

Multi-sublinear maximal operators on product total Morrey-Guliyev spaces

Sabir G. Hasanov

Received: 21.07.2025 / Revised: 07.12.2025 / Accepted: 11.02.2026

Abstract. Let \mathcal{M} be the multi-sublinear maximal operator. We study the boundedness of the multi-sublinear maximal operator \mathcal{M} on product total Morrey-Guliyev spaces $L^{p_1, \lambda_1, \mu_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m, \mu_m}(\mathbb{R}^n)$ to total Morrey-Guliyev spaces $L^{p, \lambda, \mu}(\mathbb{R}^n)$.

Keywords. total Morrey-Guliyev spaces, multi-sublinear maximal function, commutator, *BMO*.

Mathematics Subject Classification (2010): Primary 42B20, 42B25, 42B35

1 Introduction

The classical Morrey spaces were introduced by Morrey [21] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of Morrey spaces on partial differential equation, the reader is referred to [4, 25]. The total Morrey-Guliyev spaces $L^{p, \lambda, \mu}(\mathbb{R}^n)$, introduced by Guliyev [7], extend the Morrey space $L^{p, \lambda}(\mathbb{R}^n)$ by including the second parameter μ , which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L^{p, \lambda}(\mathbb{R}^n)$ and $L^{p, \mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. The total Morrey-Guliyev spaces $L^{p, \lambda, \mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L^{p, \lambda, \mu}} = \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x, t))},$$

where $B(x, t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \leq \lambda$ is equal to the maximum of the norms of $L^{p, \lambda}(\mathbb{R}^n)$ and $L^{p, \mu}(\mathbb{R}^n)$. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\tilde{L}_{p, \lambda}(\mathbb{R}^n)$, see [2, 3, 6, 8–12, 19, 23, 24].

Let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \dots \times \mathbb{R}^n$ be the m -fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r , and by ${}^c B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. We denote by \mathbf{f} the m -tuple (f_1, f_2, \dots, f_m) , $\mathbf{y} = (y_1, \dots, y_m)$ and $d\mathbf{y} = dy_1 \cdots dy_m$.

In the past twenty years, the multilinear Calderón-Zygmund theory was developed a lot and studied by many authors. Grafakos and Torres [5] introduced the multilinear Calderón-Zygmund operator and studied the boundedness of such operators. Later in [18], Lerner et al. introduced the following multi-sublinear maximal function $\mathcal{M}(\mathbf{f})(x)$ defined as (see, also [16, 30–32])

$$\mathcal{M}(\mathbf{f})(x) = \sup_{B \ni x} \prod_{i=1}^m \frac{1}{|B|} \int_B |f_i(y_i)| dy_i.$$

When $m = 1$, we get $M \equiv M_1$ the classical Hardy-Littlewood maximal operator is given by

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where f is a locally integrable function. Obviously, it is easy to see $\mathcal{M}(\mathbf{f})(x) \leq \prod_{j=1}^m M(f_j)(x)$.

In this paper, we obtain the boundedness of the multi-sublinear maximal operator \mathcal{M} on product total Morrey-Guliyev spaces $L^{p_1, \lambda_1, \mu_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m, \mu_m}(\mathbb{R}^n)$ to total Morrey-Guliyev spaces $L^{p, \lambda, \mu}(\mathbb{R}^n)$.

This paper is organized as follows: In Section 2, we give some theorems about the boundedness of multi-sublinear maximal operator \mathcal{M} on the product total Morrey-Guliyev spaces $L^{p, \lambda, \mu}(\mathbb{R}^n)$, see also [13, 17, 20, 22, 26–29].

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2 Multi-sublinear maximal operator on product total Morrey-Guliyev spaces

In this section, we investigate the boundedness of multi-sublinear maximal operator \mathcal{M} on product total Morrey-Guliyev spaces.

