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Boundedness of the fractional maximal commutator in total Morrey-Guliyev spaces for the Dunkl operator on the real line

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Abstract. *In the paper, in the setting $\mathbb{R}$ we study fractional maximal commutator operator $M_{b,\alpha,\nu}$ associated with the Dunkl operator in the total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ when b belongs to Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R})$ spaces. We give sufficient conditions for the boundedness of the operator $M_{b,\alpha,\nu}$ on total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.*

Keywords. Fractional maximal operator; Fractional maximal commutator; total D_ν-Morrey-Guliyev spaces; Dunkl operator; D_ν-Lipschitz spaces.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35

1 Introduction

Morrey spaces, introduced by Morrey [23], play an important role in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for analyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [11], extend the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|f\|_{L_p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \leq \lambda$ is equal to the maximum of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [2,3,5–8,12,16,21, 25–27].

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On the real line, the Dunkl operators Λ_ν are differential-difference operators introduced in 1989 by Dunkl [13]. For a real parameter $\nu \geq -1/2$, we consider the *Dunkl operator*, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \;\; x \in \mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

Let $\nu > -1/2$ be a fixed number and m_ν be the *weighted Lebesgue measure* on $\mathbb{R}$, given by

$$dm_\nu(x) := \left(2^{\nu+1}\Gamma(\nu+1)\right)^{-1} |x|^{2\nu+1}\, dx, \quad x \in \mathbb{R}.$$

For any $x \in \mathbb{R}$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R} : |y| \in]\max\{0, |x|-r\}, |x|+r[\ \}$ be a Dunkl-ball in $\mathbb{R}$. Then $B(0,r) =]-r, r[$ and $m_\nu B(0,r) = c_\nu r^{2\nu+2}$, where $c_\nu := \left[2^{\nu+1}(\nu+1)\,\Gamma(\nu+1)\right]^{-1}$.

The *maximal operator* M_ν associated by Dunkl operator on the real line is given by

$$M_\nu f(x) := \sup_{r>0} \left(m_\nu(B(x,r))\right)^{-1} \int_{B(x,r)} |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R}$$

and *fractional maximal operator* $M_{\alpha,\nu}$, $0 \leq \alpha < 2\nu+2$ associated by Dunkl operator on the real line is given by

$$M_{\alpha,\nu} f(x) := \sup_{r>0} \left(m_\nu B(x,r)\right)^{-1+\frac{\alpha}{2\nu+2}} \int_{B(x,r)} |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R}$$

The *fractional maximal commutator* $M_{b,\alpha,\nu}$, $0 \leq \alpha < 2\nu+2$ associated with Dunkl operator on the real line and with a locally integrable function $b \in L_1^{\mathrm{loc}}(\mathbb{R}, dm_\nu)$ is defined by

$$M_{b,\alpha,\nu} f(x) := \sup_{r>0} \left(m_\nu(B(x,r))\right)^{-1+\frac{\alpha}{2\nu+2}} \int_{B(x,r)} |b(x)-b(y)|\, |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R}.$$

We can define the (nonlinear) commutator of the fractional maximal operator $M_{\alpha,\nu}$ with a locally integrable function b by

$$[b, M_{\alpha,\nu}] f(x) = b(x) M_\nu(f)(x) - M_\nu(bf)(x).$$

For more details about the operators $M_{b,\nu}$ and $[b, M_\nu]$, we refer to [10,21] and references therein.

It is well known that maximal and fractional maximal operators play an important role in harmonic analysis (see [30]). Also the fractional maximal function and the fractional integral, associated with D_ν differential-difference Dunkl operators play an important role in Dunkl harmonic analysis, differentiation theory and PDE's. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [9,20,22]. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space. The Dunkl theory provides a useful framework for the study of multivariable analytic structures and has gained considerable interest in various fields of mathematics and in physical applications (see, for example, [14]). The maximal function, the fractional integral and related topics associated with the Dunkl differential-difference operator have been research areas for many mathematicians such as C. Abdelkefi and M. Sifi [1], V.S. Guliyev and Y.Y. Mammadov [9,10], Y.Y. Mammadov [17], L. Kamoun [15], M.A. Mourou [24], F. Soltani [28,29], K. Trimeche [31] and others. Moreover, the results on $L_\Phi(\mathbb{R}, dm_\nu)$-boundedness of fractional maximal operator and its commutators associated with D_ν were obtained in [10].

