Γ be a Carleson curve, $1 \le p < \infty$, $0 \le \lambda \le 1$ and $0 \le \mu \le 1$, then*

$$L_{p,1,\mu}(\Gamma) \subset\!\!\succ L_\infty(\Gamma) \subset\!\!\succ L_{p,\lambda,1}(\Gamma)$$

and

$$\|f\|_{L_{p,\lambda,1}(\Gamma)} \le c_0^{\frac{1}{p}} \|f\|_{L_\infty(\Gamma)} \le \|f\|_{L_{p,1,\mu}(\Gamma)}.$$

Proof. Let $f \in L_\infty(\Gamma)$. Then for all $t \in \Gamma$ and $0 < r \le 1$

$$r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))} \le c_0^{\frac{1}{p}} \|f\|_{L_\infty(\Gamma)}, \quad 0 \le \lambda \le 1$$

and for all $t \in \Gamma$ and $r \ge 1$

$$r^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,r))} \le c_0^{\frac{1}{p}} \|f\|_{L_\infty(\Gamma)}.$$

Thus $f \in L_{p,\lambda,1}(\Gamma)$ and

$$\|f\|_{L_{p,\lambda,1}(\Gamma)} \le c_0^{\frac{1}{p}} \|f\|_{L_\infty(\Gamma)}.$$

Let $f \in L_{p,1,\mu}(\Gamma)$. By the Lebesgue's differentiation theorem we have

$$\lim_{t\to 0} |\Gamma(t,r)|^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,r))} = |f(t)| \quad \text{for} \quad a.e.\ t \in \Gamma.$$

Then for a.e. $t \in \Gamma$

$$\begin{aligned} |f(t)| &= |\Gamma(t,r)|^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,r))} \\ &\le c_0^{-\frac{1}{p}} \sup_{t\in\Gamma,\, 0<r\le 1} r^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,r))} \\ &\le c_0^{-\frac{1}{p}} \|f\|_{L_{p,1,\mu}(\Gamma)}. \end{aligned}$$

Thus $f \in L_\infty(\Gamma)$ and

$$\|f\|_{L_\infty(\Gamma)} \le c_0^{-\frac{1}{p}} \|f\|_{L_{p,1,\mu}(\Gamma)}.$$

Corollary 2.2 *[26] Let Γ be a Carleson curve and $1 \le p < \infty$. Then*

$$L_{p,1}(\Gamma) = L_\infty(\Gamma)$$

and

$$\|f\|_{L_\infty(\Gamma)} \le \|f\|_{L_{p,1}(\Gamma)} \le c_0^{1/p}\|f\|_{L_\infty(\Gamma)}.$$

Lemma 2.4 *[8] Let Γ be a Carleson curve, $0 \le \lambda < 1$, $0 \le \mu < 1$, $0 \le \alpha < 1-\lambda$ and $0 \le \beta < 1-\mu$, then for $\frac{1-\lambda}{\alpha} \le p \le \frac{1-\mu}{\beta}$*

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{1,1-\alpha,1-\beta}(\Gamma)$$

and for $f \in L_{p,\lambda,\mu}(\Gamma)$ the inequality

$$\|f\|_{L_{1,1-\alpha,1-\beta}(\Gamma)} \le c_0^{\frac{1}{p'}}\|f\|_{L_{p,\lambda,\mu}(\Gamma)}$$

holds.

From Lemma 2.4 we obtain the following results.

Corollary 2.3 *Let Γ be a Carleson curve, $0 \le \mu \le \lambda < 1$, $0 \le \alpha < 1-\lambda$, then for $\frac{1-\lambda}{\alpha} \le p \le \frac{1-\mu}{\alpha}$*

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{1,1-\alpha}(\Gamma)$$

and for $f \in L_{p,\lambda,\mu}(\Gamma)$ the inequality

$$\|f\|_{L_{1,1-\alpha}(\Gamma)} \le c_0^{\frac{1}{p'}}\|f\|_{L_{p,\lambda,\mu}(\Gamma)}$$

holds.

Corollary 2.4 *[26] Let Γ be a Carleson curve, $1 \le p < \infty$ and $0 \le \lambda < 1$. If $p = \frac{1-\lambda}{\alpha}$, then*

$$L_{p,\lambda}(\Gamma) \subset L_{1,1-\alpha}(\Gamma) \quad \text{and} \quad \|f\|_{L_{1,1-\alpha}(\Gamma)} \le c_0^{1/p'}\|f\|_{L_{p,\lambda}(\Gamma)},$$

where $1/p + 1/p' = 1$.