Definition 2.1 Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L^{p, \lambda}(\mathbb{R}^n)$ the classical Morrey space, by $\tilde{L}^{p, \lambda}(\mathbb{R}^n)$ the modified Morrey space [6], and by $L^{p, \lambda, \mu}(\mathbb{R}^n)$ the total Morrey space the set of all classes of locally integrable functions f with the finite quasi-norms

$$\begin{aligned} \|f\|_{L^{p, \lambda}} &= \sup_{x \in \mathbb{R}^n, t > 0} t^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x, t))}, \quad \|f\|_{\tilde{L}^{p, \lambda}} = \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x, t))}, \\ \|f\|_{L^{p, \lambda, \mu}} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x, t))}, \end{aligned}$$

respectively.

Definition 2.2 Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $WL^{p, \lambda}(\mathbb{R}^n)$, the weak modified Morrey space $W\tilde{L}^{p, \lambda}(\mathbb{R}^n)$ [6] and the weak total Morrey space $WL^{p, \lambda, \mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite quasi-norms

$$\begin{aligned} \|f\|_{WL^{p, \lambda}} &= \sup_{x \in \mathbb{R}^n, t > 0} t^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x, t))}, \quad \|f\|_{W\tilde{L}^{p, \lambda}} = \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x, t))}, \\ \|f\|_{WL^{p, \lambda, \mu}} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL^p(B(x, t))}, \end{aligned}$$

respectively.

Lemma 2.1 [7, 9] *If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq n$, then*

$$\begin{aligned} L^{p,\lambda,\mu}(\mathbb{R}^n) &= L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n), \\ WL^{p,\lambda,\mu}(\mathbb{R}^n) &= WL^{p,\lambda}(\mathbb{R}^n) \cap WL^{p,\mu}(\mathbb{R}^n) \end{aligned}$$

and

$$\begin{aligned} \|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} &= \max \{ \|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}} \}, \\ \|f\|_{WL^{p,\lambda,\mu}(\mathbb{R}^n)} &= \max \{ \|f\|_{WL^{p,\lambda}}, \|f\|_{WL^{p,\mu}} \}, \end{aligned}$$

respectively.

The following local estimate is valid.

Lemma 2.2 [1, Lemma 3.3] *Let $1 \leq p < \infty$ and $B(x, r)$ be any ball in \mathbb{R}^n . Then, for $p > 1$ the inequality*

$$\|Mf\|_{L^p(B(x,r))} \lesssim r^{\frac{n}{p}} \sup_{t>2r} t^{-\frac{n}{p}} \|f\|_{L^p(B(x,t))} \quad (2.1)$$

holds for all $B(x, r)$ and for all $f \in L^p_{\text{loc}}(\mathbb{R}^n)$.

Moreover if $p = 1$, then the inequality

$$\|Mf\|_{WL^1(B(x,r))} \lesssim r^n \sup_{t>2r} t^{-n} \|f\|_{L^1(B(x,t))}$$

holds for all $B(x, r)$ and for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Theorem 2.1 [7] 1. *If $f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$, $0 \leq \lambda < n$ and $0 \leq \mu < n$, then $Mf \in WL^{1,\lambda,\mu}(\mathbb{R}^n)$ and*

$$\|Mf\|_{WL^{1,\lambda,\mu}} \leq C_{1,\lambda,\mu} \|f\|_{L^{1,\lambda,\mu}}, \quad (2.2)$$

where $C_{1,\lambda,\mu}$ is independent of f .

2. *If $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \leq \lambda < n$ and $0 \leq \mu < n$, then $Mf \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and*

$$\|Mf\|_{L^{p,\lambda,\mu}} \leq C_{p,\lambda,\mu} \|f\|_{L^{p,\lambda,\mu}}, \quad (2.3)$$

where $C_{p,\lambda,\mu}$ depends only on p, λ, μ and n .

Lemma 2.3 [15] *Let p be the harmonic mean of $p_1, \dots, p_m > 1$ and $\mathbf{f} \in L^1_{\text{loc}}(\mathbb{R}^n) \times \dots \times L^1_{\text{loc}}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$*

$$\mathcal{M}\mathbf{f}(x) \leq \prod_{j=1}^m \left[M(f_j^{\frac{p_j}{p}})(x) \right]^{\frac{p}{p_j}}, \quad (2.4)$$

When $m \geq 2$, we find out \mathcal{M} also have the same properties by providing the following multi-version of the Theorem 2.1.