It is well known that maximal operator play an important role in harmonic analysis (see [30]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting $\mathbb{R}$, we study the boundedness of the fractional maximal commutator operator $M_{b,\alpha,\nu}$ associated with the Dunkl operator on total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries in the Dunkl setting on $\mathbb{R}$

Definition 2.1 *Let* $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. *We denote by* $L_{p,\lambda}(\mathbb{R}, dm_\nu)$ *the Morrey space [17] (≡ D_ν-Morrey space), by* $\widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ *the modified Morrey space [17] (≡ modified D_ν-Morrey space), and by* $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ *the total Morrey-Guliyev space [19] (≡ total D_ν-Morrey-Guliyev space), associated with the Dunkl operator the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t),dm_\nu)},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t),dm_\nu)},$$

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(B(x,t),dm_\nu)},$$

respectively.

Definition 2.2 *Let* $0 < p < \infty$, $\lambda \in \mathbb{R}$ *and* $\mu \in \mathbb{R}$. *We define the weak Morrey space* $L_{p,\lambda}(\mathbb{R}, dm_\nu)$ *[17] (≡ weak D_ν-Morrey space), the weak modified Morrey space* $\widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ *[17] (≡ weak modified D_ν-Morrey space), and the weak total Morrey-Guliyev space* $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ *[19] (≡ weak total D_ν-Morrey-Guliyev space), associated with the Dunkl operator the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{WL_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t),dm_\nu)},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t),dm_\nu)},$$

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(B(x,t),dm_\nu)},$$

respectively.

Lemma 2.1 *[18, 19] If* $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq 2\nu + 2$, *then*

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = L_{p,\lambda}(\mathbb{R}, dm_\nu) \cap L_{p,\mu}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \max\left\{\|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R},dm_\nu)}\right\}.$$

Lemma 2.2 *[18,19] If $0 < p < \infty$, $0 \le \mu \le \lambda \le 2\nu+2$, then*

$$WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = WL_{p,\lambda}(\mathbb{R}, dm_\nu) \cap WL_{p,\mu}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \max\left\{\|f\|_{WL_{p,\lambda}(\mathbb{R},dm_\nu)}, \|f\|_{WL_{p,\mu}(\mathbb{R},dm_\nu)}\right\}.$$

Remark 2.1 If $0 < p < \infty$, and $\lambda > 2\nu+2$ or $\mu < 0$, then

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = \Theta(\mathbb{R}),$$

where $\Theta \equiv \Theta(\mathbb{R})$ is the set of all functions equivalent to 0 on $\mathbb{R}$.

Lemma 2.3 *[18] If $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le 2\nu+2$ and $0 \le \mu_1 \le \mu_2 \le 2\nu+2$, then*

$$L_{p,\lambda_1,\mu_1}(\mathbb{R}, dm_\nu) \subset_{\succ} L_{p,\lambda_2,\mu_2}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R},dm_\nu)} \le \|f\|_{L_{p,\lambda_1,\mu_1}(\mathbb{R},dm_\nu)}.$$

Lemma 2.4 *[18] If $0 < p < \infty$, $0 \le \lambda \le 2\nu+2$ and $0 \le \mu \le 2\nu+2$, then*

$$L_{p,2\nu+2,\mu}(\mathbb{R}, dm_\nu) \subset_{\succ} L_\infty(\mathbb{R}, dm_\nu) \subset_{\succ} L_{p,\lambda,2\nu+2}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,2\nu+2}(\mathbb{R},dm_\nu)} \le c_\nu^{1/p}\|f\|_{L_\infty(\mathbb{R},dm_\nu)} \le \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R},dm_\nu)}.$$

