Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **46** (1), 146-167 (2026).
https://doi.org/10.30546/2617-7900.46.1.2026.0064

Some new modifications of integral inequalities in the frame of AB fractional integral operator

Muhammad Tariq · **Sotiris K. Ntouyas** · **Hijaz Ahmad*** · **Rakib Efendiev**

Received: 11.04.2025 / Revised: 12.10.2025 / Accepted: 28.11.2025

Abstract. *In this paper, we aim to examine and construct a novel variant of the Hermite-Hadamard and Pachpatte type integral inequality via the Atangana-Baleanu fractional integral operator within the concept of generalized preinvex function. By employing this concept, we construct a new identity that correlates with preinvex functions. Furthermore, based on this newly derived fractional identity, some new estimations of fractional Hermite-Hadamard type inequalities involving m-preinvex via Atangana-Baleanu fractional sense are investigated. This research offers new and remarkable enhancements over previously reported findings in terms of both results and unique instances.*

Keywords. Preinvex functions; Hadamard inequality; AB-fractional operator, Pachpatte inequality

Mathematics Subject Classification (2010): 26A51; 26A33; 26D07; 26D10; 26D1535J25, 35J70, 60H15

1 Introduction

The evolution of convex theory is compelling and kicks off in the last decades of the nineteenth century. The idea of ”convex functions” is frequently employed and addressed in the popular and extensively read book ”Inequalities,” authored by G. Polya, G.H. Hardy,

* Corresponding author

Muhammad Tariq
Department of Mathematics, Capt Waqar Kakar Shaheed Balochistan Residential College, Loralai, 84800, Balochistan, Pakistan
Mathematics Research Center, Near East University, Near East Boulevard, PC: 99138, Nicosia /Mersin 10-Turkey
E-mail: captaintariq2187@gmail.com

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail: sntouyas@uoi.gr

Hijaz Ahmad
Irfan Suat Gunsel Operational Research Institute, Near East University, Nicosia/TRNC, 99138 Mersin 10, Turkey
Department of Mathematics, College of Science, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, South Korea
Scientific Research Centre, Baku Engineering University, Baku AZ0102, Azerbaijan
VIZJA University, Okopowa 59, 01-043 Warsaw, Poland
E-mail: hijaz555@gmail.com

Rakib Efendiev
Baku Engineering University, Mathematics Department, H.Aliyev, 120, Baku, AZ0101, Azerbaijan
E-mail: refendiyev@beu.edu.az

and J.E. Littlewood. This book, which focuses only on inequality, has quickly become an indispensable resource for mathematicians and offers a comprehensive overview of this fascinating field. The convex concept demonstrates the techniques and appropriate guidelines for focusing on a wide range of issues in scientific research. The significance of mathematical inequalities to the growth of numerous facets of mathematics and other scientific fields has been widely accepted in the last few decades. This framework has wonderful uses in economics, engineering, finance, and optimization. This analysis serves as a solid basis for the creation of numerical resources for the analysis and solution of challenging mathematical issues. For the literature, see [1–6].

The fractional analysis has numerous uses in control systems, transform theory, nanotechnology, modeling, fluid flow, mathematical biology, epidemiology, optimal control and physics. Because of the above-mentioned widespread viewpoints and significance, readers and academics found an investigation of fractional operators to be interesting. When describing and evaluating statistical issues and formulas that resemble quadratures, this theory is helpful. For the literature, see [7–20].

Mathematicians have recently become intrigued with fractional calculus, which concentrates on fractional integration across challenging domains and has gained attraction because of its useful applications. Researchers urge learners to become curious about implementing the fractional operator to overcome challenges in the real world. The different types of integral inequalities like Hermite-Hadamard (H-H) [21], Hermite-Hadamard-Mercer (H–H–M) [22], Simpson-type [23], and Ostrowski [24] have all been investigated in the frame of Riemann-Liouville fractional integral operator (RLFIO). The Katugampola fractional integral operator (KFIO) in [25] was utilized to address the H–H integral inequalities, whereas [26] implemented the Atangana-Baleanu fractional operator (ABFO) to examine the Simpson-Mercer integral inequality. Also, the H-H-M inequality was examined via the Caputo-Fabrizio fractional integral operator (CFFIO). The discussion explained above indicates that inequalities and fractional integral operators are related.

The Atangana-Baleanu (AB) fractional operator is a significant advancement in fractional analysis that offers enhanced modeling features for intricate systems. Unlike traditional fractional derivatives, such as Riemann-Liouville and Caputo derivatives, which rely on singular kernels, the AB operator employs a non-singular and non-local kernel based on the generalized Mittag-Leffler function. This innovative design enables more accurate capture of real-world phenomena, particularly those exhibiting memory effects and hereditary properties. As a result, the AB fractional operator has gained widespread acceptance and application across various scientific and engineering disciplines, allowing for more precise and flexible modeling of complex systems.

Hanson was the first to examine the term ”invex function” [27]. Mond and Weir investigated the concept of preinvexity [28]. One could consider Mond and Ben-Israel's [29] analysis of the preinvex and invex theory using the bifunction to be a major contribution to the optimization field.

This work's novelty and goal are to use preinvexity in the framework of ABFIO to present a novel variation of H-H and Pachpatte-type integral inequality. Additionally, we are to use ABFIO to construct some refinements of H-H type integral inequality.

This specific document is organized as follows:

First, we review some common concepts and terms in Section 2 that will aid us in the subsequent sections of our investigation. We present a new type of H–H-type inequality via ABFIO in Section 3, along with some intriguing corollaries and observations. A new integral identity is examined in Section 4, and some improvements to the H–H inequality are also constructed using this identity. We examine a new type of Pachpatte-type inequality using ABFIO in Section 5, along with a few corollaries and observations. We provide a brief conclusion and list some potential avenues for future research in the concluding Section 6.

2 Preliminaries

With so many theorems, definitions, and comments, it is best to examine and delve deeper in this section to ensure quality, reader interest, and completeness. The objective of this portion is to illustrate and analyze several common terms and definitions that we will need for our research in the following sections. First, we introduce the preinvex, m-preinvex, invex, and m-invex functions. This section is made more appealing by the addition of Condition C and extended Condition C. We conclude this section by reviewing the CFFIO and ABFIO fractional integral operators that are necessary for our evaluation.

Definition 2.1 ([30]) *$\mathbb{X} \subset \mathbb{R}^n$ is invex w.r.t $\Omega(.,.)$, if*

$$\upsilon_1 + \wp\Omega(\upsilon_2, \upsilon_1) \in \mathbb{X},$$

$\forall\ \upsilon_1, \upsilon_2 \in \mathbb{X}$ and $\wp \in [0,1]$.

There are numerous uses for the term "invexity" in the fields of pure and applied sciences, as well as in nonlinear optimization and variational inequalities.

Definition 2.2 ([31]) *Let $\Omega : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}^n$ and $\mathbb{X} \subseteq \mathbb{R}^n$. Then $\mathbb{X}$ is m–invex w.r.t. Ω, if*

$$m\upsilon_2 + \wp\Omega(\upsilon_1, \upsilon_2, m) \in \mathbb{X}$$

holds $\forall\ \upsilon_1, \upsilon_2 \in \mathbb{X}$, $m \in (0,1]$ and $\wp \in [0,1]$.

Example 1 ([31]) Let $m = \frac{1}{4}$, $\mathbb{X} = [\frac{-\pi}{2}, 0) \bigcup (0, \frac{1}{2}]$ and

$$\Omega(\upsilon_2, \upsilon_1, m) = \begin{cases} m\cos(\upsilon_2 - \upsilon_1) \quad if\ \upsilon_1 \in (0, \frac{\pi}{2}], \upsilon_2 \in (0, \frac{\pi}{2}]; \\ -m\cos(\upsilon_2 - \mu_1) \quad if\ \upsilon_1 \in [\frac{-\pi}{2}, 0), \upsilon_2 \in [\frac{-\pi}{2}, 0); \\ m\cos(\upsilon_1) \quad if\ \upsilon_1 \in (0, \frac{\pi}{2}], \upsilon_2 \in [\frac{-\pi}{2}, 0); \\ -m\cos(\upsilon_1) \quad if\ \upsilon_1 \in [-\frac{\pi}{2}, 0), \upsilon_2 \in (0, \frac{\pi}{2}]. \end{cases}$$

Then, $\mathbb{X}$ is an m–invex set but not convex $\forall\ \wp \in [0,1]$.

In 1988, Weir and Mond [28], employed the idea of invexity and investigated the idea of the preinvex function.

Definition 2.3 ([28]) *Let $\Omega : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$ and $\mathbb{X} \subseteq \mathbb{R}^n$. Then $\Psi : \mathbb{X} \to \mathbb{R}$ is preinvex w.r.t Ω if*

$$\Psi\left(\upsilon_2 + \wp\Omega\left(\upsilon_1, \upsilon_2\right)\right) \le \wp\,\Psi\left(\upsilon_1\right) + \left(1 - \wp\right)\Psi\left(\upsilon_2\right), \qquad \forall \upsilon_1,\ \upsilon_2 \in \mathbb{X},\ \ \wp \in [0,1].$$

Over the past decade, many scholars have worked to refine the idea of preinvex function in various directions. The generalized m-preinvex function was introduced by Kalsoom [32] and is expressed by

Definition 2.4 *Let $\Omega : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}^n$ and $\mathbb{X} \subseteq \mathbb{R}^n$. Then $\Psi : \mathbb{X} \to \mathbb{R}$ is generalized m-preinvex w.r.t. Ω if*

$$\Psi(m\upsilon_2 + \wp\Omega(\upsilon_1, \upsilon_2, m)) \le \wp\Psi\left(\upsilon_1\right) + m\left(1 - \wp\right)\Psi\left(\upsilon_2\right), \tag{2.1}$$

holds for every $\upsilon_1, \upsilon_2 \in \mathbb{X}$, $m \in (0,1]$ and $\wp \in [0,1]$.