Theorem 2.2 *Let p be the harmonic mean of $p_1, \dots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^m \frac{\lambda_j}{p_j} \text{ for } 0 \leq \lambda_j < n, \quad \frac{\mu}{p} = \sum_{j=1}^m \frac{\mu_j}{p_j} \text{ for } 0 \leq \mu_j < n. \quad (2.5)$$

(i) If $p > 1$, then the operator \mathcal{M} is bounded from product total Morrey space $L^{p_1, \lambda_1, \mu_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m, \mu_m}(\mathbb{R}^n)$ to total Morrey space $L^{p, \lambda, \mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\mathbf{f} \in L^{p_1, \lambda_1, \mu_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m, \mu_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\mathbf{f}\|_{L^{p, \lambda, \mu}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j, \mu_j}}.$$

(ii) If $p = 1$, then the operator \mathcal{M} is bounded from product total Morrey space $L^{p_1, \lambda_1, \mu_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m, \mu_m}(\mathbb{R}^n)$ to weak total Morrey space $WL^{p, \lambda, \mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\mathbf{f} \in L^{p_1, \lambda_1, \mu_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m, \mu_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\mathbf{f}\|_{WL^{p, \lambda, \mu}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j, \mu_j}}.$$

Proof. (i) If $p > 1$, by (2.4) and the Hölder inequality, we get

$$\begin{aligned} & \left([t]_1^{-\lambda} [1/t]_1^\mu \int_{B(x,t)} |\mathcal{M}\mathbf{f}(y)|^p dy \right)^{\frac{1}{p}} \\ & \leq \left([t]_1^{-\lambda} [1/t]_1^\mu \int_{B(x,t)} \left(\prod_{j=1}^m \left[M(f_j^{\frac{p_j}{p}})(y) \right]^{\frac{p}{p_j}} \right)^p dy \right)^{\frac{1}{p}} \\ & \leq \prod_{j=1}^m \left([t]_1^{-\lambda_j} [1/t]_1^{\mu_j} \int_{B(x,t)} \left[M(f_j^{\frac{p_j}{p}})(y) \right]^p dy \right)^{\frac{1}{p_j}}. \end{aligned}$$

Taking the p -th root of both sides and applying Theorem 2.1 with $p > 1$ and $|f_j|^{\frac{p_j}{p}} \in L^{p, \lambda_j, \mu_j}(\mathbb{R}^n)$, we get

$$\begin{aligned} \|\mathcal{M}\mathbf{f}\|_{L^{p, \lambda, \mu}} &= \sup_{x \in \mathbb{R}^n, t > 0} \left([t]_1^{-\lambda} [1/t]_1^\mu \int_{B(x,t)} |\mathcal{M}\mathbf{f}(y)|^p dy \right)^{\frac{1}{p}} \\ &= \prod_{j=1}^m \left\| M(f_j^{\frac{p_j}{p}}) \right\|_{L^{p, \lambda_j, \mu_j}} \leq \prod_{j=1}^m \left\| f_j^{\frac{p_j}{p}} \right\|_{L^{p, \lambda_j, \mu_j}} = \prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j, \mu_j}}, \end{aligned}$$

which is the desired inequality.

(ii) If $p = 1$, for any $\tau > 0$, let $\varepsilon_0 = \tau$, $\varepsilon_m = 1$ and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1} > 0$ be arbitrary which will be chosen later. From the pointwise estimate (2.4), we get

$$\begin{aligned} & \{y \in B(x, t) : \mathcal{M}\mathbf{f}(y) > \tau\} \\ & \subset \bigcup_{j=1}^m \left\{ y \in B(x, t) : \left[M(f_j^{\frac{p_j}{p}})(y) \right]^{\frac{p}{p_j}} > \frac{\varepsilon_{j-1}}{\varepsilon_j [t]_1^{\frac{\lambda-\lambda_j}{p_j}} [1/t]_1^{\frac{\mu-\mu_j}{p_j}}} \right\}. \end{aligned}$$