Lemma 2.5 *[18] If $0 \le \lambda < 2\nu+2$, $0 \le \mu < 2\nu+2$, $0 \le \alpha < 2\nu+2-\lambda$ and $0 \le \beta < 2\nu+2-\mu$, then for $\frac{2\nu+2-\lambda}{\alpha} \le p \le \frac{2\nu+2-\mu}{\beta}$*

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) \subset_{\succ} L_{1,2\nu+2-\alpha,2\nu+2-\beta}(\mathbb{R}, dm_\nu)$$

and for $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ the following inequality

$$\|f\|_{L_{1,2\nu+2-\alpha,2\nu+2-\beta}(\mathbb{R},dm_\nu)} \le c_\nu^{1/p'}\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}$$

is valid.

Lemma 2.6 *Let $0 \le \alpha < 2\nu+2$, $1 \le p < \frac{2\nu+2}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$ and $B(x,r)$ be any Dunkl-ball in $\mathbb{R}$. If $p > 1$, then the inequality*

$$\|M_{\alpha,\nu}f\|_{L_q(B(x,r),dm_\nu)} \lesssim r^{\frac{2\nu+2}{q}} \sup_{t>2r} t^{-\frac{2\nu+2}{q}} \|f\|_{L_p(B(x,t),dm_\nu)} \tag{2.1}$$

holds for all $f \in L_p^{\text{loc}}(\mathbb{R}, dm_\nu)$.

Moreover if $p = 1$, then the inequality

$$\|M_{\alpha,\nu}f\|_{WL_q(B(x,r),dm_\nu)} \lesssim r^{\frac{2\nu+2}{q}} \sup_{t>2r} t^{-\frac{2\nu+2}{q}} \|f\|_{L_1(B(x,t),dm_\nu)} \tag{2.2}$$

holds for all $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$.

Theorem 2.1 *[19] Let $0 \le \alpha < 2\nu+2$, $0 \le \lambda, \mu < 2\nu+2$, $1 \le p < \min\{\frac{2\nu+2-\lambda}{\alpha}, \frac{2\nu+2-\mu}{\alpha}\}$, and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$.*

1. If $f \in L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$, then $M_{\alpha,\nu}f \in WL_{q,\lambda,\mu}(\mathbb{R}, dm_\nu)$ and

$$\|M_{\alpha,\nu}f\|_{WL_{q,\lambda q,\mu q}(\mathbb{R},dm_\nu)} \le C_{1,\lambda,\mu}\, \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)}, \tag{2.3}$$

where $C_{q,\lambda,\mu}$ is independent of f.

2. If $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$, $1 < p < \infty$, then $M_\nu f \in L_{q,\lambda,\mu}(\mathbb{R}, dm_\nu)$ and

$$\|M_{\alpha,\nu}f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R},dm_\nu)} \le C_{p,q,\lambda,\mu}\, \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}, \tag{2.4}$$

where $C_{p,\lambda,\mu}$ depends only on p,λ,μ and ν.

3 Fractional maximal commutator $M_{b,\alpha,\nu}$ in total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$

In this section, we study the boundedness of the fractional maximal commutator operator $M_{b,\alpha,\nu}$ in the total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

At first, we give the definition of the Lipschitz spaces on $\mathbb{R}$.

Definition 3.1 *Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R})$ if there exists a constant C such that for all $x, y \in \mathbb{R}$,*

$$|b(x) - b(y)| \le C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R})$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta}$.

