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Cubature formula for a class of vector potentials with weak singularities

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Abstract. *This paper presents a cubature formula for a class of vector potentials with weak singularities. In addition, error estimates for the constructed cubature formulas are provided.*

Keywords. electrical boundary problems, magnetic boundary problems, vector potentials, cubature formula.

Mathematics Subject Classification (2010): 65R20, 45E05

1 Introduction

It is known that (see [1, p. 153-155]) the internal and external electric boundary problems, as well as the internal and external magnetic boundary problems, lead to a system of integral equations that depend on vector potentials:

$$(Af)(x) = 2\int_{\Omega} \Phi_k(x,\, y)\ [n(x), [n(y), f(y)]]\, d\Omega_y, x = (x_1, x_2, x_3) \in \Omega, \tag{1.1}$$

and

$$(Bg)(x) = 2\int_{\Omega} [n(x), rot_x \{\Phi_k(x,\, y)\, g(y)\}]\, d\Omega_y, x = (x_1, x_2, x_3) \in \Omega, \tag{1.2}$$

where $\Omega \subset R^3$ is the Lyapunov surface, $n(x) = (n_1(x), n_2(x), n_3(x))$ is the outer unit normal at the point $x \in \Omega$, vector function $f(x) = (f_1(x), f_2(x), f_3(x))$ belongs to the class $C(\Omega)$ −the space of all continuous functions on the surface Ω with the norm $\|f\|_\infty = \max\limits_{x\in\Omega} |f(x)|$, the vector function $g(x) = (g_1(x), g_2(x), g_3(x))$ belongs to the class $C_\perp(\Omega) = \{g \in C(\Omega) \mid (g(x), n(x)) = 0,\ \forall x \in \Omega\}$, the notation $[a, b]$ means the cross product of the vectors a and b,the notation (a, b) −the dot product,

$$\Phi_k(x,\, y) = \frac{\exp(i\,k\,|x-y|)}{4\pi\,|x-y|}, x, y \in R^3, x \neq y,$$

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the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$, Δ is the Laplace operator, and k is a wave number with $\operatorname{Im} k \geq 0$.

Since in many cases it is impossible to find exact solutions to integral equations, the study of approximate solutions to these integral equations becomes of interest. To find an approximate solution, it is primarily necessary to construct cubature formulas for the integrals involved in these equations. It should be noted that in work [7], a quadrature formula for a class of weakly singular curvilinear integrals was constructed; in work [8], a quadrature formula for the normal derivative of the double layer potential was developed; in work [3], a cubature formula for the normal derivative of the acoustic potential of a simple layer was presented; and in work [5], using the result of the work [4] a new method for constructing a cubature formula for the normal derivative of the acoustic potential of a double layer was proposed. The present work is dedicated to constructing cubature formulas for the integrals (1.1) and (1.2).

2 Cubature formula for integral (1.1)

We partition Ω into "regular" elementary parts: $\Omega = \bigcup\limits_{l=1}^{N} \Omega_l$. By a regular elementary part we mean a set of points subordinate to the following requirements:

(1) for each $l \in \{1, 2, ..., N\}$ the elementary part Ω_l is closed and the set $\overset{0}{\Omega}_l$ of its interior points with respect to Ω is not empty; moreover, $mes\,\overset{0}{\Omega}_l = mes\Omega_l$ and $\overset{0}{\Omega}_l \bigcap \overset{0}{\Omega}_j = \emptyset$ for $j \in \{1, 2, ...N\}, j \neq l$;

(2) for each $l \in \{1, 2, ..., N\}$ the elementary part Ω_l is a connected piece of the surface Ω and the boundary of the elementary part Ω_l is a continuous curve;

(3) for each $l \in \{1, 2, ..., N\}$ there exists a so-called supporting point $x(l) = \big(x_1(l), x_2(l), x_3(l)\big) \in \Omega_l$ such that

(3.1) $r_l(N) \sim R_l(N)$ (the expression $r_l(N) \sim R_l(N)$ means that $r_l(N)$ and $R_l(N)$ are equivalent, i.e., there exist numbers $C_1 > 0$ and $C_2 < +\infty$ such that $C_1 \leq \frac{r_l(N)}{R_l(N)} \leq C_2$ for any N), where $r_l(N) = \min\limits_{x \in \partial\Omega_l} |x - x(l)|$ and $R_l(N) = \max\limits_{x \in \partial\Omega_l} |x - x(l)|$;

(3.2) $R_l(N) \leq \frac{d}{2}$, where d is the radius of the standard sphere (see [9, p. 400]);

(3.3) $r_j(N) \sim r_l(N)$ for each $j \in \{1, 2, ..., N\}$.