Corollary 2.5 *Let Γ be a Carleson curve, $0 < \alpha < 1$, $0 \le \lambda \le 1-\alpha$. Then for $\frac{1-\lambda}{\alpha} \le p \le \frac{1}{\alpha}$*

$$\widetilde{L}_{p,\lambda}(\Gamma) \subset_{\succ} L_{1,1-\alpha}(\Gamma) \quad \text{and} \quad \|f\|_{L_{1,1-\alpha}(\Gamma)} \le c_0^{1/p'}\|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)}.$$

For the $0 \le \alpha < 1$ we define the following fractional maximal functions on Γ

$$M^\alpha_{\Gamma,p} f(t) \equiv \left(M^\alpha_\Gamma |f|^p\right)^{1/p}(t) = \sup_{r>0}\left(|\Gamma(t,r)|^{-1+\alpha}\int_{\Gamma(t,r)}|f(\tau)|^p d\nu(\tau)\right)^{1/p}.$$

In the case $\alpha = 0$ we denote $M^0_{\Gamma,p} f$ by $M_{\Gamma,p} f$ and in the case $p = 1$ we denote $M^\alpha_{\Gamma,1} f$ by $M^\alpha_\Gamma f$.

Lemma 2.5 *Let $1 \le p < \infty$, $0 \le \alpha < 1$ and $f \in L_{p,1-\alpha}(\Gamma)$. Then $M^\alpha_{\Gamma,p} f \in L_\infty(\Gamma)$ and*

$$\left\|M^\alpha_{\Gamma,p} f\right\|_{L_\infty(\Gamma)} = \|f\|_{L_{p,1-\alpha}(\Gamma)}.$$

Proof.

$$\left\|M^{\alpha}_{\Gamma,p}f\right\|_{L_\infty(\Gamma)} = \sup_{t\in\Gamma, r>0}\left(r^{\alpha-1}\int_{\Gamma(t,r)}|f(\tau)|^p d\nu(\tau)\right)^{1/p} = \|f\|_{L_{p,1-\alpha}(\Gamma)}.$$

Lemma 2.6 *Let* $1 \le p < \infty$, $0 \le \alpha < 1$ *and* $f \in \widetilde{L}_{p,1-\alpha}(\Gamma)$. *Then* $M^{\alpha}_{\Gamma,p}f \in L_\infty(\Gamma)$ *and*

$$\left\|M^{\alpha}_{\Gamma,p}f\right\|_{L_\infty(\Gamma)} \le \|f\|_{\widetilde{L}_{p,1-\alpha}(\Gamma)}.$$

Proof.

$$\begin{aligned}
\left\|M^{\alpha}_{\Gamma,p}f\right\|_{L_\infty(\Gamma)} &= \sup_{t\in\Gamma, r>0}\left(r^{\alpha-1}\int_{\Gamma(t,r)}|f(\tau)|^p d\nu(\tau)\right)^{1/p} \\
&= \sup_{t\in\Gamma, r>0}\left(r^{-1}[r]_1\right)^{\frac{1-\alpha}{p}}\left([r]_1^{\alpha-1}\int_{\Gamma(t,r)}|f(\tau)|^p d\nu(\tau)\right)^{1/p} \\
&\le \|f\|_{\widetilde{L}_{p,1-\alpha}}\sup_{r>0}\left(r^{-1}[r]_1\right)^{\frac{1-\alpha}{p}} \\
&= \|f\|_{\widetilde{L}_{p,1-\alpha}(\Gamma)}.
\end{aligned}$$

In the case $\alpha = 0$ from Lemma 2.5 for $M_{\Gamma,p}f$ the following property is valid.

Corollary 2.6 *Let* $1 \le p < \infty$ *and* $f \in L_\infty(\Gamma)$. *Then* $M_{\Gamma,p}f \in L_\infty(\Gamma)$ *and*

$$\|M_{\Gamma,p}f\|_{L_\infty(\Gamma)} = \|f\|_{L_\infty(\Gamma)}.$$

In the case $p = 1$ from Corollary 2.4 and Lemma 2.5 for $M^{\alpha}_{\Gamma}f$ the following property is valid.