Lemma 3.1 *Let $0 < \beta < 1$, $0 \le \alpha < \alpha + \beta < 2\nu + 2$ and $b \in \dot{\Lambda}_\beta(\mathbb{R})$, then the following pointwise estimate holds:*

$$M_{b,\alpha,\nu} f(x) \lesssim \|b\|_{\dot{\Lambda}_\beta} M_{\alpha+\beta,\nu} f(x).$$

Proof. If $b \in \dot{\Lambda}_\beta(\mathbb{R})$, then

$$\begin{aligned} M_{b,\alpha,\nu} f(x) &= \sup_{r>0} \big(m_\nu B(x,r)\big)^{-1+\frac{\alpha}{2\nu+2}} \int_{B(x,r)} |b(x) - b(y)|\, |f(y)|\, dm_\nu(y) \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta} \sup_{r>0} \big(m_\nu B(x,r)\big)^{-1+\frac{\alpha+\beta}{2\nu+2}} \int_{B(x,r)} |f(y)|\, dm_\nu(y) \\ &= \|b\|_{\dot{\Lambda}_\beta} M_{\alpha+\beta} f(x). \end{aligned}$$

The following Spanne's type result for the fractional maximal commutator operators $M_{b,\alpha,\nu}$ on total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Theorem 3.1 *Let $0 < \beta < 1$, $0 \le \alpha < \alpha + \beta < 2\nu + 2$ and $b \in \dot{\Lambda}_\beta(\mathbb{R})$. Let also $0 \le \lambda, \mu < 2\nu + 2$, $1 \le p < \min\{\frac{2\nu+2-\lambda}{\alpha}, \frac{2\nu+2-\mu}{\alpha}\}$, and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{2\nu+2}$.*

1. If $f \in L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$, then $M_{b,\alpha,\nu} f \in WL_{q,\lambda,\mu}(\mathbb{R}, dm_\nu)$ and

$$\|M_{b,\alpha,\nu} f\|_{WL_{q,\lambda q,\mu q}(\mathbb{R},dm_\nu)} \le C_{1,\lambda,\mu} \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)}, \tag{3.1}$$

where $C_{q,\lambda,\mu}$ is independent of f.

2. If $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$, $1 < p < \infty$, then $M_{b,\alpha,\nu} f \in L_{q,\lambda,\mu}(\mathbb{R}, dm_\nu)$ and

$$\|M_{b,\alpha,\nu} f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R},dm_\nu)} \le C_{p,q,\lambda,\mu} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}, \tag{3.2}$$

where $C_{p,\lambda,\mu}$ depends only on p,λ,μ and ν.

Proof. Let $p = 1$. From the inequality (2.2) we get

$$\begin{aligned} &\|M_{b,\alpha,\nu} f\|_{WL_{q,\lambda q,\mu q}(\mathbb{R},dm_\nu)} \lesssim \|b\|_{\dot{\Lambda}_\beta} \|M_{\alpha+\beta,\nu} f\|_{WL_{q,\lambda q,\mu q}(\mathbb{R},dm_\nu)} \\ &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda} [1/t]_1^{\mu} \|M_{\alpha+\beta,\nu} f\|_{WL_q(B(x,t),dm_\nu)} \\ &\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda} [1/t]_1^{\mu}\, t^{\frac{2\nu+2}{q}} \sup_{\tau>2t} \tau^{-\frac{2\nu+2}{q}} \|f\|_{L_1(B(x,\tau),dm_\nu)} \\ &\lesssim \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda} [1/t]_1^{\mu}\, t^{-\alpha-\beta+2\nu+2} \sup_{\tau>t} \tau^{\alpha+\beta-2\nu+2} [\tau]_1^{\lambda} [1/\tau]_1^{-\mu} \\ &= \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\alpha-\beta+2\nu+2-\lambda} [1/t]_1^{\alpha+\beta+\mu-2\nu-2} \\ &\times \sup_{\tau>t} [\tau]_1^{\alpha+\beta+\lambda-2\nu-2} [1/\tau]_1^{-\alpha-\beta+2\nu+2-\mu} \lesssim \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)}, \end{aligned}$$