Condition C: Suppose that $\mathbb{X} \subset \mathbb{R}$ is an open invex subset w.r.t. $\Omega : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. We say that the function Ω satisfies the condition C, if for any $\upsilon_1, \upsilon_2 \in \mathbb{X}$ and $\wp \in [0,1]$,

$$\Omega(\upsilon_1, \upsilon_1 + \wp\, \Omega(\upsilon_2, \upsilon_1)) = -\wp\, \Omega(\upsilon_2, \upsilon_1)$$
$$\Omega(\upsilon_2, \upsilon_1 + \wp\, \Omega(\upsilon_2, \upsilon_1)) = (1-\wp)\ \Omega(\upsilon_2, \upsilon_1).$$

For any $\upsilon_1, \upsilon_2 \in \mathbb{X}$, $\wp_1, \wp_2 \in [0,1]$, then according to the above equations, we have

$$\Omega(\upsilon_1 + \wp_2\, \Omega(\upsilon_2, \upsilon_1),\ \upsilon_1 + \wp_1\, \Omega(\upsilon_2, \upsilon_1)) = (\wp_2 - \wp_1)\, \Omega(\upsilon_2, \upsilon_1).$$

The optimization and development of the theory of inequalities depend heavily on this condition (see [33,34]).

Extended Condition C: [35] Let $\mathbb{X} \subset \mathbb{R}$ be an open invex subset w.r.t. $\Omega : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}$. We say that the function Ω satisfies the Extended Condition C, if for any $\upsilon_1, \upsilon_2 \in \mathbb{X}$, $\wp \in [0,1]$, we have

$$\Omega(\upsilon_2, m\upsilon_2 + \wp\, \Omega(\upsilon_1, \upsilon_2, m), m) = -\wp\, \Omega(\upsilon_1, \upsilon_2, m)$$
$$\Omega(\upsilon_1, m\upsilon_2 + \wp\, \Omega(\upsilon_1, \upsilon_2, m), m) = (1-\wp)\ \Omega(\upsilon_1, \upsilon_2, m)$$
$$\Omega(\upsilon_1, \upsilon_2, m) = -\Omega(\upsilon_2, \upsilon_1, m).$$

Theorem 2.1 *[36] Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\Psi_1, \Psi_2 : [x_1, x_2] \to \mathbb{R}$ such that $|\Psi_1|^p$ and $|\Psi_2|^q$ are integrable on $[x_1, x_2]$. Then*

$$\int_{x_1}^{x_2} |\Psi_1(x)\Psi_2(x)|dx \le \Big(\int_{x_1}^{x_2} |\Psi_1(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_{x_1}^{x_2} |\Psi_2(x)|^q dx\Big)^{\frac{1}{q}}.$$

Theorem 2.2 *[36] Let $p \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\Psi_1, \Psi_2 : [x_1, x_2] \to \mathbb{R}$ such that $|\Psi_1|^p$ and $|\Psi_2|^q$ are integrable on $[x_1, x_2]$. Then*

$$\int_{x_1}^{x_2} |\Psi_1(x)\Psi_2(x)|dx \le \Big(\int_{x_1}^{x_2} |\Psi_1(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_{x_1}^{x_2} |\Psi_1(x)|dx \int_{x_1}^{x_2} |\Psi_2(x)|^q dx\Big)^{\frac{1}{q}}.$$

Definition 2.5 *[37] Let $\Psi \in H^1(\upsilon_1, \upsilon_2) := \{y \in L^2(\upsilon_1, \upsilon_2) : y' \in L^2(\upsilon_1, \upsilon_2)\}, \upsilon_2 > \upsilon_1, \alpha \in [0,1]$ then, Caputo-Fabrizio fractional integral is:*

$$^{CF}_{\upsilon_1}I^{\alpha}(\Psi(t)) = \frac{1-\alpha}{B(\alpha)}\Psi(t) + \frac{\alpha}{B(\alpha)}\int_{\upsilon_1}^{t} \Psi(y)dy$$

and

$$^{CF}I^{\alpha}_{\upsilon_2}(\Psi(t)) = \frac{1-\alpha}{B(\alpha)}\Psi(t) + \frac{\alpha}{B(\alpha)}\int_{t}^{\upsilon_2} \Psi(y)dy,$$

where $B(\alpha)$ is normalization function.

Tariq et al. [38] investigated the new form of H-H type inequality pertaining to C-FFIO involving preinvexity as follows.

Theorem 2.3 *Let $\Psi : I = [\upsilon_1, \upsilon_1 + \Omega(\upsilon_2, \upsilon_1)] \to (0, \infty)$ be a preinvex function and $\Psi \in L[\upsilon_1, \upsilon_1 + \Omega(\upsilon_2, \upsilon_1)]$. If $\alpha \in [0,1]$, then*

$$\begin{aligned}&\Psi\left(\frac{2\upsilon_1 + \Omega(\upsilon_2, \upsilon_1)}{2}\right)\\ \le&\frac{B(\alpha)}{\alpha\Omega(\upsilon_2, \upsilon_1)}\left[^{CF}_{\upsilon_1}I^{\alpha}\{\Psi(k)\} + {}^{CF}I^{\alpha}_{\upsilon_1+\Omega(\upsilon_2,\upsilon_1)}\{\Psi(k)\} - \frac{2(1-\alpha)}{B(\alpha)}\Psi(k)\right]\\ \le&\frac{\Psi(\upsilon_1) + \Psi(\upsilon_2)}{2},\end{aligned}$$

where $k \in [\upsilon_1, \upsilon_1 + \Omega(\upsilon_2, \upsilon_1)]$.

Atangana and Baleanu use the Mittag-Leffler function to study the new derivative operators in the C-F derivative operator.

Definition 2.6 *[39] Suppose $\Psi \in H^1(\upsilon_1, \upsilon_2), \upsilon_2 > \upsilon_1, \alpha \in [0,1)$ then, the CFD is defined by*

$$ {}^{ABC}_{\upsilon_1}D^{\alpha}_{t}[\Psi(t)] = \frac{B(\alpha)}{1-\alpha}\int_{\upsilon_1}^{t}\Psi'(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{(1-\alpha)}\right]dx. \tag{2.2}$$

Definition 2.7 *[39] Suppose $\Psi \in H^1(\upsilon_1, \upsilon_2), \upsilon_2 > \upsilon_1, \alpha \in [0,1)$ then, the CFD is defined by*

$$ {}^{ABR}_{\upsilon_1}D^{\alpha}_{t}[\Psi(t)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{\upsilon_1}^{t}\Psi(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{(1-\alpha)}\right]dx. \tag{2.3}$$

Definition 2.8 *[39] The fractional integral associated to the new fractional derivative with non-local kernel of a function $\Psi \in H^1(\upsilon_1, \upsilon_2)$ is defined:*

$$ {}^{AB}_{\upsilon_1}I^{\alpha}\{\Psi(t)\} = \frac{1-\alpha}{B(\alpha)}\Psi(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\upsilon_1}^{t}\Psi(y)(t-y)^{\alpha-1}dy,$$

where $b > a, \alpha \in (0,1]$.

Abdeljawad and Baleanu presented the right-hand side of the integral operator as follows in [40]: The definition of the right fractional new integral with ML kernel of order $\alpha \in (0,1]$ is

$$ {}^{AB}I^{\alpha}_{\upsilon_2}\{\Psi(t)\} = \frac{1-\alpha}{B(\alpha)}\Psi(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t}^{\upsilon_2}\Psi(y)(y-t)^{\alpha-1}dy.$$

3 Hermite–Hadamard Inequality via AB Fractional Integral Operator

Hadamard's inequality is considered to be the most prominent and important inequality in the realm of inequality investigations. First to establish this inequality were Hermite and Hadamard [41]. It has several applications and a very intriguing geometric interpretation. The H–H inequality, which adhere to the Jensen's inequality, is an extension of the concept of convexity. Many mathematicians became motivated by the premise of this inequality to assess and examine classical inequalities using different convexity definitions.

The main aim of this section is to offer a new kind of the H–H-type inequality for a m-preinvex function via ABFIO.

Theorem 3.1 *Suppose that $I \subseteq \mathbb{R}$ is an open and non-empty m-invex subset w.r.t. $\Omega : I \times I \to \mathbb{R}$ and $\upsilon_1, \upsilon_2 \in I$ with $m\upsilon_1 < m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)$. If $\Psi : [m\upsilon_1, m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)] \to \mathbb{R}$ is a m-preinvex function, $\Psi \in L[m\upsilon_1, m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)]$ and Ω satisfies extended condition C, then for $\alpha \in (0,1]$, we have*

$$\begin{aligned}
&\Psi\left(\frac{2m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)}{2}\right)\\
&\leq \frac{B(\alpha)\Gamma(\alpha)}{2[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi(m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m))\} + {}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\{\Psi(m\upsilon_1)\}\right]\\
&- \frac{(1-\alpha)\Gamma(\alpha)}{2[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}[\Psi(m\upsilon_1) + \Psi(\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m))]\\
&\leq \frac{m\Psi(\upsilon_1) + \Psi(\upsilon_2)}{2},
\end{aligned}\tag{3.1}$$

where $B(\alpha) > 0$ is the normalization function and $\Gamma(.)$ is Gamma function.