Obviously, $r(N) \sim R(N)$ and $\lim\limits_{N\to\infty} r(N) = \lim\limits_{N\to\infty} R(N) = 0$, where $R(N) = \max\limits_{l=\overline{1,N}} R_l(N)$, $r(N) = \min\limits_{l=\overline{1,N}} r_l(N)$.

The following lemmas are true.

Lemma 2.1 *([6]). There exist constants $C_0' > 0$ and $C_1' > 0$ not depending on N such that, for all $l, j \in \{1, 2, ..., N\}$, $j \neq l$, and all $y \in \Omega_j$, the following inequalities hold:*

$$C_0' \, |y - x(l)| \leq |x(j) - x(l)| \leq C_1' \, |y - x(l)|,$$

where the $x(l)$, $l \in \{1, 2, ..., N\}$, are supporting points.

Lemma 2.2 *([6]). For a partition $\Omega = \bigcup\limits_{l=1}^{N} \Omega_l$ of the surface Ω into regular elementary parts, the following relation holds: $R(N) \sim \frac{1}{\sqrt{N}}$.*

Let us introduce the modulus of continuity of the vector function $f \in C(\Omega)$:

$$\omega(f,\delta) = \delta \sup_{\tau \geq \delta} \frac{\bar{\omega}(f,\tau)}{\tau}, \delta > 0,$$

where

$$\bar{\omega}(f,\tau) = \max_{\substack{|x-y| \leq \tau \\ x,\, y \in \Omega}} |f(x) - f(y)|,$$

$$|f(x) - f(y)| = \sqrt{(f_1(x) - f_1(y))^2 + (f_2(x) - f_2(y))^2 + (f_3(x) - f_3(y))^2}.$$

Moreover, let

$$\left(A^N f\right)(x(l)) = e_1\left(\left(A_{11}^N f\right)(x(l)) + \left(A_{12}^N f\right)(x(l))\right)$$

$$+e_2\left(\left(A_{21}^N f\right)(x(l)) + \left(A_{22}^N f\right)(x(l))\right) + e_3\left(\left(A_{31}^N f\right)(x(l)) + \left(A_{32}^N f\right)(x(l))\right),$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$,

$$\left(A_{11}^N f\right)(x(l)) = 2n_2(x(l))$$

$$\times \sum_{\substack{j=1, \\ j \neq l}}^{N} \Phi_k(x(l), x(j)) (n_1(x(j)) f_2(x(j)) - n_2(x(j)) f_1(x(j))) mes\Omega_j,$$

$$\left(A_{12}^N f\right)(x(l)) = 2n_3(x(l)) \times$$

$$\times \sum_{\substack{j=1, \\ j \neq l}}^{N} \Phi_k(x(l), x(j)) (n_1(x(j)) f_3(x(j)) - n_3(x(j)) f_1(x(j))) mes\Omega_j,$$

$$\left(A_{21}^N f\right)(x(l)) = 2n_3(x(l))$$

$$\times \sum_{\substack{j=1, \\ j \neq l}}^{N} \Phi_k(x(l), x(j)) (n_2(x(j)) f_3(x(j)) - n_3(x(j)) f_2(x(j))) mes\Omega_j,$$

$$\left(A_{22}^N f\right)(x(l)) = 2n_1(x(l))$$

$$\times \sum_{\substack{j=1, \\ j \neq l}}^{N} \Phi_k(x(l), x(j)) (n_2(x(j)) f_1(x(j)) - n_1(x(j)) f_2(x(j))) mes\Omega_j,$$

$$\left(A_{31}^N f\right)(x(l)) = 2n_1(x(l))$$

$$\times \sum_{\substack{j=1, \\ j \neq l}}^{N} \Phi_k(x(l), x(j)) (n_3(x(j)) f_1(x(j)) - n_1(x(j)) f_3(x(j))) mes\Omega_j$$

and

$$\left(A_{32}^N f\right)(x(l)) = 2n_2(x(l)) \times$$

$$\times \sum_{\substack{j=1, \\ j \neq l}}^{N} \Phi_k(x(l), x(j)) (n_3(x(j)) f_2(x(j)) - n_2(x(j)) f_3(x(j))) mes\Omega_j.$$