Let us now take $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1} > 0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j-1}} = \frac{\left[\prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j, \mu_j}} \right]^{s'/p_j}}{\tau^{s'/p_j} \|f_j\|_{L^{p_j, \lambda_j, \mu_j}}}, \quad j = 1, 2, \dots, m.$$

Then, applying Theorem 2.1 with $p = 1$ and the fact $|f_j|^{p_j} \in L^{1, \lambda_j, \mu_j}(\mathbb{R}^n)$, we get

$$\begin{aligned}
& \left| \{y \in B(x, t) : \mathcal{M}\mathbf{f}(y) > \tau\} \right| \\
& \lesssim \sum_{j=1}^m \left| \left\{ y \in B(x, t) : M(f_j^{p_j})(y) > \left(\frac{\varepsilon_{j-1}}{\varepsilon_j [t]_1^{\frac{\lambda-\lambda_j}{p_j}} [1/t]_1^{-\frac{\mu-\mu_j}{p_j}}} \right)^{p_j} \right\} \right| \\
& \leq \sum_{j=1}^m [t]_1^{\lambda_j} [1/t]_1^{-\mu_j} \left(\frac{\varepsilon_j [t]_1^{\frac{\lambda-\lambda_j}{p_j}} [1/t]_1^{-\frac{\mu-\mu_j}{p_j}}}{\varepsilon_{j-1}} \right)^{p_j} \|f_j^{p_j}\|_{L^{1, \lambda_j, \mu_j}} \\
& = \sum_{j=1}^m [t]_1^{\lambda} [1/t]_1^{-\mu} \left(\frac{\varepsilon_j}{\varepsilon_{j-1}} \right)^{p_j} \|f_j\|_{L^{p_j, \lambda_j, \mu_j}}^{p_j} = \sum_{j=1}^m [t]_1^{\lambda} [1/t]_1^{-\mu} \left[\left(\frac{\varepsilon_j}{\varepsilon_{j-1}} \right) \|f_j\|_{L^{p_j, \lambda_j, \mu_j}} \right]^{p_j} \\
& = \sum_{j=1}^m [t]_1^{\lambda} [1/t]_1^{-\mu} \left(\frac{1}{\tau} \prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j, \mu_j}} \right)^{s'} = [t]_1^{\lambda} [1/t]_1^{-\mu} \left(\frac{1}{\tau} \prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j, \mu_j}} \right)^p.
\end{aligned}$$

Hence, we obtain the following inequality

$$\begin{aligned}
& \|\mathcal{M}\mathbf{f}\|_{WL^{p, \lambda, \mu}} \\
& = \sup_{\tau > 0} \tau \sup_{x \in \mathbb{R}^n, t > 0} \left([t]_1^{-\lambda} [1/t]_1^{\mu} \left| \{y \in B(x, t) : \mathcal{M}\mathbf{f}(y) > \tau\} \right| \right)^{\frac{1}{p}} \\
& \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j, \mu_j}}.
\end{aligned}$$

This is the conclusion (ii) of Theorem 2.2.

In the case $\lambda = \mu$, $\lambda_j = \mu_j$, $j = 1, \dots, m$ from Theorem 2.2 we get the following corollary

Corollary 2.1 [14, Theorem 2] *Let p be the harmonic mean of $p_1, \dots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^m \frac{\lambda_j}{p_j} \text{ for } 0 \leq \lambda_j < n. \quad (2.6)$$

(i) *If $p > 1$, then the operator \mathcal{M} is bounded from product Morrey space $L^{p_1, \lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m}(\mathbb{R}^n)$ to Morrey space $L^{p, \lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\mathbf{f} \in L^{p_1, \lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m}(\mathbb{R}^n)$*

$$\|\mathcal{M}\mathbf{f}\|_{L^{p, \lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j}}.$$

