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Complex error function and Weyl-Titchmarsh theory for a perturbed discrete Hermite operator

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Abstract. *The discrete Hermite operator and its perturbation are considered. Using the spectral theory of the discrete Hermite operator has studied asymptotics of a complex error function in the upper half -plane. The expansionion formulas are obtained in terms of eigenfunctions for the continuous spectrum of perturbed discrete Hermite operator.*

Keywords. Hermite operator · complex error function · Weyl function · expansionion formulas · Hamburger-Nevanlinna theorem.

Mathematics Subject Classification (2010): 34A55, 34B24, 58C40, 47E05

1 Introduction

The complex complementary error function

$$w\left(z\right)=e^{-z^{2}}\left(1+\frac{2i}{\sqrt{\pi}}\int\limits_{0}^{z}e^{t^{2}}dt\right) \tag{1.1}$$

is often called the Faddeeva function or Kramp function. It is related to the Fresnel integral, to Dawson's integral, and to the Voigt function (see [1], [8]). The Faddeeva function arises in various physical problems, typically relating to electromagnetic responses in complicated media([2], [6]). This includes the problems associated with small-amplitude waves propagating through Maxwellian plasmas (see [6], [7]). This function in appears in the plasmas permittivity from which dispersion relations are derived, hence it is sometimes referred to as the plasma dispersion function (see [6], [7], [10], [14]).

This paper is devoted to the study of the asymptotics of function (1.1) in the upper half-plane. To our knowledge, this issue has not been studied before. In this paper, it is

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established that the complex complementary error function coincides in the upper plane with the Weyl function of the discrete Hermite operator. Therefore, the results obtained are also of interest from the point of view of the spectral theory of discrete operators.

2 Asymtotics of the complex error function

Theorem 2.1 *When* $z \to \infty$, $0 \leq \arg z \leq \pi$ *the following relation holds:*

$$w(z) \sim \frac{i}{\sqrt{\pi}} \frac{1}{z}, z \to \infty. \tag{2.1}$$

Proof. Let us consider the operator H generated in the space $\ell_2 [0, \infty)$ by the difference expression

$$(ly)_n = \sqrt{\frac{n}{2}} y_{n-1} + \sqrt{\frac{n+1}{2}} y_{n+1}, n = 0, 1, 2, ...$$

and boundary condution

$$y_{-1} = 0.$$

In the work of Yu.M.Berezansky [5] it was established that the operator H is a positive self-adjoint operator. Moreover, the spectrum of H is purely absolutely continuous and coincides with $(-\infty, +\infty)$ and the corresponding spectral measure is given by $d\rho(\lambda) = \frac{1}{\sqrt{\pi}} e^{-\lambda^2} d\lambda$.

We denote by $P_n(z), Q_n(z)$ the solutions of the equation

$$\sqrt{\frac{n}{2}} y_{n-1} + \sqrt{\frac{n+1}{2}} y_{n+1} = z y_n, n = 0, 1, 2, ... \tag{2.2}$$

with initial data $P_{-1}(z) = Q_0(z) = 0,\ P_0(z) = 1,\ Q_1(z) = \sqrt{2}$ (“sine” and “cosine” type solutions). It is evident that every solution of this equation is their linear combination. It should be noted that $P_n(z), n = 0, 1, 2, ...$ is a polynomial of degree n. Moreover, $P_n(z), n = 0, 1, 2, ...$ are normalized Hermite polynomials:

$$P_n(z) = \frac{H_n(z)}{\sqrt{2^n n!}}, n = 0, 1, 2, ...,$$

$$H_0(z) = 1,\ H_1(z) = 2z,\ H_2(z) = 4z^2 - 2,\ H_3(z) = 12z^3 - 8z, ...$$

It is known (see, for example, [4], [12]) that for $Imz \neq 0$ the equation (2.2) has solution (the Weyl solution of the operator H):

$$\Psi_n(z) = Q_n(z) + m(z) P_n(z), n = -1, 0, 1, ..., \tag{2.3}$$

such that $\sum\limits_{n=-1}^{\infty} |\Psi_n(z)|^2 < \infty$. In this notation $m(z)$ is the Weyl function of the operator H represented in the form

$$m(z) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-t^2}}{t - z} dt. \tag{2.4}$$

On other hand, the Faddeeva function occurs as (see [7])

$$w(z) = \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{e^{-t^2}}{t - z} dt, Imz > 0.$$