Proof. Since Ψ is m-preinvex function on $[m\upsilon_1, \upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)]$, we can write (see, e.g., [42,43])

$$2\Psi\left(\frac{2m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)}{2}\right) \leq \Psi(m\upsilon_1 + \varphi\Omega(\upsilon_2, \upsilon_1, m)) + \Psi(m\upsilon_1 + (1-\varphi)\Omega(\upsilon_2, \upsilon_1, m)). \tag{3.2}$$

Multiplying both sides of the above inequality (3.2) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\varphi^{\alpha-1}$ then integrating the resulting inequality with respect to φ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{B(\alpha)\Gamma(\alpha)}\Psi\left(\frac{2m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)}{2}\right)\\
&\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 \varphi^{\alpha-1}\Psi(m\upsilon_1 + \varphi\Omega(\upsilon_2, \upsilon_1, m))\,d\varphi\\
&\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\Psi(m\upsilon_1 + (1-\varphi)\Omega(\upsilon_2, \upsilon_1, m))\,d\varphi\\
&= \frac{\alpha}{B(\alpha)\Gamma(\alpha)[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)} (x - m\upsilon_1)^{\alpha-1}\Psi(x)dx\\
&\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)} (m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m) - y)^{\alpha-1}\Psi(y)dy.
\end{aligned}$$

Then we can write

$$\begin{aligned}
&\frac{2}{B(\alpha)\Gamma(\alpha)}\Psi\left(\frac{2m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)}{2}\right)\\
&\leq \frac{1}{[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}\left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)} (x - m\upsilon_1)^{\alpha-1}\Psi(x)dx + \frac{(1-\alpha)}{B(\alpha)}\Psi(m\upsilon_1)\right]\\
&\quad-\frac{(1-\alpha)}{B(\alpha)[\Omega(m\upsilon_2, \upsilon_1)]^{\alpha}}\Psi(m\upsilon_1)\\
&\quad+\frac{1}{[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}\left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)} (m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m) - y)^{\alpha-1}\Psi(y)dy\right.\\
&\quad\left.+\frac{(1-\alpha)}{B(\alpha)}\Psi(m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m))\right] - \frac{(1-\alpha)}{B(\alpha)[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}\Psi(m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)).
\end{aligned}$$

So, using ABFIO, we get

$$\begin{aligned}
&\frac{2}{B(\alpha)\Gamma(\alpha)}\Psi\left(\frac{2m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)}{2}\right)\\
&\leq \frac{1}{[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi(m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m))\} + {}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\{\Psi(m\upsilon_1)\}\right]\\
&\quad-\frac{(1-\alpha)}{B(\alpha)[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}}[\Psi(m\upsilon_1) + \Psi(m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m))]
\end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality, we first mark that if Ψ is a m-preinvex function, then

$$\Psi(m\upsilon_1 + \varphi\Omega(\upsilon_2, \upsilon_1, m)) \leq m(1-\varphi)\Psi(\upsilon_1) + \varphi\Psi(\upsilon_2)$$

and

$$\Psi\left(mv_1+(1-\varphi)\Omega\left(v_2,v_1,m\right)\right)\leq m\varphi\Psi\left(v_1\right)+(1-\varphi)\Psi\left(v_2\right).$$

By adding these inequalities side by side, we have

$$\Psi\left(mv_1+\varphi\Omega\left(v_2,v_1,m\right)\right)+\Psi\left(mv_1+(1-\varphi)\Omega\left(v_2,v_1,m\right)\right)\leq m\Psi\left(v_1\right)+\Psi\left(v_2\right). \tag{3.3}$$

Then, multiplying both sides of the above inequality (3.3) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\varphi^{\alpha-1}$ and integrating the resulting inequality with respect to φ over $[0,1]$, we obtain

$$\begin{aligned}&\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1\varphi^{\alpha-1}\Psi\left(mv_1+\varphi\Omega\left(v_2,v_1,m\right)\right)d\varphi\\&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1\varphi^{\alpha-1}\Psi\left(mv_1+(1-\varphi)\Omega\left(v_2,v_1,m\right)\right)d\varphi\\&\leq\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[m\Psi\left(v_1\right)+\Psi\left(v_2\right)\right]\int_0^1\varphi^{\alpha-1}d\varphi.\end{aligned}$$

Then, we can write

$$\begin{aligned}&\frac{1}{\left[\Omega\left(v_2,v_1,m\right)\right]^{\alpha}}\left[{}^{AB}_{mv_1}I^{\alpha}\left\{\Psi\left(mv_1+\Omega\left(v_2,v_1,m\right)\right)\right\}+{}^{AB}I^{\alpha}_{mv_1+\Omega(v_2,v_1,m)}\left\{\Psi\left(mv_1\right)\right\}\right]\\&-\frac{(1-\alpha)}{B(\alpha)\left[\Omega\left(v_2,v_1,m\right)\right]^{\alpha}}\left[\Psi\left(mv_1\right)+\Psi\left(mv_1+\Omega\left(v_2,v_1,m\right)\right)\right]\\&\leq\frac{m\Psi\left(v_1\right)+\Psi\left(v_2\right)}{B(\alpha)\Gamma(\alpha)}.\end{aligned}$$

So, the proof of this theorem is completed.

Remark 3.1 Choosing $m=1$ and $\Omega\left(v_2,v_1\right)=v_2-v_1$, we attain the result in [44, Proposition 2.1], inequality (13).

Remark 3.2 Choosing $m=1$ and $\alpha=1$ in the above theorem, we have the result in the paper [45].

Note that, if in Theorem 3.1, we put $m=1$, then we attain the novel sort of H-H integral inequality over preinvexity via ABFIO.

Remark 3.3 Let $I\subseteq\mathbb{R}$ be an open invex subset w.r.t. $\Omega:I\times I\to\mathbb{R}$ and $v_1,v_2\in I$ with $v_1<v_1+\Omega\left(v_2,v_1\right)$. If $\Psi:\left[v_1,v_1+\Omega\left(v_2,v_1\right)\right]\to\mathbb{R}$ is a preinvex function, $\Psi\in L\left[v_1,v_1+\Omega\left(v_2,v_1\right)\right]$ and Ω satisfies extended condition C, then for $\alpha\in(0,1]$, we have

$$\begin{aligned}&\Psi\left(\frac{2v_1+\Omega\left(v_2,v_1\right)}{2}\right)\\&\leq\frac{B(\alpha)\Gamma(\alpha)}{2\left[\Omega\left(v_2,v_1\right)\right]^{\alpha}}\left[{}^{AB}_{v_1}I^{\alpha}\left\{\Psi\left(v_1+\Omega\left(v_2,v_1\right)\right)\right\}+{}^{AB}I^{\alpha}_{v_1+\Omega(v_2,v_1)}\left\{\Psi\left(v_1\right)\right\}\right]\\&-\frac{(1-\alpha)\Gamma(\alpha)}{2\left[\Omega\left(v_2,v_1\right)\right]^{\alpha}}\left[\Psi\left(v_1\right)+\Psi\left(v_1+\Omega\left(v_2,v_1\right)\right)\right]\\&\leq\frac{\Psi\left(v_1\right)+\Psi\left(v_2\right)}{2}.\end{aligned}$$

Also, if in Theorem 3.1, we put $\Omega\left(v_2,v_1,m\right)=v_2-mv_1$, then we attain the novel sort of H-H inequality over convexity via ABFIO.

Remark 3.4 If $\Psi : [m\upsilon_1, \upsilon_2] \to \mathbb{R}$ is a m-convex function, $\Psi \in L[m\upsilon_1, \upsilon_2]$ and Ω satisfies extended condition C, then

$$\begin{aligned}&\Psi\left(\frac{m\upsilon_1+\upsilon_2}{2}\right)\\ &\leq \frac{B(\alpha)\Gamma(\alpha)}{2\left[(\upsilon_2-m\upsilon_1)\right]^{\alpha}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\left\{\Psi(\upsilon_2)\right\}+{}^{AB}I^{\alpha}_{\upsilon_2}\left\{\Psi(m\upsilon_1)\right\}\right]-\frac{(1-\alpha)\Gamma(\alpha)}{2\left[(\upsilon_2-m\upsilon_1)\right]^{\alpha}}\left[\Psi(m\upsilon_1)+\Psi(\upsilon_2)\right]\\ &\leq \frac{m\Psi(\upsilon_1)+\Psi(\upsilon_2)}{2}.\end{aligned}$$

4 Generalizations of H–H-Type Inequality via AB Fractional Integral Operator

Many scholars in the realm of convex theory have started to collaborate jointly on novel approaches to this challenge from different angles. Convex function and inequality are a fundamental concept in pure and practical mathematics. In recent decades, H–H inequalities in the form of of convexity have attracted a lot of attention, leading to a significant number of incremental advancements and findings.