Theorem 2.1 *Let $f \in C(\Omega)$ and $\Omega \subset R^3$ be a Lyapunov surface with exponent $0 < \alpha \leq 1$. Then the expression $\left(A^N f\right)(x(l))$ at the support points $x(l)$, $l = \overline{1, N}$, is the cubature formula for the integral* (1.1)*, and*

$$\max_{l=\overline{1,N}} \left|(Af)(x(l)) - \left(A^N f\right)(x(l))\right| \leq M^1 \left(\|f\|_\infty N^{-\frac{\alpha}{2}} + \omega\left(f, N^{-\frac{1}{2}}\right)\right) \text{ at } 0 < \alpha < 1,$$

$$\max_{l=\overline{1,N}} \left|(Af)(x(l)) - \left(A^N f\right)(x(l))\right| \leq M \left(\|f\|_\infty N^{-\frac{1}{2}} \ln N + \omega\left(f, N^{-\frac{1}{2}}\right)\right) \text{ at } \alpha = 1.$$

Proof. It is easy to calculate that

$$(Af)(x) = e_1\left((A_{11}f)(x) + (A_{12}f)(x)\right) +$$
$$+ e_2\left((A_{21}f)(x) + (A_{22}f)(x)\right) + e_3\left((A_{31}f)(x) + (A_{32}f)(x)\right),$$

where

$$(A_{11}f)(x) = 2n_2(x) \int_\Omega \Phi_k(x, y)\left(n_1(y) f_2(y) - n_2(y) f_1(y)\right) d\Omega_y,$$

$$(A_{12}f)(x) = 2n_3(x) \int_\Omega \Phi_k(x, y)\left(n_1(y) f_3(y) - n_3(y) f_1(y)\right) d\Omega_y,$$

$$(A_{21}f)(x) = 2n_3(x) \int_\Omega \Phi_k(x, y)\left(n_2(y) f_3(y) - n_3(y) f_2(y)\right) d\Omega_y,$$

$$(A_{22}f)(x) = 2n_1(x) \int_\Omega \Phi_k(x, y)\left(n_2(y) f_1(y) - n_1(y) f_2(y)\right) d\Omega_y,$$

$$(A_{31}f)(x) = 2n_1(x) \int_\Omega \Phi_k(x, y)\left(n_3(y) f_1(y) - n_1(y) f_3(y)\right) d\Omega_y,$$

and

$$(A_{32}f)(x) = 2n_2(x) \int_\Omega \Phi_k(x, y)\left(n_3(y) f_2(y) - n_2(y) f_3(y)\right) d\Omega_y.$$

As can be seen, to prove the theorem, it is sufficient to show that the expression $\left(A_{11}^N f\right)(x(l))$ at the points $x(l)$, $l = \overline{1, N}$, is a cubature formula for the integral $(A_{11}f)(x)$ and estimate the errors of this cubature formula. It is obvious that

$$(A_{11}f)(x(l)) - \left(A_{11}^N f\right)(x(l))$$
$$= \frac{n_2(x(l))}{2\pi} \int_{\Omega_l} \frac{\left(n_1(y) f_2(y) - n_2(y) f_1(y)\right) \exp\left(ik|x(l) - y|\right)}{|x(l) - y|} d\Omega_y$$
$$+ \frac{n_2(x(l))}{2\pi} \sum_{\substack{j=1,\\ j\neq l}}^{N} \int_{\Omega_j} \left(\frac{\left(n_1(y) f_2(y) - n_2(y) f_1(y)\right) \exp\left(ik|x(l) - y|\right)}{|x(l) - y|}\right.$$
$$\left. - \frac{\left(n_1(x(j)) f_2(x(j)) - n_2(x(j)) f_1(x(j))\right) \exp\left(ik|x(l) - x(j)|\right)}{|x(l) - y|}\right) d\Omega_y$$
$$+ \frac{n_2(x(l))}{2\pi} \sum_{\substack{j=1,\\ j\neq l}}^{N} \int_{\Omega_j} \left(\frac{1}{|x(l) - y|} - \frac{1}{|x(l) - x(j)|}\right)$$