Corollary 2.7 *Let* Γ *be a Carleson curve,* $0 \le \alpha < 1$, $0 \le \lambda \le 1-\alpha$ *and* $f \in L_{\frac{1-\lambda}{\alpha},\lambda}(\Gamma)$. *Then* $M^{\alpha}_{\Gamma}f \in L_\infty(\Gamma)$ *and*

$$\|M^{\alpha}_{\Gamma}f\|_{L_\infty(\Gamma)} = \|f\|_{L_{1,1-\alpha}(\Gamma)} \le c_0^{1-\frac{\alpha}{1-\lambda}}\|f\|_{L_{\frac{1-\lambda}{\alpha},\lambda}(\Gamma)}.$$

From Corollary 2.5 and Lemma 2.6 for $M^{\alpha}_{\Gamma}f$ the following property is valid.

Corollary 2.8 *Let* Γ *be a Carleson curve,* $0 \le \alpha < 1$, $0 \le \lambda \le 1-\alpha$ *and* $f \in \widetilde{L}_{p,\lambda}(\Gamma)$. *Then* $M^{\alpha}_{\Gamma}f \in L_\infty(\Gamma)$ *for* $\frac{1-\lambda}{\alpha} \le p \le \frac{1}{\alpha}$ *and*

$$\|M^{\alpha}_{\Gamma}f\|_{L_\infty(\Gamma)} = \|f\|_{L_{1,1-\alpha}(\Gamma)} \le c_0^{1/p'}\|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)}.$$

3 $L_{p,\lambda,\mu}(\Gamma)$-boundedness of the maximal operator M_Γ

In this section we study the $L_{p,\lambda,\mu}(\Gamma)$-boundedness of the maximal operator M_Γ.

The following Guliyev type local estimates are valid (see also [10, 11]).

Lemma 3.1 *[4] Let Γ be a Carleson curve and $1 \le p < \infty$. If $p > 1$, then the inequality*

$$\|M_\Gamma f\|_{L_p(\Gamma(t,r))} \lesssim r^{\frac{1}{p}} \sup_{\tau > 2r} \tau^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,\tau))} \tag{3.1}$$

holds for all $f \in L_p^{\rm loc}(\Gamma)$.

Moreover if $p = 1$, then the inequality

$$\|M_\nu f\|_{WL_1(\Gamma(t,r))} \lesssim r \sup_{\tau > 2r} \tau^{-1} \|f\|_{L_1(\Gamma(t,\tau))} \tag{3.2}$$

holds for all $f \in L_1^{\rm loc}(\Gamma)$.

Applying Lemma 3.1, one obtains the following result.

Theorem 3.1 *Let Γ be a Carleson curve, , $0 \le \lambda < 1$ and $0 \le \mu < 1$.*

1. If $f \in L_{1,\lambda,\mu}(\Gamma)$, then $M_\Gamma f \in WL_{1,\lambda,\mu}(\Gamma)$ and

$$\|M_\Gamma f\|_{WL_{1,\lambda,\mu}(\Gamma)} \le C_{1,\lambda,\mu} \, \|f\|_{L_{1,\lambda,\mu}(\Gamma)}, \tag{3.3}$$

where $C_{1,\lambda,\mu}$ is independent of f.

2. If $f \in L_{p,\lambda,\mu}(\Gamma)$, $1 < p < \infty$, then $M_\Gamma f \in L_{p,\lambda,\mu}(\Gamma)$ and

$$\|M_\Gamma f\|_{L_{p,\lambda,\mu}(\Gamma)} \le C_{p,\lambda,\mu} \, \|f\|_{L_{p,\lambda,\mu}(\Gamma)}, \tag{3.4}$$

where $C_{p,\lambda,\mu}$ depends only on p,λ and μ.