The purpose of this section is to investigate and present a novel equality. Based on this recently studied equality, we use an ABFIO to derive some new improvements of H–H-type inequalities. We add a few remarks to enhance the content and draw readers in. We start by proving a lemma in the ABFIO frame.

In this section, the normalization function is denoted by $B(\alpha)$ and the gamma function by $\Gamma(.)$.

Lemma 4.1 *Suppose $I \subseteq \mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega : I\times I \to \mathbb{R}$ and $\upsilon_1, \upsilon_2 \in I$ with $m\upsilon_1 < m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)$. Let $\Psi : I \to \mathbb{R}$ be a differentiable function. If $\Psi' \in L[m\upsilon_1, m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)]$, then*

$$\begin{aligned}&\frac{B(\alpha)\Gamma(\alpha)}{\left[\Omega(\upsilon_2,\upsilon_1,m)\right]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\left\{\Psi\left(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)\right)\right\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left\{\Psi(m\upsilon_1)\right\}\right]\\ &-\left(\frac{\left[\Omega(\upsilon_2,\upsilon_1,m)\right]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{\left[\Omega(\upsilon_2,\upsilon_1,m)\right]^{\alpha+1}}\right)\left[\Psi(m\upsilon_1)+\Psi\left(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)\right)\right]\\ &=\int_0^1(1-\varphi)^{\alpha}\Psi'\left(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m)\right)d\varphi-\int_0^1\varphi^{\alpha}\Psi'\left(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m)\right)d\varphi\end{aligned}$$

where $\alpha \in (0,1], \varphi \in [0,1]$.

Proof. By using integration, we have

$$\begin{aligned}&\int_0^1(1-\varphi)^{\alpha}\Psi'\left(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m)\right)d\varphi\\ &=\left.\frac{(1-\varphi)^{\alpha}\Psi\left(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m)\right)}{\Omega(\upsilon_2,\upsilon_1,m)}\right|_0^1+\frac{\alpha}{\Omega(\upsilon_2,\upsilon_1,m)}\int_0^1\Psi\left(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m)\right)(1-\varphi)^{\alpha-1}d\varphi\\ &=-\frac{m\Psi(\upsilon_1)}{\Omega(\upsilon_2,\upsilon_1,m)}+\frac{\alpha}{\Omega(\upsilon_2,\upsilon_1,m)}\int_0^1(1-\varphi)^{\alpha-1}\Psi\left(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m)\right)d\varphi\\ &=-\frac{m\Psi(\upsilon_1)}{\Omega(\upsilon_2,\upsilon_1,m)}+\frac{\alpha}{\left[\Omega(\upsilon_2,\upsilon_1,m)\right]^{\alpha+1}}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)-x\right)^{\alpha-1}\Psi(x)dx.\end{aligned} \tag{4.1}$$

If we multiply both sides of the above inequality (4.1) by $\frac{1}{B(\alpha)\Gamma(\alpha)}$, we get

$$\frac{1}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-\varphi)^{\alpha}\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi=-\frac{\Psi\left(\upsilon_1\right)}{B(\alpha)\Gamma(\alpha)\Omega\left(\upsilon_2,\upsilon_1,m\right)}$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1)}\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)-x\right)^{\alpha-1}\Psi(x)dx.$$

Then, we can write

$$\begin{aligned}&\frac{1}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-\varphi)^{\alpha}\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi\\ =&-\frac{\Psi\left(m\upsilon_1\right)}{B(\alpha)\Gamma(\alpha)\Omega\left(\upsilon_2,\upsilon_1,m\right)}\\ &+\frac{1}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)-x\right)^{\alpha-1}\Psi(x)dx\right.\\ &\left.+\frac{(1-\alpha)}{B(\alpha)}\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]-\frac{(1-\alpha)}{B(\alpha)\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right).\end{aligned}$$

Using ABFIO, we have

$$\begin{aligned}&\frac{1}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-\varphi)^{\alpha}\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi\\ =&-\frac{\Psi\left(m\upsilon_1\right)}{B(\alpha)\Gamma(\alpha)\Omega\left(\upsilon_2,\upsilon_1,m\right)}+\frac{1}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[{}_{\upsilon_1}^{AB}I^{\alpha}\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}\right]\\ &-\frac{(1-\alpha)}{B(\alpha)\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right).\end{aligned}\tag{4.2}$$

Similarly, using integration, we get

$$\begin{aligned}&\int_0^1 \varphi^{\alpha}\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi\\ =&\left.\frac{\varphi^{\alpha}\Psi\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)}{\Omega\left(\upsilon_2,\upsilon_1,m\right)}\right|_0^1-\frac{\alpha}{\Omega\left(\upsilon_2,\upsilon_1,m\right)}\int_0^1\Psi\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\varphi^{\alpha-1}d\varphi\\ =&\frac{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)}{\Omega\left(\upsilon_2,\upsilon_1\right)}-\frac{\alpha}{\Omega\left(\upsilon_2,\upsilon_1,m\right)}\int_0^1\varphi^{\alpha-1}\Psi\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi\\ =&\frac{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)}{\Omega\left(\upsilon_2,\upsilon_1,m\right)}-\frac{\alpha}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left(u-m\upsilon_1\right)^{\alpha-1}\Psi(u)du.\end{aligned}\tag{4.3}$$

If we multiply both sides of the above inequality (4.3) by $-\frac{1}{B(\alpha)\Gamma(\alpha)}$, we have

$$\begin{aligned}&-\frac{1}{B(\alpha)\Gamma(\alpha)}\int_0^1 \varphi^{\alpha}\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi\\ =&-\frac{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)}{B(\alpha)\Gamma(\alpha)\Omega\left(\upsilon_2,\upsilon_1,m\right)}+\frac{\alpha}{B(\alpha)\Gamma(\alpha)\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left(u-m\upsilon_1\right)^{\alpha-1}\Psi(u)du.\end{aligned}$$

Then we can write

$$-\frac{1}{B(\alpha)\Gamma(\alpha)}\int_0^1 \varphi^\alpha \Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi$$
$$=-\frac{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)}{B(\alpha)\Gamma(\alpha)\Omega\left(\upsilon_2,\upsilon_1,m\right)}+\frac{1}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{m\upsilon_1}^{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left(u-m\upsilon_1\right)^{\alpha-1}\Psi(u)du\right.$$
$$\left.+\frac{(1-\alpha)}{B(\alpha)}\Psi\left(m\upsilon_1\right)\right]-\frac{(1-\alpha)}{B(\alpha)\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\Psi\left(m\upsilon_1\right).$$

Using ABFIO, we have

$$\begin{aligned}
&-\frac{1}{B(\alpha)\Gamma(\alpha)}\int_0^1 \varphi^\alpha \Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi\\
&=-\frac{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)}{B(\alpha)\Gamma(\alpha)\Omega\left(\upsilon_2,\upsilon_1,m\right)}+\frac{1}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[{}^{AB}I^\alpha_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\\
&-\frac{(1-\alpha)}{B(\alpha)\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\Psi\left(m\upsilon_1\right).
\end{aligned}\tag{4.4}$$

By adding identities (4.2) and (4.4), we obtain the proof of Lemma 4.1.

Remark 4.1 If we put $\Omega\left(\upsilon_2,\upsilon_1,m\right)=\upsilon_2-m\upsilon_1$ and $m=1$ in Lemma 4.1 then we have the result in [44, Theorem 3.1], equality (29).

Theorem 4.1 *Let $I\subseteq\mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega: I\times I\to\mathbb{R}$ and $\upsilon_1,\upsilon_2\in I$ with $m\upsilon_1<m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)$. Suppose that $\Psi: I\to\mathbb{R}$ is a differentiable function and $\Psi'\in L\left[m\upsilon_1,m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]$. If $|\Psi'|$ is m-preinvex function, then*

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^\alpha\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}+{}^{AB}I^\alpha_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.\\
&\left.-\left(\frac{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^\alpha+(1-\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]\right|\\
&\leq\frac{m\left|\Psi'\left(\upsilon_1\right)\right|+\left|\Psi'\left(\upsilon_2\right)\right|}{\alpha+1},
\end{aligned}$$

where $\alpha\in(0,1]$.

Proof. From Lemma 4.1, we have

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^\alpha\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}+{}^{AB}I^\alpha_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.\\
&\left.-\left(\frac{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^\alpha+(1-\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]\right|\\
&=\left|\int_0^1(1-\varphi)^\alpha\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi-\int_0^1\varphi^\alpha\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)d\varphi\right|\\
&\leq\int_0^1(1-\varphi)^\alpha\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|d\varphi+\int_0^1\varphi^\alpha\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|d\varphi.
\end{aligned}$$

Since $|\Psi'|$ is m-preinvex function, we obtain

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\{\Psi(m\upsilon_1)\}\right]\right.\\
&\left.-\left(\frac{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\right)[\Psi(m\upsilon_1)+\Psi(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))]\right|\\
\leq&\int_0^1(1-\varphi)^{\alpha}\left[m(1-\varphi)\left|\Psi'(\upsilon_1)\right|+\varphi\left|\Psi'(\upsilon_2)\right|\right]d\varphi\\
&+\int_0^1\varphi^{\alpha}\left[m(1-\varphi)\left|\Psi'(\upsilon_1)\right|+\varphi\left|\Psi'(\upsilon_2)\right|\right]d\varphi\\
=&\frac{m\left|\Psi'(\upsilon_1)\right|+\left|\Psi'(\upsilon_2)\right|}{\alpha+1}.
\end{aligned}$$

So, the proof is completed.