[1] From here on we will denote by M positive constants that are different in different inequalities.

$$\times \left(n_1 \left(x \left(j\right)\right) f_2 \left(x \left(j\right)\right) - n_2 \left(x \left(j\right)\right) f_1 \left(x \left(j\right)\right)\right) \exp \left(ik \left|x \left(l\right) - x \left(j\right)\right|\right) d\Omega_y.$$

We denote the terms in the last equality by $h_1^N \left(x \left(l\right)\right)$, $h_2^N \left(x \left(l\right)\right)$ and $h_3^N \left(x \left(l\right)\right)$, respectively.

Applying the formula for reducing a surface integral to a double integral (see [2, p. 276]) and taking into account Lemmas 2.1 and 2.2, we obtain:

$$\left|h_1^N \left(x \left(l\right)\right)\right| \leq M \left\|f\right\|_\infty \int_{\Omega_l} \frac{d\Omega_y}{\left|x \left(l\right) - y\right|} \leq M \left\|f\right\|_\infty \int_0^{R(N)} dt \leq M \left\|f\right\|_\infty \frac{1}{\sqrt{N}}.$$

Let $y \in \Omega_j$ and $j \neq l$. Since

$$\left(n_1 \left(y\right) f_2 \left(y\right) - n_2 \left(y\right) f_1 \left(y\right)\right) \exp \left(ik \left|x \left(l\right) - y\right|\right)$$
$$- \left(n_1 \left(x \left(j\right)\right) f_2 \left(x \left(j\right)\right) - n_2 \left(x \left(j\right)\right) f_1 \left(x \left(j\right)\right)\right) \exp \left(ik \left|x \left(l\right) - x \left(j\right)\right|\right)$$
$$= \left(n_1 \left(y\right) f_2 \left(y\right) - n_2 \left(y\right) f_1 \left(y\right)\right) \left(\exp \left(ik \left|x \left(l\right) - y\right|\right) - \exp \left(ik \left|x \left(l\right) - x \left(j\right)\right|\right)\right)$$
$$+ \left(\left(n_1 \left(y\right) - n_1 \left(x \left(j\right)\right)\right) f_2 \left(y\right) + n_1 \left(x \left(j\right)\right) \left(f_2 \left(y\right) - f_2 \left(x \left(j\right)\right)\right)\right) \exp \left(ik \left|x \left(l\right) - x \left(j\right)\right|\right)$$
$$+ \left(\left(n_2 \left(x \left(j\right)\right) - n_2 \left(y\right)\right) f_1 \left(x \left(j\right)\right) + n_2 \left(y\right) \left(f_1 \left(x \left(j\right)\right) - f_1 \left(y\right)\right)\right) \exp \left(ik \left|x \left(l\right) - x \left(j\right)\right|\right),$$

then taking into account the inequalities

$$\left|\exp \left(ik \left|x - y\right|\right) - \exp \left(ik \left|x - z\right|\right)\right| \leq M \left|y - z\right|, \forall x, y, z \in \Omega, \tag{2.1}$$

and

$$\left|n \left(y\right) - n \left(x\right)\right| \leq M \left|y - x\right|^\alpha, \forall x, y \in \Omega, \tag{2.2}$$

we have

$$\left|h_2^N \left(x \left(l\right)\right)\right| \leq M \sum_{\substack{j=1, \\ j \neq l}}^N \int_{\Omega j} \frac{\left\|f\right\|_\infty \left(R \left(N\right)\right)^\alpha + \omega \left(f, R \left(N\right)\right)}{\left|x \left(l\right) - y\right|} d\Omega_y$$
$$\leq M \left(\left\|f\right\|_\infty \left(R \left(N\right)\right)^\alpha + \omega \left(f, R \left(N\right)\right)\right) \int_\Omega \frac{1}{\left|x \left(l\right) - y\right|} d\Omega_y$$
$$\leq M \left(\frac{\left\|f\right\|_\infty}{\sqrt{N^\alpha}} + \omega \left(f, \frac{1}{\sqrt{N}}\right)\right).$$