Therefore,

$$w(z) = \frac{m(z)}{\sqrt{\pi}i} \tag{2.5}$$

for $Imz > 0$. Note that the increasing function $\rho(\lambda) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\lambda} e^{-t^2} dt$ for each $n = 0, 1, , , ,$ satisfies the inequality

$$s_n = \int\limits_{-\infty}^{+\infty} |\lambda|^n \, d\rho(\lambda) < \infty$$

and $s_0 = 1$. Then, according to the Hamburger-Nevanlinna theorem (see [3], theorem 3.2.1) namely the function $m(z) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-t^2}}{t-z} dt$ such that given any fixed $\delta > 0$ however small it is true uniformly in the range of angles $\delta \leq \arg z \leq \pi - \delta$ that

$$m(z) \sim -\frac{1}{z}, z \to \infty. \tag{2.6}$$

Let us consider the Weyl function $m(z)$, defined by formula (2.4). Let $z = x + iy$. By virtue of formula (2.4), we have

$$m(z) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-t^2}}{t - x - iy} dt = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{[(t-x) + iy] \, e^{-t^2}}{(t-x)^2 + y^2} dt =$$

$$= \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{(t-x) \, e^{-t^2}}{(t-x)^2 + y^2} dt + i \frac{1}{\sqrt{\pi}} y \int\limits_{-\infty}^{\infty} \frac{e^{-t^2}}{(t-x)^2 + y^2} dt = I_1 + iI_2.$$

It is known (see, for example, [9]) existence of the principal value

$$\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-t^2}}{t-x} dt = \frac{1}{\sqrt{\pi}} \lim_{y \to +0} \int\limits_{|t-x| \geq y} \frac{e^{-t^2}}{t-x} dt$$

is here equivalent almost everywhere to existence of

$$\lim_{y \to +0} I_1 = \frac{1}{\sqrt{\pi}} \lim_{y \to +0} \int\limits_{-\infty}^{\infty} \frac{(t-x) \, e^{-t^2}}{(t-x)^2 + y^2} dt.$$

On other hand [9], at any x (where $\rho'(x)$ exists and finite)

$$\lim_{y \to +0} I_2 = \sqrt{\pi} e^{-x^2}.$$

Therefore, if $z = x$, then

$$m(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-t^2}}{t-x} dt + \sqrt{\pi} i e^{-x^2} = -2e^{-x^2} \int\limits_{0}^{x} e^{t^2} dt + \sqrt{\pi} i e^{-x^2}. \tag{2.7}$$

Note that the validity of the last equality is also ensured by formulas (2.4), (2.5) and the fact that $w\left(z\right)$ is an entire function. By virtue of formula (2.7) we get

$$m\left(x\right)\sim-\frac{1}{x},x\to\pm\infty. \tag{2.8}$$

From these considerations and formula (2.5) it follows that, the function $w\left(z\right)$ such that given any fixed $\delta>0$ however small it is true uniformly in the range of angles $\delta\leq\arg z\leq\pi-\delta$ that

$$w\left(z\right)\sim\frac{i}{\sqrt{\pi}}\frac{1}{z},z\to\infty. \tag{2.9}$$

Moreover,

$$w\left(x\right)\sim\frac{i}{\sqrt{\pi}}\frac{1}{x},x\to\pm\infty. \tag{2.10}$$

Now let us set $z=re^{i\varphi}$. From formula (1.1) it follows that for $r\to\infty$the relation

$$zw\left(z\right)=O\left(e^{3r^2}\right)$$

is satisfied uniformly in the entire upper half-plane. Since relation (2.9) is valid for sufficiently small δ, then by applying the Phragmen and Lindelof [13] theorem to angles $0\leq\arg z\leq\delta$ and $\pi-\delta\leq\arg z\leq\pi$ we obtain relation (2.1).

The theorem is proved.

3 Weyl-Titchmarsh theory for discrete Hermite operator

From the above reasoning it follows that for real values of $\lambda$the equation

$$\begin{aligned}&y_{-1}+\sqrt{\tfrac{1}{2}}y_1=\lambda y_0,\\&\sqrt{\tfrac{n}{2}}y_{n-1}+\sqrt{\tfrac{n+1}{2}}y_{n+1}=\lambda y_n, n=1,2,....\end{aligned} \tag{3.1}$$

has solutions in the form

$$\Psi_n\left(\lambda\right)=Q_n\left(\lambda\right)+m\left(\lambda\right)P_n\left(\lambda\right),$$

Clearly $\Psi_n\left(\lambda\right)$ is an analytic function in the upper half-plane and is continuous up to the real axis. As is known [4], for $\Psi_n\left(\lambda\right)$ the following integral representation is valid:

$$\Psi_n\left(\lambda\right)=\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\frac{P_n\left(t\right)e^{-t^2}}{t-\lambda}dt.$$

On other hand [11], at any t

$$P_n\left(t\right)=\frac{\sqrt{2}}{\sqrt[4]{\pi}}\frac{e^{\frac{t^2}{2}}}{\left(2n\right)^{\frac{1}{4}}}\left[\cos\left(\sqrt{2n+1}t-\frac{n\pi}{2}\right)+O\left(n^{-1}\right)+O\left(n^{-\frac{1}{4}}\left|t\right|^{\frac{5}{2}}\right)\right],n\to\infty$$