Remark 4.2 In the above Theorem 4.1, if we choose $m=1$, then we obtain

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha+1}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\{\Psi(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))\}+{}^{AB}I^{\alpha}_{\upsilon_1+\Omega(\upsilon_2,\upsilon_1)}\{\Psi(\upsilon_1)\}\right]\right.\\
&\left.-\left(\frac{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha+1}}\right)[\Psi(\upsilon_1)+\Psi(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))]\right|\\
&\leq\frac{\left|\Psi'(\upsilon_1)\right|+\left|\Psi'(\upsilon_2)\right|}{\alpha+1}.
\end{aligned}$$

Remark 4.3 In the above Theorem 4.1, if we choose $\Omega(\upsilon_2,\upsilon_1,m)=\upsilon_2-m\upsilon_1$, then we obtain

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{(\upsilon_2-m\upsilon_1)^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi(\upsilon_2)\}+{}^{AB}I^{\alpha}_{\upsilon_2}\{\Psi(m\upsilon_1)\}\right]\right.\\
&\left.-\left(\frac{(\upsilon_2-m\upsilon_1)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{(\upsilon_2-m\upsilon_1)^{\alpha+1}}\right)[\Psi(m\upsilon_1)+\Psi(\upsilon_2)]\right|\\
&\leq\frac{m\left|\Psi'(\upsilon_1)\right|+\left|\Psi'(\upsilon_2)\right|}{\alpha+1}.
\end{aligned}$$

Remark 4.4 In the above Theorem 4.1, if we choose $\Omega(\upsilon_2,\upsilon_1,m)=\upsilon_2-m\upsilon_1$ and $m=1$, then we obtain

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{(\upsilon_2-\upsilon_1)^{\alpha+1}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\{\Psi(\upsilon_2)\}+{}^{AB}I^{\alpha}_{\upsilon_2}\{\Psi(\upsilon_1)\}\right]\right.\\
&\left.-\left(\frac{(\upsilon_2-\upsilon_1)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{(\upsilon_2-\upsilon_1)^{\alpha+1}}\right)[\Psi(\upsilon_1)+\Psi(\upsilon_2)]\right|\\
&\leq\frac{\left|\Psi'(\upsilon_1)\right|+\left|\Psi'(\upsilon_2)\right|}{\alpha+1}.
\end{aligned}$$

Theorem 4.2 *Suppose $I\subseteq\mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega: I\times I\to\mathbb{R}$ and $\upsilon_1,\upsilon_2\in I$ with $m\upsilon_1<m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)$. Let $\Psi: I\to\mathbb{R}$ is a differentiable*

function and $\Psi' \in L\left[m\upsilon_1, m\upsilon_1 + \Omega\left(\upsilon_2, \upsilon_1, m\right)\right]$. *If* $|\Psi'|^q$ *is a* m*-preinvex function, then*

$$\left|\frac{B(\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.$$
$$\left.-\left(\frac{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]\right|$$
$$\leq 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{m\left|\Psi'\left(\upsilon_1\right)\right|^q+\left|\Psi'\left(\upsilon_2\right)\right|^q}{2}\right)^{\frac{1}{q}},$$

where $p^{-1}+q^{-1}=1, q>1, \alpha\in(0,1]$.

Proof. From Lemma 4.1, we have

$$\left|\frac{B(\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.$$
$$\left.-\left(\frac{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]\right|$$
$$\leq\int_0^1(1-\varphi)^{\alpha}\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|d\varphi+\int_0^1\varphi^{\alpha}\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|d\varphi.$$

By employing Hölder inequality, we get

$$\left|\frac{B(\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.$$
$$\left.-\left(\frac{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]\right|$$
$$\leq\left(\int_0^1(1-\varphi)^{\alpha p}d\varphi\right)^{\frac{1}{p}}\left(\int_0^1\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|^q d\varphi\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\varphi^{\alpha p}d\varphi\right)^{\frac{1}{p}}\left(\int_0^1\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|^q d\varphi\right)^{\frac{1}{q}}.$$

By utilizing m-preinvexity of $|\Psi'|^q$, we have

$$\left|\frac{B(\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1\right)\right]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.$$
$$\left.-\left(\frac{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{\left[\Omega\left(\upsilon_2,\upsilon_1,m\right)\right]^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]\right|$$
$$\leq\left(\int_0^1(1-\varphi)^{\alpha p}d\varphi\right)^{\frac{1}{p}}\left(\int_0^1\left[m(1-\varphi)\left|\Psi'\left(\upsilon_1\right)\right|^q+\varphi\left|\Psi'\left(\upsilon_2\right)\right|^q\right]d\varphi\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\varphi^{\alpha p}d\varphi\right)^{\frac{1}{p}}\left(\int_0^1\left[m(1-\varphi)\left|\Psi'\left(\upsilon_1\right)\right|^q+\varphi\left|\Psi'\left(\upsilon_2\right)\right|^q\right]d\varphi\right)^{\frac{1}{q}}.$$

By calculating the integrals in the above inequality, we get the desired result.

Remark 4.5 In the above Theorem, if we choose $m = 1$, then we obtain

$$\begin{aligned}&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha+1}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\{\Psi(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))\}+{}^{AB}I^{\alpha}_{\upsilon_1+\Omega(\upsilon_2,\upsilon_1)}\{\Psi(\upsilon_1)\}\right]\right.\\&\left.-\left(\frac{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha+1}}\right)[\Psi(\upsilon_1)+\Psi(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))]\right|\\&\leq 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\upsilon_1)|^q+|\Psi'(\upsilon_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Remark 4.6 In the above Theorem, if we choose $\Omega(\upsilon_2,\upsilon_1,m) = \upsilon_2 - m\upsilon_1$, then we obtain

$$\begin{aligned}&\left|\frac{B(\alpha)\Gamma(\alpha)}{(\upsilon_2-m\upsilon_1)^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi(\upsilon_2)\}+{}^{AB}I^{\alpha}_{\upsilon_2}\{\Psi(m\upsilon_1)\}\right]\right.\\&\left.-\left(\frac{(\upsilon_2-m\upsilon_1)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{(\upsilon_2-m\upsilon_1)^{\alpha+1}}\right)[\Psi(m\upsilon_1)+\Psi(\upsilon_2)]\right|\\&\leq 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(m\upsilon_1)|^q+|\Psi'(\upsilon_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Remark 4.7 In the above Theorem, if we choose $\Omega(\upsilon_2,\upsilon_1,m) = \upsilon_2 - m\upsilon_1$ and $m = 1$, then we obtain

$$\begin{aligned}&\left|\frac{B(\alpha)\Gamma(\alpha)}{(\upsilon_2-\upsilon_1,m)^{\alpha+1}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\{\Psi(\upsilon_2)\}+{}^{AB}I^{\alpha}_{\upsilon_2}\{\Psi(m\upsilon_1)\}\right]\right.\\&\left.-\left(\frac{(\upsilon_2-\upsilon_1)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{(\upsilon_2-\upsilon_1,m)^{\alpha+1}}\right)[\Psi(\upsilon_1)+\Psi(\upsilon_2)]\right|\\&\leq 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\upsilon_1)|^q+|\Psi'(\upsilon_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 4.3 *Suppose $I \subseteq \mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega : I \times I \to \mathbb{R}$ and $\upsilon_1, \upsilon_2 \in I$ with $m\upsilon_1 < m\upsilon_1 + \Omega(\upsilon_2,\upsilon_1,m)$. Let $\Psi : I \to \mathbb{R}$ is a differentiable function and $\Psi' \in L[m\upsilon_1, m\upsilon_1 + \Omega(\upsilon_2,\upsilon_1,m)]$. If $|\Psi'|^q$ is a m-preinvex function, then we have the following inequality for ABFIO*

$$\begin{aligned}&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\{\Psi(m\upsilon_1)\}\right]\right.\\&\left.-\left(\frac{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\right)[\Psi(m\upsilon_1)+\Psi(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))]\right|\\&\leq\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{m|\Psi'(\upsilon_1)|^q}{\alpha+2}+\frac{|\Psi'(\upsilon_2)|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{m|\Psi'(\upsilon_1)|^q}{(\alpha+1)(\alpha+2)}+\frac{|\Psi'(\upsilon_2)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right],\end{aligned}$$

where $\alpha \in (0,1], q \geq 1$.