which implies that the operator M_ν is bounded from $L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$ to $WL_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Let $1 < p < \min\{\frac{2\nu+2-\lambda}{\alpha}, \frac{2\nu+2-\mu}{\alpha}\}$. From the inequality (2.1) we get

$$\|M_{b,\alpha,\nu}f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R},dm_\nu)} \lesssim \|b\|_{\dot{\Lambda}_\beta} \|M_{\alpha+\beta,\nu}f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R},dm_\nu)}$$
$$= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|M_{\alpha+\beta,\nu}f\|_{L_q(B(x,t),dm_\nu)}$$
$$\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} t^{\frac{2\nu+2}{q}} \sup_{\tau>2t} \tau^{-\frac{2\nu+2}{q}} \|f\|_{L_p(B(x,\tau))}$$
$$\lesssim \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} t^{-\alpha-\beta+\frac{2\nu+2}{q}} \sup_{\tau>t} \tau^{\alpha+\beta-\frac{2\nu+2}{q}} [\tau]_1^{\frac{\lambda}{p}} [1/\tau]_1^{-\frac{\mu}{p}}$$
$$= \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\alpha-\beta+\frac{2\nu+2-\lambda}{p}} [1/t]_1^{\alpha+\beta+\frac{\mu-2\nu+2}{p}}$$
$$\times \sup_{\tau>t} [\tau]_1^{\alpha+\beta+\frac{\lambda-2\nu-2}{p}} [1/\tau]_1^{-\alpha-\beta+\frac{2\nu+2-\mu}{p}}$$
$$\lesssim \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}$$

which implies that the operator $M_{\alpha,\nu}$ is bounded from $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ to $L_{q,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

From Theorem 3.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3.1 *[1,28] Let* $0 < \beta < 1$, $0 \le \alpha < \alpha + \beta < 2\nu + 2$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R})$. *Let also* $0 \le \lambda < 2\nu + 2$, $1 \le p < \frac{2\nu+2-\lambda}{\alpha}$, *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{2\nu+2}$.

1. If $f \in L_{1,\lambda}(\mathbb{R}, dm_\nu)$, *then* $M_{b,\alpha,\nu}f \in WL_{q,\lambda}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_{b,\alpha,\nu}f\|_{WL_{q,\lambda}(\mathbb{R},dm_\nu)} \le C_{q,\lambda} \|f\|_{L_{1,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{q,\lambda}$ *is independent of* f.

2. If $f \in L_{p,\lambda}(\mathbb{R}, dm_\nu)$, $p > 1$, *then* $M_{b,\alpha,\nu}f \in L_{q,\lambda}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_{b,\alpha,\nu}f\|_{L_{q,\lambda}(\mathbb{R},dm_\nu)} \le C_{p,q,\lambda} \|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{p,q,\lambda}$ *depends only on* p, q, λ *and* ν.

Corollary 3.2 *[17] Let* $0 < \beta < 1$, $0 \le \alpha < \alpha + \beta < 2\nu + 2$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R})$. *Let also* $0 \le \lambda < 2\nu + 2$, $1 \le p < \frac{2\nu+2-\lambda}{\alpha}$, *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$.

1. If $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}, dm_\nu)$, *then* $M_{b,\alpha,\nu}f \in W\widetilde{L}_{q,\lambda}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_{b,\alpha,\nu}f\|_{W\widetilde{L}_{q,\lambda}(\mathbb{R},dm_\nu)} \le C_{q,\lambda} \|f\|_{\widetilde{L}_{1,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{1,\lambda}$ *is independent of* f.

2. If $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$, $p > 1$, *then* $M_{b,\alpha,\nu}f \in \widetilde{L}_{q,\lambda}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_{b,\alpha,\nu}f\|_{\widetilde{L}_{q,\lambda}(\mathbb{R},dm_\nu)} \le C_{p,q,\lambda} \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{p,q,\lambda}$ *depends only on* p, q, λ *and* ν.

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