Moreover, from Lemma 2.1 and 2.2, we obtain that for any$y \in \Omega_j$, $j \neq l$,

$$\left|\frac{1}{\left|x \left(l\right) - y\right|} - \frac{1}{\left|x \left(l\right) - x \left(j\right)\right|}\right| \leq M \frac{\left|y - x \left(j\right)\right|}{\left|x \left(l\right) - y\right| \left|x \left(l\right) - x \left(j\right)\right|}$$
$$\leq M \frac{R \left(n\right)}{\left|x \left(l\right) - y\right|^2} \leq \frac{M}{\left|x \left(l\right) - y\right|^2 \sqrt{N}}.$$

Then

$$\left|h_3^N \left(x \left(l\right)\right)\right| \leq M \left\|f\right\|_\infty \frac{1}{\sqrt{N}} \int_{\Omega \backslash \Omega_l} \frac{1}{\left|x \left(l\right) - y\right|^2} d\Omega_y$$
$$\leq M \left\|f\right\|_\infty \frac{1}{\sqrt{N}} \int_{r(N)}^{diam\, \Omega} \frac{dt}{t} dt \leq M \left\|f\right\|_\infty \frac{\ln N}{\sqrt{N}}.$$

As a result, summing up the obtained estimates for the expressions $h_1^N(x(l))$, $h_2^N(x(l))$ and $h_3^N(x(l))$, we obtain that the expression $\left(A_{11}^N f\right)(x(l))$ at the points$x(l)$, $l=\overline{1,N}$, is a cubature formula for the integral $(A_{11}f)(x)$, and

$$\max_{l=\overline{1,N}} \left|(A_{11}f)(x(l)) - \left(A_{11}^N f\right)(x(l))\right|$$

$$\leq M\left(\|f\|_\infty \frac{1}{\sqrt{N^\alpha}} + \|f\|_\infty \frac{\ln N}{\sqrt{N}} + \omega\left(f, \frac{1}{\sqrt{N}}\right)\right).$$

The theorem is proven.

3 Cubature formula for integral (1.2)

Since, $(g(x), n(x)) = 0$, $\forall x \in \Omega$, it is obvious that

$$[n(x), rot_x\{\Phi_k(x, y)\, g(y)\}]$$

$$= (n(x) - n(y), g(y))\, grad_x \Phi_k(x, y) - g(y)\frac{\partial \Phi_k(x, y)}{\partial n(x)}.$$

Then the integral (1.2) can be represented as

$$(Bg)(x) = e_1(B_1 g)(x) + e_2(B_2 g)(x) + e_3(B_3 g)(x),$$

where

$$(B_1 g)(x) = 2\int_\Omega \left((n(x) - n(y), g(y))\frac{\partial \Phi_k(x, y)}{\partial x_1} - g_1(y)\frac{\partial \Phi_k(x, y)}{\partial n(x)}\right) d\Omega_y,$$

$$(B_2 g)(x) = 2\int_\Omega \left((n(x) - n(y), g(y))\frac{\partial \Phi_k(x, y)}{\partial x_2} - g_2(y)\frac{\partial \Phi_k(x, y)}{\partial n(x)}\right) d\Omega_y,$$

and

$$(B_3 g)(x) = 2\int_\Omega \left((n(x) - n(y), g(y))\frac{\partial \Phi_k(x, y)}{\partial x_3} - g_3(y)\frac{\partial \Phi_k(x, y)}{\partial n(x)}\right) d\Omega_y.$$

Let us divide the surface Ω into "regular" elementary parts $\Omega = \bigcup_{l=1}^N \Omega_l$ and let

$$\left(B^N g\right)(x) = e_1\left(B_1^N g\right)(x) + e_2\left(B_2^N g\right)(x) + e_3\left(B_3^N g\right)(x),$$

where

$$\left(B_1^N g\right)(x) = 2\sum_{\substack{j=1,\\ j\neq l}}^N \left((n(x(l)) - n(x(j)), g(x(j)))\frac{\partial \Phi_k(x(l), x(j))}{\partial x_1(l)}\right.$$