Proof. From Lemma 4.1 and the power mean inequality, we get

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.\\
&\left.-\left(\frac{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]\right|\\
\le&\int_0^1(1-\varphi)^{\alpha}\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|d\varphi+\int_0^1\varphi^{\alpha}\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|d\varphi\\
\le&\left(\int_0^1(1-\varphi)^{\alpha}d\varphi\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\varphi)^{\alpha}\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|^q d\varphi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\varphi^{\alpha}d\varphi\right)^{1-\frac{1}{q}}\left(\int_0^1\varphi^{\alpha}\left|\Psi'\left(m\upsilon_1+\varphi\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right|^q d\varphi\right)^{\frac{1}{q}}.
\end{aligned}$$

By using m-preinvexity of $|\Psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\left\{\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.\\
&\left.-\left(\frac{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(m\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1,m\right)\right)\right]\right|\\
\le&\left(\int_0^1(1-\varphi)^{\alpha}d\varphi\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\varphi)^{\alpha}\left[m(1-\varphi)\left|\Psi'\left(\upsilon_1\right)\right|^q+\varphi\left|\Psi'\left(\upsilon_2\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\varphi^{\alpha}d\varphi\right)^{1-\frac{1}{q}}\left(\int_0^1\varphi^{\alpha}\left[m(1-\varphi)\left|\Psi'\left(\upsilon_1\right)\right|^q+\varphi\left|\Psi'\left(\upsilon_2\right)\right|^q\right]d\varphi\right)^{\frac{1}{q}}\\
=&\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{m\left|\Psi'\left(\upsilon_1\right)\right|^q}{\alpha+2}+\frac{\left|\Psi'\left(\upsilon_2\right)\right|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{m\left|\Psi'\left(\upsilon_1\right)\right|^q}{(\alpha+1)(\alpha+2)}+\frac{\left|\Psi'\left(\upsilon_2\right)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Th proof is completed.

Remark 4.8 In the above Theorem, if we choose $m=1$, we obtain

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha+1}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\left\{\Psi\left(\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1\right)\right)\right\}+{}^{AB}I^{\alpha}_{\upsilon_1+\Omega(\upsilon_2,\upsilon_1)}\left\{\Psi\left(\upsilon_1\right)\right\}\right]\right.\\
&\left.-\left(\frac{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha+1}}\right)\left[\Psi\left(\upsilon_1\right)+\Psi\left(\upsilon_1+\Omega\left(\upsilon_2,\upsilon_1\right)\right)\right]\right|\\
\le&\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|\Psi'\left(\upsilon_1\right)\right|^q}{\alpha+2}+\frac{\left|\Psi'\left(\upsilon_2\right)\right|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{\left|\Psi'\left(\upsilon_1\right)\right|^q}{(\alpha+1)(\alpha+2)}+\frac{\left|\Psi'\left(\upsilon_2\right)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 4.9 In the above Theorem, if we choose $\Omega\left(\upsilon_2,\upsilon_1,m\right)=\upsilon_2-m\upsilon_1$, we obtain

$$\begin{aligned}
&\left|\frac{B(\alpha)\Gamma(\alpha)}{(\upsilon_2-m\upsilon_1)^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\left\{\Psi\left(\upsilon_2\right)\right\}+{}^{AB}I^{\alpha}_{\upsilon_2}\left\{\Psi\left(m\upsilon_1\right)\right\}\right]\right.\\
&\left.-\left(\frac{(\upsilon_2-m\upsilon_1)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{(\upsilon_2-m\upsilon_1)^{\alpha+1}}\right)\left[\Psi\left(m\upsilon_1\right)+\Psi\left(\upsilon_2\right)\right]\right|\\
\le&\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|\Psi'\left(m\upsilon_1\right)\right|^q}{\alpha+2}+\frac{\left|\Psi'\left(\upsilon_2\right)\right|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{\left|\Psi'\left(m\upsilon_1\right)\right|^q}{(\alpha+1)(\alpha+2)}+\frac{\left|\Psi'\left(\upsilon_2\right)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 4.10 In the above Theorem, if we choose $\Omega(\upsilon_2,\upsilon_1,m)=\upsilon_2-m\upsilon_1$ and $m=1$, we obtain

$$\begin{aligned}&\Bigg|\frac{B(\alpha)\Gamma(\alpha)}{(\upsilon_2-\upsilon_1)^{\alpha+1}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\{\Psi(\upsilon_2)\}+{}^{AB}I^{\alpha}_{\upsilon_2}\{\Psi(\upsilon_1)\}\right]\\&-\left(\frac{(\upsilon_2-\upsilon_1)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{(\upsilon_2-\upsilon_1)^{\alpha+1}}\right)[\Psi(\upsilon_1)+\Psi(\upsilon_2)]\Bigg|\\\le&\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\Psi'(\upsilon_1)|^q}{\alpha+2}+\frac{|\Psi'(\upsilon_2)|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{|\Psi'(\upsilon_1)|^q}{(\alpha+1)(\alpha+2)}+\frac{|\Psi'(\upsilon_2)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 4.4 *Suppose $I\subseteq\mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega: I\times I\to\mathbb{R}$ and $\upsilon_1,\upsilon_2\in I$ with $m\upsilon_1<m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)$. Let $\Psi: I\to\mathbb{R}$ is a differentiable function and $\Psi'\in L[m\upsilon_1,m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)]$. If $|\Psi'|^q$ is a m-preinvex function, then we have the following inequality for ABFIO*

$$\begin{aligned}&\Bigg|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1)}\{\Psi(m\upsilon_1)\}\right]\\&-\left(\frac{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\right)[\Psi(m\upsilon_1)+\Psi(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))]\Bigg|\\\le&\frac{2}{p(\alpha p+1)}+\frac{m|\Psi'(\upsilon_1)|^q+|\Psi'(\upsilon_2)|^q}{q},\end{aligned}$$

where $p^{-1}+q^{-1}=1, q>1, \alpha\in(0,1]$.

Proof. From Lemma 4.1 and utilizing the Young inequality $xy\le\frac{1}{p}x^p+\frac{1}{q}y^q$, we have

$$\begin{aligned}&\Bigg|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\{\Psi(m\upsilon_1)\}\right]\\&-\left(\frac{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha+1}}\right)[\Psi(m\upsilon_1)+\Psi(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))]\Bigg|\\\le&\int_0^1(1-\varphi)^{\alpha}\left|\Psi'(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m))\right|d\varphi+\int_0^1\varphi^{\alpha}\left|\Psi'(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m))\right|d\varphi.\\\le&\frac{1}{p}\int_0^1(1-\varphi)^{\alpha p}d\varphi+\frac{1}{q}\int_0^1\left|\Psi'(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m))\right|^q d\varphi\\&+\frac{1}{p}\int_0^1\varphi^{\alpha p}d\varphi+\frac{1}{q}\int_0^1\left|\Psi'(m\upsilon_1+\varphi\Omega(\upsilon_2,\upsilon_1,m))\right|^q d\varphi.\end{aligned}$$

By using preinvexity of $|\Psi'|^q$ and by a simple computation, we have the desired result.

Remark 4.11 In the above Theorem, if we choose $m=1$, then we obtain

$$\begin{aligned}&\Bigg|\frac{B(\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha+1}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\{\Psi(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))\}+{}^{AB}I^{\alpha}_{\upsilon_1+\Omega(\upsilon_2,\upsilon_1)}\{\Psi(\upsilon_1)\}\right]\\&-\left(\frac{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha+1}}\right)[\Psi(\upsilon_1)+\Psi(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))]\Bigg|\\\le&\frac{2}{p(\alpha p+1)}+\frac{|\Psi'(\upsilon_1)|^q+|\Psi'(\upsilon_2)|^q}{q}.\end{aligned}$$

Remark 4.12 In the above Theorem, if we choose $\Omega(\upsilon_2, \upsilon_1, m) = \upsilon_2 - m\upsilon_1$, we obtain

$$\left| \frac{B(\alpha)\Gamma(\alpha)}{(\upsilon_2 - m\upsilon_1)^{\alpha+1}} \left[{}_{m\upsilon_1}^{AB} I^{\alpha} \{\Psi(\upsilon_2)\} + {}^{AB}I_{\upsilon_2}^{\alpha} \{\Psi(m\upsilon_1)\} \right] - \left(\frac{(\upsilon_2 - m\upsilon_1)^{\alpha} + (1-\alpha)\Gamma(\alpha)}{(\upsilon_2 - m\upsilon_1)^{\alpha+1}} \right) [\Psi(m\upsilon_1) + \Psi(\upsilon_2)] \right| \leq \frac{2}{p(\alpha p + 1)} + \frac{|\Psi'(m\upsilon_1)|^q + |\Psi'(\upsilon_2)|^q}{q}.$$

Remark 4.13 In the above Theorem, if we choose $\Omega(\upsilon_2, \upsilon_1, m) = \upsilon_2 - m\upsilon_1$ and $m = 1$, we obtain

$$\left| \frac{B(\alpha)\Gamma(\alpha)}{(\upsilon_2 - \upsilon_1)^{\alpha+1}} \left[{}_{\upsilon_1}^{AB} I^{\alpha} \{\Psi(\upsilon_2)\} + {}^{AB}I_{\upsilon_2}^{\alpha} \{\Psi(\upsilon_1)\} \right] - \left(\frac{(\upsilon_2 - \upsilon_1)^{\alpha} + (1-\alpha)\Gamma(\alpha)}{(\upsilon_2 - \upsilon_1)^{\alpha+1}} \right) [\Psi(\upsilon_1) + \Psi(\upsilon_2)] \right| \leq \frac{2}{p(\alpha p + 1)} + \frac{|\Psi'(\upsilon_1)|^q + |\Psi'(\upsilon_2)|^q}{q}.$$

5 Pachpatte-Type Inequality via AB Fractional Integral Operator

The combined study of convex analysis and integral inequality has become more and more popular in recent years. In the practical sciences, convexity applications frequently lead to an accounting of various inequalities. A lot of research papers have been published on the famous inequality, namely, the Pachpatte-type inequality via fractional integral operators. Numerous generalizations to and refinements of this inequality were first introduced by Pachpatte [46] and Cristescu [47].