(ii) *If $p = 1$, then the operator \mathcal{M} is bounded from product Morrey space $L^{p_1, \lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m}(\mathbb{R}^n)$ to weak Morrey space $WL^{p, \lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\mathbf{f} \in L^{p_1, \lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m, \lambda_m}(\mathbb{R}^n)$*

$$\|\mathcal{M}\mathbf{f}\|_{WL^{p, \lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j, \lambda_j}}.$$

In the case $\mu = \mu_j = 0$, $j = 1, \dots, m$ from Theorem 2.2 we get the following corollary

Corollary 2.2 [14, Theorem 4] *Let p be the harmonic mean of $p_1, \dots, p_m > 1$ and satisfy (2.6).*

(i) *If $p > 1$, then the operator \mathcal{M} is bounded from product modified Morrey space $\tilde{L}^{p_1, \lambda_1}(\mathbb{R}^n) \times \dots \times \tilde{L}^{p_m, \lambda_m}(\mathbb{R}^n)$ to modified Morrey space $\tilde{L}^{p, \lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\mathbf{f} \in \tilde{L}^{p_1, \lambda_1}(\mathbb{R}^n) \times \dots \times \tilde{L}^{p_m, \lambda_m}(\mathbb{R}^n)$*

$$\|\mathcal{M}\mathbf{f}\|_{\tilde{L}^{p, \lambda}} \leq C \prod_{j=1}^m \|f_j\|_{\tilde{L}^{p_j, \lambda_j}}.$$

(ii) *If $p = 1$, then the operator \mathcal{M} is bounded from product modified Morrey space $\tilde{L}^{p_1, \lambda_1}(\mathbb{R}^n) \times \dots \times \tilde{L}^{p_m, \lambda_m}(\mathbb{R}^n)$ to weak modified Morrey space $W\tilde{L}^{p, \lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\mathbf{f} \in \tilde{L}^{p_1, \lambda_1}(\mathbb{R}^n) \times \dots \times \tilde{L}^{p_m, \lambda_m}(\mathbb{R}^n)$*

$$\|\mathcal{M}\mathbf{f}\|_{W\tilde{L}^{p, \lambda}} \leq C \prod_{j=1}^m \|f_j\|_{\tilde{L}^{p_j, \lambda_j}}.$$

Acknowledgements.

The author would like to express his gratitude to the referee for his/her very valuable comments and kindly suggestion.

References

1. Akbulut A., Guliyev V.S., Mustafayev R.Ch.: *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. **137** (1), 27-43 (2012).
2. Akbulut, A., Isayev, F.A., Omarova, M.N.: *Commutator of parabolic fractional integral operators with parabolic Lipschitz functions on parabolic total Morrey-Guliyev spaces*, Azerb. J. Math. **16**(1), 287-300 (2026).
3. Celik, S., Akbulut, A., Omarova, M.N.: *Characterizations of anisotropic Lipschitz functions via the commutators of anisotropic maximal function in total anisotropic Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **45**(1), 25-37 (2025).
4. Fazio GD, Palagachev D.K., Ragusa M.A.: *Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients*, J. Funct. Anal. **166** (2), 179-196 (1999).
5. Grafakos L., Torres R.: *Multilinear Calderón-Zygmund theory*, Adv. Math. **165**, 124-164 (2002).
6. Guliyev, V.S., Hasanov, J.J., Zeren, Y.: *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. **5**(4), 491-506 (2011).
7. Guliyev, V.S.: *Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal. **16**(4), 1509-1524 (2022).
8. Guliyev, V.S.: *Characterizations of Lipschitz functions via the commutators of maximal function in total Morrey spaces*, Math. Meth. Appl. Sci, **47**(11), 8669-8682 (2024).