$$\left. - g_1(x(j))\frac{\partial \Phi_k(x(l), x(j))}{\partial n(x(l))}\right) mes\Omega_j,$$

$$\left(B_2^N g\right)(x) = 2\sum_{\substack{j=1,\\ j\neq l}}^N \left((n(x(l)) - n(x(j)), g(x(j)))\frac{\partial \Phi_k(x(l), x(j))}{\partial x_2(l)}\right.$$

$$-g_2(x(j))\,\frac{\partial\Phi_k(x(l),\,x(j))}{\partial n(x(l))}\Big)mes\Omega_j,$$

$$\left(B_3^N g\right)(x)=2\sum_{\substack{j=1,\\ j\neq l}}^{N}\Big((n(x(l))-n(x(j)),g(x(j)))\,\frac{\partial\Phi_k(x(l),x(j))}{\partial x_3(l)}$$

$$-g_3(x(j))\,\frac{\partial\Phi_k(x(l),\,x(j))}{\partial n(x(l))}\Big)mes\Omega_j$$

and $x(l)=(x_1(l),\,x_2(l),\,x_3(l))\in\Omega_l$ are supporting points.

Theorem 3.1 *Let $g\in C_\perp(\Omega)$ and $\Omega\subset R^3$ be a Lyapunov surface with exponent $0<\alpha\leq 1$. Then the expression $\left(B^N g\right)(x(l))$ at the support points $x(l)$, $l=\overline{1,\,N}$, is the cubature formula for the integral* (1.2)*, and*

$$\max_{l=\overline{1,\,N}}\left|(Bg)(x(l))-\left(B^N g\right)(x(l))\right|\leq M\left(\|g\|_\infty N^{-\frac{\alpha}{2}}\ln N+\omega\left(g,N^{-\frac{1}{2}}\right)\right).$$

Proof. First, we show that the expression $\left(B_1^N g\right)(x(l))$ at the points $x(l)\,,\ \ l=\overline{1,\,N}$, is a cubature formula for the integral $(B_1 g)(x)$ and estimate the errors of this cubature formula. It is obvious that

$$(B_1 g)(x(l))-\left(B_1^N g\right)(x)$$

$$=2\int_{\Omega_l}\left((n(x(l))-n(y)\,,g(y))\,\frac{\partial\Phi_k(x(l)\,,\,y)}{\partial x_1(l)}-g_1(y)\,\frac{\partial\Phi_k(x(l)\,,\,y)}{\partial n(x(l))}\right)d\Omega_y$$

$$+2\sum_{\substack{j=1,\\ j\neq l}}^{N}\int_{\Omega_j}(n(x(l))-n(x(j))\,,g(y)-g(x(j)))\,\frac{\partial\Phi_k(x(l)\,,\,y)}{\partial x_1(l)}d\Omega_y$$

$$+2\sum_{\substack{j=1,\\ j\neq l}}^{N}\int_{\Omega_j}(n(x(j))-n(y)\,,g(y))\,\frac{\partial\Phi_k(x(l)\,,\,y)}{\partial x_1(l)}d\Omega_y$$

$$+2\sum_{\substack{j=1,\\ j\neq l}}^{N}\int_{\Omega_j}(n(x(l))-n(x(j))\,,g(x(j)))$$

$$\times\left(\frac{\partial\Phi_k(x(l)\,,\,y)}{\partial x_1(l)}-\frac{\partial\Phi_k(x(l)\,,\,x(j))}{\partial x_1(l)}\right)d\Omega_y$$

$$+2\sum_{\substack{j=1,\\ j\neq l}}^{N}\int_{\Omega_j}g_1(y)\left(\frac{\partial\Phi_k(x(l)\,,\,x(j))}{\partial n(x(l))}-\frac{\partial\Phi_k(x(l)\,,\,y)}{\partial n(x(l))}\right)d\Omega_y$$

$$+2\sum_{\substack{j=1,\\ j\neq l}}^{N}\int_{\Omega_j}(g_1(x(j))-g_1(y))\,\frac{\partial\Phi_k(x(l)\,,\,x(j))}{\partial n(x(l))}d\Omega_y.$$

We denote the terms in the last equality by $\delta_1^N(x(l))$, $\delta_2^N(x(l))$, $\delta_3^N(x(l))$, $\delta_4^N(x(l))$, $\delta_5^N(x(l))$ and $\delta_6^N(x(l))$, respectively.