We study and explore this inequality via ABFIO, in light of the aforementioned results and literature.

Theorem 5.1 *Suppose $I \subseteq \mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega : I \times I \to \mathbb{R}$ and $\upsilon_1, \upsilon_2 \in I$ with $m\upsilon_1 < m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)$. If $\Psi_1, \Psi_2 : [m\upsilon_1, m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)] \to \mathbb{R}$ are m-preinvex functions, $\Psi_1, \Psi_2 \in L[m\upsilon_1, m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)]$, then the following inequality for ABFIO holds*

$$\begin{aligned} &\frac{1}{[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}} \left[{}_{m\upsilon_1}^{AB} I^{\alpha} \{\Psi_1\Psi_2(m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m))\} + {}^{AB}I_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}^{\alpha} \{\Psi_1\Psi_2(m\upsilon_1)\} \right] \\ &\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[[\Psi_1(m\upsilon_1)\Psi_2(m\upsilon_1) + \Psi_1(\upsilon_2)\Psi_2(\upsilon_2)] \left(\frac{2}{\alpha(\alpha+1)(\alpha+2)} + \frac{1}{\alpha+2} \right) \right. \\ &\quad \left. + 2\frac{m[\Psi_1(\upsilon_1)\Psi_2(\upsilon_2) + \Psi_1(\upsilon_2)\Psi_2(\upsilon_1)]}{(\alpha+1)(\alpha+2)} \right] \\ &\quad + \frac{(1-\alpha)}{B(\alpha)[\Omega(\upsilon_2, \upsilon_1, m)]^{\alpha}} [\Psi_1(m\upsilon_1)\Psi_2(m\upsilon_1) + \Psi_1(m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m))\Psi_2(m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m))], \end{aligned}$$

where $\alpha \in (0, 1]$.

Proof. Since Ψ_1 and Ψ_2 are preinvex functions on $[m\upsilon_1, m\upsilon_1 + \Omega(\upsilon_2, \upsilon_1, m)]$, we get

$$\Psi_1(m\upsilon_1 + \varphi\Omega(\upsilon_2, \upsilon_1, m)) \leq m(1-\varphi)\Psi_1(\upsilon_1) + \varphi\Psi_1(\upsilon_2) \tag{5.1}$$

and

$$\Psi_2(mv_1+\varphi\Omega(v_2,v_1,m)) \le m(1-\varphi)\Psi_2(v_1)+\varphi\Psi_2(v_2). \tag{5.2}$$

By multiplying both inequalities 5.1 and 5.2 side by side, we get

$$\begin{aligned}&\Psi_1(mv_1+\varphi\Omega(v_2,v_1,m))\Psi_2(mv_1+\varphi\Omega(v_2,v_1,m))\\ &\le m^2(1-\varphi)^2\Psi_1(v_1)\Psi_2(v_1)+\varphi^2\Psi_1(v_2)\Psi_2(v_2)\\ &\quad+m\varphi(1-\varphi)\left[\Psi_1(v_1)\Psi_2(v_2)+\Psi_1(v_2)\Psi_2(v_1)\right].\end{aligned} \tag{5.3}$$

By multiplying both sides of (5.3) with $(1-\varphi)^{\alpha-1}$ and integrating the resulting inequality w.r.t. φ over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1(1-\varphi)^{\alpha-1}\Psi_1(mv_1+\varphi\Omega(v_2,v_1,m))\Psi_2(mv_1+\varphi\Omega(v_2,v_1,m))\,d\varphi\\ &\le\int_0^1(1-\varphi)^{\alpha-1}\left[m^2(1-\varphi)^2\Psi_1(v_1)\Psi_2(v_1)+\varphi^2\Psi_1(v_2)\Psi_2(v_2)\right.\\ &\quad\left.+m\varphi(1-\varphi)\left[\Psi_1(v_1)\Psi_2(v_2)+\Psi_1(v_2)\Psi_2(v_1)\right]\right]d\varphi\\ &=\frac{m^2\Psi_1(v_1)\Psi_2(v_1)}{\alpha+2}+2\frac{\Psi_1(v_2)\Psi_2(v_2)}{\alpha(\alpha+1)(\alpha+2)}+\frac{m\left[\Psi_1(v_1)\Psi_2(v_2)+\Psi_1(v_2)\Psi_2(v_1)\right]}{(\alpha+1)(\alpha+2)}.\end{aligned}$$

By changing the variable $mv_1+\varphi\Omega(v_2,v_1,m)=x$, we can write the inequality in (5.4) as

$$\begin{aligned}&\frac{1}{[\Omega(v_2,v_1,m)]^\alpha}\int_{mv_1}^{mv_1+\Omega(v_2,v_1,m)}(mv_1+\Omega(v_2,v_1,m)-x)^{\alpha-1}\Psi_1(x)\Psi_2(x)dx\\ &\le\frac{m^2\Psi_1(v_1)\Psi_2(v_1)}{\alpha+2}+2\frac{\Psi_1(v_2)\Psi_2(v_2)}{\alpha(\alpha+1)(\alpha+2)}+\frac{m\left[\Psi_1(v_1)\Psi_2(v_2)+\Psi_1(v_2)\Psi_2(v_1)\right]}{(\alpha+1)(\alpha+2)}\end{aligned} \tag{5.4}$$

By multiplying the both sides of (5.4) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}$ and then adding the term $\frac{(1-\alpha)}{B(\alpha)[\Omega(v_2,v_1,m)]^\alpha}\Psi_1(mv_1+\Omega(v_2,v_1,m))\Psi_2(mv_1+\Omega(v_2,v_1,m))$ to both sides of (5.4) and finally using ABFIO, we get

$$\begin{aligned}&\frac{1}{[\Omega(v_2,v_1,m)]^\alpha}\left[{}^{AB}_{mv_1}I^\alpha\left\{\Psi_1\Psi_2(mv_1+\Omega(v_2,v_1,m))\right\}\right]\\ &\le\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[\frac{m^2\Psi_1(v_1)\Psi_2(v_1)}{\alpha+2}+2\frac{\Psi_1(v_2)\Psi_2(v_2)}{\alpha(\alpha+1)(\alpha+2)}+\frac{m\left[\Psi_1(v_1)\Psi_2(v_2,m)+\Psi_1(v_2)\Psi_2(v_1)\right]}{(\alpha+1)(\alpha+2)}\right]\\ &\quad+\frac{(1-\alpha)}{B(\alpha)\left[\Omega(v_2,v_1)\right]^\alpha}\Psi_1(mv_1+\Omega(v_2,v_1,m))\Psi_2(mv_1+\Omega(v_2,v_1,m)).\end{aligned} \tag{5.5}$$

Similarly, by multiplying both sides of (5.3) with $\varphi^{\alpha-1}$ and integrating the resulting inequality w.r.t. φ over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1\varphi^{\alpha-1}\Psi_1(mv_1+\varphi\Omega(v_2,v_1,m))\Psi_2(mv_1+\varphi\Omega(v_2,v_1,m))\,d\varphi\\ &\le\int_0^1\varphi^{\alpha-1}\left[m^2(1-\varphi)^2\Psi_1(v_1)\Psi_2(v_1)+\varphi^2\Psi_1(v_2)\Psi_2(v_2)\right.\\ &\quad\left.+m\varphi(1-\varphi)\left[\Psi_1(v_1)\Psi_2(v_2)+\Psi_1(v_2)\Psi_2(v_1)\right]\right]d\varphi\\ &=2\frac{m^2\Psi_1(v_1)\Psi_2(v_1)}{\alpha(\alpha+1)(\alpha+2)}+\frac{\Psi_1(v_2)\Psi_2(v_2)}{\alpha+2}+\frac{m\left[\Psi_1(v_1)\Psi_2(v_2)+\Psi_1(v_2)\Psi_2(v_1)\right]}{(\alpha+1)(\alpha+2)}.\end{aligned}$$

By making calculations similar to those in the proof of (5.5), we obtain

$$\frac{1}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}}\left[{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\{\Psi_1\Psi_2(m\upsilon_1)\}\right]$$
$$\le \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[2\frac{m^2\Psi_1(\upsilon_1)\Psi_2(\upsilon_1)}{\alpha(\alpha+1)(\alpha+2)}+2\frac{\Psi_1(\upsilon_2)\Psi_2(\upsilon_2)}{\alpha+2}+\frac{m[\Psi_1(\upsilon_1)\Psi_2(\upsilon_2)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_1)]}{(\alpha+1)(\alpha+2)}\right]$$
$$+\frac{(1-\alpha)}{B(\alpha)[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}}\Psi_1(m\upsilon_1)\Psi_2(m\upsilon_1). \tag{5.6}$$

Adding (5.5) and (5.6) side by side, we get

$$\frac{1}{[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi_1\Psi_2(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))\}+{}^{AB}I^{\alpha}_{m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m)}\{\Psi_1\Psi_2(m\upsilon_1)\}\right]$$
$$\le \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[\left[m^2\Psi_1(\upsilon_1)\Psi_2(\upsilon_1)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_2)\right]\left(\frac{2}{\alpha(\alpha+1)(\alpha+2)}+\frac{1}{\alpha+2}\right)\right.$$
$$\left.+2\frac{m[\Psi_1(\upsilon_1)\Psi_2(\upsilon_2)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_1)]}{(\alpha+1)(\alpha+2)}\right]$$
$$+\frac{(1-\alpha)}{B(\alpha)[\Omega(\upsilon_2,\upsilon_1,m)]^{\alpha}}\left[\Psi_1(m\upsilon_1)\Psi_2(m\upsilon_1)+\Psi_1(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))\Psi_2(m\upsilon_1+\Omega(\upsilon_2,\upsilon_1,m))\right].$$

The proof is completed.