Proof. Let $p = 1$. From the inequality (3.2) we get

$$\begin{aligned}
\|M_\Gamma f\|_{WL_{1,\lambda,\mu}(\Gamma)} &= \sup_{t\in\Gamma,\, r>0} [r]_1^{-\lambda} \, [1/r]_1^{\mu} \, \|M_\Gamma f\|_{WL_1(\Gamma(t,r))} \\
&\lesssim \sup_{t\in\Gamma,\, r>0} [r]_1^{-\lambda} \, [1/r]_1^{\mu} \, r \sup_{\tau>2r} \tau^{-1} \, \|f\|_{L_1(\Gamma(t,\tau))} \\
&\lesssim \|f\|_{L_{1,\lambda,\mu}(\Gamma)} \sup_{t\in\Gamma,\, r>0} [r]_1^{-\lambda} \, [1/r]_1^{\mu} \, r \sup_{\tau>r} \tau^{-1} \, [\tau]_1^{\lambda} \, [1/\tau]_1^{-\mu} \\
&= \|f\|_{L_{1,\lambda,\mu}(\Gamma)} \sup_{t\in\Gamma,\, r>0} [r]_1^{1-\lambda} \, [1/r]_1^{\mu-1} \sup_{\tau>r} [\tau]_1^{\lambda-1} \, [1/\tau]_1^{1-\mu} \\
&\lesssim \|f\|_{L_{1,\lambda,\mu}(\Gamma)}
\end{aligned}$$

which implies that the operator M_Γ is bounded from $L_{1,\lambda,\mu}(\Gamma)$ to $WL_{1,\lambda,\mu}(\Gamma)$.

Let $1 < p < \infty$. From the inequality (3.1) we get

$$\begin{aligned}
\|M_\Gamma f\|_{L_{p,\lambda,\mu}(\Gamma)} &= \sup_{t\in\Gamma,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, \|M_\Gamma f\|_{L_p(\Gamma(t,r))} \\
&\lesssim \sup_{t\in\Gamma,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-\frac{1}{p}} \, \|f\|_{L_p(\Gamma(t,\tau))} \\
&\lesssim \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \sup_{t\in\Gamma,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, r^{\frac{1}{p}} \sup_{\tau>r} \tau^{-\frac{1}{p}} \, [\tau]_1^{\frac{\lambda}{p}} \, [1/\tau]_1^{-\frac{\mu}{p}} \\
&= \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \sup_{t\in\Gamma,\, r>0} [r]_1^{\frac{1-\lambda}{p}} \, [1/t]_1^{\frac{\mu-1}{p}} \sup_{\tau>r} [\tau]_1^{\frac{\lambda-1}{p}} \, [1/\tau]_1^{\frac{1-\mu}{p}} \\
&\lesssim \|f\|_{L_{p,\lambda,\mu}(\Gamma)},
\end{aligned}$$

which implies that the operator M_Γ is bounded in $L_{p,\lambda,\mu}(\Gamma)$.

From Theorem 3.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3.1 *[34] 1. If $f \in L_{1,\lambda}(\Gamma)$ and $0 \le \lambda < 1$, then $M_\Gamma f \in WL_{1,\lambda}(\Gamma)$ and*

$$\|M_\Gamma f\|_{WL_{1,\lambda}(\Gamma)} \le C_{1,\lambda} \, \|f\|_{L_{1,\lambda}(\Gamma)},$$

where $C_{1,\lambda}$ is independent of f.

2. If $f \in L_{p,\lambda}(\Gamma)$, $1 < p < \infty$ and $0 \le \lambda < 1$, then $M_\Gamma f \in L_{p,\lambda}(\Gamma)$ and

$$\|M_\Gamma f\|_{L_{p,\lambda}(\Gamma)} \le C_{p,\lambda} \, \|f\|_{L_{p,\lambda}(\Gamma)},$$

where $C_{p,\lambda}$ depends only on p and λ.

Corollary 3.2 *[9, Theorem 1] 1. If $f \in \widetilde{L}_{1,\lambda}(\Gamma)$ and $0 \le \lambda < 1$, then $M_\Gamma f \in W\widetilde{L}_{1,\lambda}(\Gamma)$ and*

$$\|M_\Gamma f\|_{W\widetilde{L}_{1,\lambda}(\Gamma)} \le C_{1,\lambda} \, \|f\|_{\widetilde{L}_{1,\lambda}(\Gamma)},$$

where $C_{1,\lambda}$ is independent of f.

2. If $f \in \widetilde{L}_{p,\lambda}(\Gamma)$, $1 < p < \infty$ and $0 \le \lambda < n$, then $M_\Gamma f \in \widetilde{L}_{p,\lambda}(\Gamma)$ and

$$\|M_\Gamma f\|_{\widetilde{L}_{p,\lambda}(\Gamma)} \le C_{p,\lambda} \, \|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)},$$

where $C_{p,\lambda}$ depends only on p and λ.

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