It is not difficult to calculate that

$$\frac{\partial \Phi_k(x,\,y)}{\partial x_1}=\frac{(ik\;|x-y|-1)\;(x_1-y_1)\exp(ik\,|x-y|)}{4\pi\;|x-y|^3}$$

and

$$\frac{\partial \Phi_k(x,\,y)}{\partial n(x)}=\frac{(ik\;|x-y|-1)\;(x-y,n(x))\exp(ik\,|x-y|)}{4\pi\;|x-y|^3}.$$

Then, taking into account inequalities (2.2) and

$$|(x-y,n(x))|\le M\,|x-y|^{1+\alpha}\,,\forall x,y\in\Omega, \tag{3.1}$$

we get that

$$\left|\delta_1^N(x(l))\right|\le M\,\|g\|_\infty\int_{\Omega_l}\frac{d\Omega_y}{|y-x(l)|^{2-\alpha}}$$

$$\le M\,\|g\|_\infty\int_0^{R(N)}\frac{dt}{t^{1-\alpha}}\le M\,\|g\|_\infty\,(R(N))^\alpha.$$

Moreover, taking into account Lemma 2.1, we have

$$\left|\delta_2^N(x(l))\right|\le M\int_{\Omega\backslash\Omega_l}\frac{\omega(g,R(N))}{|y-x(l)|^{2-\alpha}}d\Omega_y$$

$$\le M\omega(g,R(N))\int_{\Omega}\frac{d\Omega_y}{|y-x(l)|^{2-\alpha}}\le M\omega(g,R(N))$$

and

$$\left|\delta_3^N(x(l))\right|\le M\,\|g\|_\infty\int_{\Omega\backslash\Omega_l}\frac{|y-x(j)|^\alpha}{|y-x(l)|^2}d\Omega_y\le M\,\|g\|_\infty\,(R(N))^\alpha\int_{r(N)}^{diam\Omega}\frac{dt}{t}$$

$$\le M\,\|g\|_\infty\,(R(N))^\alpha\,|\ln R(N)|\,.$$

Since

$$\frac{\partial\Phi_k(x(l)\,,\,y)}{\partial x_1(l)}-\frac{\partial\Phi_k(x(l)\,,\,x(j))}{\partial x_1(l)}$$

$$=\frac{ik\,(\;|x(l)-y|-|x(l)-x(j)|)\;(x_1(l)-y_1)\exp(ik\,|x(l)-y|)}{4\pi\;|x(l)-y|^3}$$

$$+\frac{(\,ik\,|x(l)-x(j)|-1)\;(x_1(j)-y_1)\exp(ik\,|x(l)-y|)}{4\pi\;|x(l)-y|^3}$$

$$+\frac{(\,ik\,|x(l)-x(j)|-1)\;(x_1(l)-x_1(j))}{4\pi\;|x(l)-y|^3}$$

$$\times(\exp(ik\,|x(l)-y|)-\exp(ik\,|x(l)-x(j)|))$$

$$+(\,ik\,|x(l)-x(j)|-1)\;(x_1(l)-x_1(j))$$

$$\times\exp(ik\,|x(l)-x(j)|)\left(\frac{1}{4\pi\;|x(l)-y|^3}-\frac{1}{4\pi\;|x(l)-x(j)|^3}\right),$$

then, taking into account inequality (2.1) and Lemma 2.1, we find

$$\left|\frac{\partial\Phi_k(x(l)\,,\,y)}{\partial x_1(l)}-\frac{\partial\Phi_k(x(l)\,,\,x(j))}{\partial x_1(l)}\right|\le M\frac{R(N)}{|x(l)-y|^3},\forall y\in\Omega_j,j\ne l.$$