9. Guliyev, V.S.: *Characterizations for the fractional maximal operator and its commutators on total Morrey spaces*, *Positivity* **28**(4), Article no. 51, 1-20 (2024).
10. Guliyev, V.S., Isayev, F.A., Serbetci, A.: *Boundedness of the anisotropic fractional maximal operator in total anisotropic Morrey spaces*, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics* **44**(1), 41-50 (2024).
11. Guliyev, V.S.: *Characterizations of commutators of the maximal function in total Morrey spaces on stratified Lie groups*, *Anal. Math. Phys.* **15**, Article number: 42 (2025).
12. Guliyev, V.S., Akbulut, A., Isayev, F.A., Serbetci, A.: *Commutators of maximal function with BMO functions on total mixed Morrey spaces*, *Journal of Contemporary Applied Mathematics* **16**(1), 119-138 (2026).
13. Guliyev, V.S.: *Hardy-Littlewood-Sobolev inequality in total Morrey spaces on stratified Lie groups*, *Appl. Comput. Math.* **25**(1), 1-20 (2026).
14. Hasanov S.G.: *Multi-sublinear rough maximal operator on product Morrey and product modified Morrey spaces*, *Journal of Contemporary Applied Mathematics*, **4** (2), 57-65 (2014).
15. Hasanov S.G.: *Multi-sublinear rough fractional maximal operator on product modified Morrey spaces*, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **41** (1), 77-87 (2015).
16. Hu, G., Yang, D.: *Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type*, *J. Math. Anal. Appl.* **354**, 249-262 (2009).
17. Kpata, B.A., Nagacy, P.: *Total Fofana-Guliyev spaces with applications to norm inequalities for classical operators*, *Khayyam Journal of Mathematics*, **11**(2), 285-301 (2026).
18. Lerner A.K., Ombrosi S., Perez C., Torres R.H., Trujillo-Gonzalez R.: *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, *Adv. Math.* **220**, 1222-1264 (2009).
19. Mammadov, Y.Y., Guliyev, V.S., Muslimova, F.A.: *Commutator of the maximal function in total Morrey spaces for the Dunkl operator on the real line*, *Azerb. J. Math.* **15**(2), 85-104 (2025).
20. Mammadov, Y.Y., Akbulut, A., Guliyev, V.S., Omarova, M.N.: *Commutator of the maximal function in total Morrey spaces in the Dunkl setting*, *TWMS Journal of Pure and Applied Mathematics*, **17**(1), 1-18 (2026).
21. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, *Trans. Amer. Math. Soc.* **43**, 126-166 (1938).
22. Nagacy, P. *Boundedness of some commutators in total Fofana spaces*, *European Journal of Mathematical Analysis*, **4**(22), 1-10 (2024).
23. Omarova, M.N.: *Commutators of parabolic fractional maximal operators on parabolic total Morrey spaces*, *Math. Meth. Appl. Sci.* **48**(11), 11037-11044 (2025).
24. Omarova, M.N.: *Commutators of anisotropic maximal operators with BMO functions on anisotropic total Morrey spaces*, *Azerb. J. Math.* **15**(2), 150-162 (2025).
25. Ragusa M.A.: *Regularity for weak solutions to the Dirichlet problem in Morrey space*, *Rev. Mat. Univ. Parma*, **5** (3), 355-369 (1994).
26. Serbetci, A.: *Boundedness of the B-maximal operator in B-total Morrey-Guliyev spaces*, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics* **45**(4), 145-154 (2025).
27. Xue, F.: *Commutators for the fractional maximal and sharp functions on total Morrey spaces*, *Proceedings of the Romanian Academy. Series A*, **25**(2), 103-110 (2024).
28. Wei, M., Li, S., Liu, X., Hou, Y.: *Three geometric constants for total Morrey spaces*, *Mathematical Inequalities and Applications*, **28**(2), 271-277 (2025).
29. Wei, M., Hou, Y., Yan, D.: *Sharp bounds for the m-linear Hardy operator on central total Morrey spaces*, *Mediterranean Journal of Mathematics*, **22**, Paper No. 47, (2025).

-
30. Zhang P.: *Multiple weighted estimates for commutators of multilinear maximal function*, Acta Math. Sin. (Engl. Ser.) **31** (6), 973-994 (2015).
 31. Zhang P.: *Commutators of multi-sublinear maximal functions with Lipschitz functions*, Results Math. **74** (1), Paper No. 49, 21 pp. (2019).
 32. Yu X., Zhang P., Li H.: *Some estimates for the commutators of multilinear maximal function on Morrey-type space*, Open Math. **19** (1), 515-530 (2021).