Using the last two relations it is established that for each $\lambda\in(-\infty,+\infty)$ the sequence $\Psi_n\left(\lambda\right)$ is bounded. Further, for each $\lambda\in(-\infty,+\infty)$ the solution of equation (3.1) is also

$$\overline{\Psi_n\left(\lambda\right)}=Q_n\left(\lambda\right)+\overline{m\left(\lambda\right)}P_n\left(\lambda\right).$$

These solutions $\mathit{\Psi}_n(\lambda)$ and $\overline{\mathit{\Psi}_n(\lambda)}$ are linearly independent, since their Wronskian

$$W\left[\mathit{\Psi}_n(\lambda),\overline{\mathit{\Psi}_n(\lambda)}\right]=\sqrt{\frac{n+1}{2}}\left(\mathit{\Psi}_n(\lambda)\overline{\mathit{\Psi}_{n+1}(\lambda)}-\mathit{\Psi}_{n+1}(\lambda)\overline{\mathit{\Psi}_n(\lambda)}\right)$$

is equal to

$$m(\lambda)-\overline{m(\lambda)}=2\sqrt{\pi}ie^{-\lambda^2}.$$

It follows that the identity holds

$$\frac{P_n(\lambda)}{a_0(\lambda)}=\overline{\mathit{\Psi}_n(\lambda)}+S_0(\lambda)\mathit{\Psi}_n(\lambda),n=-1,0,...$$

where

$$a_0(\lambda)=\frac{i}{2\sqrt{\pi}}e^{\lambda^2},\ S_0(\lambda)=\frac{\overline{a_0(\lambda)}}{a_0(\lambda)}=-1$$

The expansion formula holds

$$\frac{1}{2\sqrt{\pi^3}}\int\limits_{-\infty}^{+\infty}e^{\lambda^2}Re\left\{\left(\overline{\mathit{\Psi}_n(\lambda)}+S_0(\lambda)\mathit{\Psi}_n(\lambda)\right)\mathit{\Psi}_m(\lambda)\right\}d\lambda=\delta_{nm}, \tag{3.2}$$

where δ_{nm} is the Kronecker delta. The last formula is a modification of the well-known [5] expansion formula

$$\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}e^{-\lambda^2}P_n(\lambda)P_m(\lambda)d\lambda=\delta_{nm}.$$

Now we consider the equation

$$\begin{aligned}&y_{-1}+b_0y_0+\left(\sqrt{\tfrac{1}{2}}+a_0\right)y_1=\lambda y_0,\\&\left(\sqrt{\tfrac{n}{2}}+a_{n-1}\right)y_{n-1}+b_ny_n+\left(\sqrt{\tfrac{n+1}{2}}+a_n\right)y_{n+1}=\lambda y_n,n=1,2,....\end{aligned} \tag{3.3}$$

where the sequences a_n,b_n tend to zero fairly quickly. The operator$\tilde{H}$ generated by equation (3.3) and the boundary condition $y_{-1}=0$ is a compact perturbation of the operator H. Therefore, the continuous spectrum of the operator $\tilde{H}$ also fills the entire real axis $(-\infty,+\infty)$. It can be proven that equation (3.3) for $Im\lambda\geq 0$ has a solution with the asymptotics

$$f_n(\lambda)=\mathit{\Psi}_n(\lambda)[1+o(1)],n\to\infty.$$

Let us denote by $\varphi_n(\lambda)$ the solution of equation (3.3) with initial conditions $\varphi_{-1}(\lambda)=0$, $\varphi_0(\lambda)=1$. Then on the real axis the identity

$$\frac{\varphi_n(\lambda)}{a(\lambda)}=\overline{f_n(\lambda)}+S(\lambda)f_n(\lambda),n=-1,0,...$$

holds, wheree

$$a(\lambda)=-\frac{i}{2\sqrt{\pi}}f_{-1}(\lambda)e^{\lambda^2},S(\lambda)=-\frac{\overline{f_{-1}(\lambda)}}{f_{-1}(\lambda)}$$

The formula expansion in terms of eigenfunctions for the continuous spectrum of the operator $\tilde{H}$ will take the form

$$\frac{1}{2\sqrt{\pi^3}}\int\limits_{-\infty}^{+\infty}e^{\lambda^2}Re\left\{\left(\overline{f_n(\lambda)}+S(\lambda)f_n(\lambda)\right)f_m(\lambda)\right\}d\lambda=\delta_{nm}. \tag{3.4}$$

Formulas (3.2), (3.4) can be used to solve the inverse scattering problem for equation (3.3).

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