As a result, taking into account inequality (2.2), we obtain that if $0 < \alpha < 1$, then

$$\left|\delta_4^N(x(l))\right| \le M\|g\|_\infty R(N)\int_{\Omega\backslash\Omega_l}\frac{d\Omega_y}{|y-x(l)|^{3-\alpha}}$$

$$\le M\|g\|_\infty R(N)\int_{r(N)}^{diam\Omega}\frac{dt}{t^{2-\alpha}} \le M\|g\|_\infty (R(N))^\alpha,$$

and if $\alpha = 1$, then

$$\left|\delta_4^N(x(l))\right| \le M\|g\|_\infty R(N)\left|\ln R(N)\right|.$$

Let $y \in \Omega_j$ and $j \ne l$. It is obvious that

$$\frac{\partial\Phi_k(x(l),y)}{\partial n(x(l))} - \frac{\partial\Phi_k(x(l),x(j))}{\partial n(x(l))}$$

$$= \frac{ik(\,|x(l)-y| - |x(l)-x(j)|)\,(x(l)-y, n(x(l)))\exp(ik|x(l)-y|)}{4\pi\,|x(l)-y|^3}$$

$$+\frac{(\,ik|x(l)-x(j)|-1)\,(x(j)-y,n(x(l)))\exp(ik|x(l)-y|)}{4\pi\,|x(l)-y|^3}$$

$$+\frac{(\,ik|x(l)-x(j)|-1)\,(x(l)-x(j),n(x(l)))}{4\pi\,|x(l)-y|^3}$$

$$\times\left(\exp(ik|x(l)-y|) - \exp(ik|x(l)-x(j)|)\right)$$

$$+(\,ik|x(l)-x(j)|-1)\,(x(l)-x(j),n(x(l)))$$

$$\times\exp(ik|x(l)-x(j)|)\left(\frac{1}{4\pi\,|x(l)-y|^3} - \frac{1}{4\pi\,|x(l)-x(j)|^3}\right).$$

Moreover, taking into account inequalities (2.2) and (3.1), we find

$$|(x(j)-y,n(x(l)))| = |(x(j)-y,n(x(l))-n(y)) + (x(j)-y,n(y))|$$

$$\le |(x(j)-y,n(x(l))-n(y))| + |(x(j)-y,n(y))|$$

$$\le R(N)\,|x(l)-y|^\alpha + (R(N))^{1+\alpha}.$$

Then, taking into account inequalities (2.1) and (3.1) and Lemma 2.1, we obtain that

$$\left|\delta_5^N(x(l))\right| \le M\|g\|_\infty\int_{\Omega\backslash\Omega_l}\left(\frac{R(N)}{|x(l)-y|^{3-\alpha}} + \frac{(R(N))^{1+\alpha}}{|x(l)-y|^3}\right)d\Omega_y$$

$$\le M\|g\|_\infty (R(N))^\alpha.$$

Taking into account inequality (3.1), we have

$$\left|\delta_6^N(x(l))\right| \le M\omega(g,R(N))\int_{\Omega\backslash\Omega_l}\frac{1}{|y-x(l)|^{2-\alpha}}d\Omega_y \le M\omega(g,R(N)).$$

As a result, summing up the obtained estimates for the expressions $\delta_1^N(x(l))$, $\delta_2^N(x(l))$, $\delta_3^N(x(l))$, $\delta_4^N(x(l))$, $\delta_5^N(x(l))$ and $\delta_6^N(x(l))$, and taking into account Lemma 2.2, we obtain that

$$\left|(B_1g)(x(l)) - \left(B_1^Ng\right)(x)\right| \le M\left(\|g\|_\infty N^{-\frac{\alpha}{2}}\ln N + \omega\left(g,N^{-\frac{1}{2}}\right)\right).$$

Similarly, it can be shown that the expressions $\left(B_2^N g\right)(x(l))$ and $\left(B_3^N g\right)(x(l))$ at the points $x(l)$, $\quad l = \overline{1, N}$, is a cubature formula for the integrals $(B_2 g)(x)$ and $(B_3 g)(x)$, respectively, and

$$\left|(B_2 g)(x(l)) - \left(B_2^N g\right)(x)\right| \leq M \left(\|g\|_\infty N^{-\frac{\alpha}{2}} \ln N + \omega\left(g, N^{-\frac{1}{2}}\right)\right),$$

$$\left|(B_3 g)(x(l)) - \left(B_3^N g\right)(x)\right| \leq M \left(\|g\|_\infty N^{-\frac{\alpha}{2}} \ln N + \omega\left(g, N^{-\frac{1}{2}}\right)\right).$$

The theorem is proven.

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