Note that in the above Theorem 5.1, if we put the value of $m=1$, then

Remark 5.1 Suppose $I\subseteq\mathbb{R}$ be an open and non-empty invex subset w.r.t. $\Omega: I\times I\to\mathbb{R}$ and $\upsilon_1,\upsilon_2\in I$ with $\upsilon_1<\upsilon_1+\Omega(\upsilon_2,\upsilon_1)$. If $\Psi_1,\Psi_2:[\upsilon_1,\upsilon_1+\Omega(\upsilon_2,\upsilon_1)]\to\mathbb{R}$ are preinvex functions, $\Psi_1,\Psi_2\in L[\upsilon_1,\upsilon_1+\Omega(\upsilon_2,\upsilon_1,)]$, then the following inequality for ABFIO holds

$$\frac{1}{[\Omega(\upsilon_2,\upsilon_1)]^{\alpha}}\left[{}^{AB}_{\upsilon_1}I^{\alpha}\{\Psi_1\Psi_2(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))\}+{}^{AB}I^{\alpha}_{\upsilon_1+\Omega(\upsilon_2,\upsilon_1)}\{\Psi_1\Psi_2(\upsilon_1)\}\right]$$
$$\le \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[\left[\Psi_1(\upsilon_1)\Psi_2(\upsilon_1)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_2)\right]\left(\frac{2}{\alpha(\alpha+1)(\alpha+2)}+\frac{1}{\alpha+2}\right)\right.$$
$$\left.+2\frac{[\Psi_1(\upsilon_1)\Psi_2(\upsilon_2)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_1)]}{(\alpha+1)(\alpha+2)}\right]$$
$$+\frac{(1-\alpha)}{B(\alpha)[\Omega(\upsilon_2,\upsilon_1)]^{\alpha}}\left[\Psi_1(\upsilon_1)\Psi_2(\upsilon_1)+\Psi_1(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))\Psi_2(\upsilon_1+\Omega(\upsilon_2,\upsilon_1))\right].$$

Also, in the above Theorem 5.1, if we put $\Omega(\upsilon_2,\upsilon_1,m)=\upsilon_2-m\upsilon_1$, then

Remark 5.2 If $\Psi_1,\Psi_2:[m\upsilon_1,\upsilon_2]\to\mathbb{R}$ are m-convex functions, $\Psi_1,\Psi_2\in L[m\upsilon_1,\upsilon_2]$, then

$$\frac{1}{(\upsilon_2-m\upsilon_1)^{\alpha}}\left[{}^{AB}_{m\upsilon_1}I^{\alpha}\{\Psi_1\Psi_2(\upsilon_2)\}+{}^{AB}I^{\alpha}_{\upsilon_2}\{\Psi_1\Psi_2(m\upsilon_1)\}\right]$$
$$\le \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[\left[\Psi_1(m\upsilon_1)\Psi_2(m\upsilon_1)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_2)\right]\left(\frac{2}{\alpha(\alpha+1)(\alpha+2)}+\frac{1}{\alpha+2}\right)\right.$$
$$\left.+2\frac{m[\Psi_1(\upsilon_1)\Psi_2(\upsilon_2)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_1)]}{(\alpha+1)(\alpha+2)}\right]$$
$$+\frac{(1-\alpha)}{B(\alpha)(\upsilon_2-m\upsilon_1)^{\alpha}}\left[\Psi_1(m\upsilon_1)\Psi_2(m\upsilon_1)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_2)\right].$$

Finally, if we put $\Omega(\upsilon_2, \upsilon_1, m) = \upsilon_2 - m\upsilon_1$ and $m = 1$ inTheorem 5.1, then

Remark 5.3 If $\Psi_1, \Psi_2 : [\upsilon_1, \upsilon_2] \to \mathbb{R}$ are m-convex functions, $\Psi_1, \Psi_2 \in L[\upsilon_1, \upsilon_2]$, then

$$\begin{aligned}
&\frac{1}{(\upsilon_2-\upsilon_1)^{\alpha}}\left[{}_{\upsilon_1}^{AB}I^{\alpha}\left\{\Psi_1\Psi_2(\upsilon_2)\right\}+{}^{AB}I_{\upsilon_2}^{\alpha}\left\{\Psi_1\Psi_2(\upsilon_1)\right\}\right]\\
&\le \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[\left[\Psi_1(\upsilon_1)\Psi_2(\upsilon_1)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_2)\right]\left(\frac{2}{\alpha(\alpha+1)(\alpha+2)}+\frac{1}{\alpha+2}\right)\right.\\
&\left.+2\frac{\left[\Psi_1(\upsilon_1)\Psi_2(\upsilon_2)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_1)\right]}{(\alpha+1)(\alpha+2)}\right]\\
&+\frac{(1-\alpha)}{B(\alpha)(\upsilon_2-\upsilon_1)^{\alpha}}\left[\Psi_1(\upsilon_1)\Psi_2(\upsilon_1)+\Psi_1(\upsilon_2)\Psi_2(\upsilon_2)\right].
\end{aligned}$$

6 Conclusions

Fractional calculus has garnered significant attention from scholars and authors across diverse disciplines. Meanwhile, convexity theory has emerged as a powerful tool for developing innovative numerical models, enabling the solution of complex problems in both pure and applied sciences. As a result, convex analysis and its associated inequalities are experiencing a surge in academic interest and popularity, driven by ongoing advancements, extensions, and applications.

In this work:

(1) First, using ABFIO and a few comments and corollaries, we looked into a novel type of H–H inequality.
(2) We presented a novel lemma. We also talked about some new improvements to the H–H inequality based on a recently developed lemma.
(3) We used a recently developed idea in the framework of the fractional integral operator to introduce a new kind of Pachpatte-type inequality.

This work offers intriguing methods and insightful information that can be applied to Raina function research. Additionally, the discussed inequalities can be investigated using interval analysis and quantum calculus as frameworks. Notably, research on integral inequality is developing quickly. The application of quantum calculus and interval-valued analysis to the study of integral inequalities is set to enthrall researchers and open up fascinating new research directions.

Abbreviations

The following abbreviations are used in this manuscript:

AB	Atangana–Baleanu
H–H	Hermite–Hadamard
R–L	Riemann–Liouville
KFIO	Katugampola fractional integral operator
ABFIO	Atangana–Baleanu fractional integral operator
RLFIO	Riemann–Liouville fractional integral operator
CFFIO	Caputo–Fibrizio fractional integral operator
H–H–M	Hermite–Hadamard–Mercer
CFD	Caputo-Fabrizio derivative

References

1. Green, J., Walter P.H.: *Mathematical Analysis and Convexity with Applications to Economics*, In Handbook of Mathematical Economics; Handbooks in economics; North-Holland Publishing Co.: Amsterdam, The Netherlands. **1**, 15–52 (1981).
2. Pennanen. T.: *Convex duality in stochastic optimization and mathematical finance*, Math. Oper. Res. **36**, 340–362 (2011).
3. Pelczynski, J.: *Application of the theory of convex sets for engineering structures with uncertain parameters*, Appl. Sci. **10**, 6864 (2020).
4. Föllmer, A., Schied, A.: *Convex measures of risk and trading constraints*, Financ. Stoch. **6**, 429–447 (2006).
5. Tariq, M., Sahoo, S.K., Nasir, J., Aydi, H., Alsamir, H.: *Some Ostrowski-type inequalities via n-polynomial exponentially s-convex functions and their applications*, AIMS Math. **6**(12), 13272-13290 (2021).
6. Tariq, M., Sahoo, S. K., Nasir, J., Awan, S. K.: *Some Ostrowski-Type integral inequalities using hypergeometric functions*, J. Frac. Cal. Nonlinear Syst. **2**(1), 2441 (2021).
7. Oliveira, E.C., Jarosz, S., Vaz, J.: *Fractional calculus via Laplace transform and its application in relaxation processes*, Commun. Nonlinear Sci. Numer. Simul. **69**, 58–72 (2019).
8. Baleanu, D., Güvenç, Z.B., Machado, J.T.: *New Trends in Nanotechnology and Fractional Calculus Applications*; Springer: New York, NY, USA, (2010).
9. Hoan, L.V.C., Akinlar, M.A., Inc, M., Gomez-Aguilar, J.F., Chu, Y.M., Almohsen, B.: *A new fractional-order compartmental disease model*, Alex. Eng. J. **59**, 3187–3196 (2020).
10. Kulish, V.V., Lage, J.L.: *Application of fractional calculus to fluid mechanics*, J. Fluids Eng. **124**, 803–806 (2002).
11. Magin, R.L.: *Fractional Calculus in Bio-Engineering*; Begell House Inc. Publishers: Danbury, CN, USA, (2006).
12. Atangana, A.: *Application of fractional calculus to epidemiology*, In Fractional Dynamics; De Gruyter Open Poland: Warsaw, Poland, 174–190 (2016).
13. Baleanu, D., Jajarmi, A., Sajjadi, S.S., Mozyrska, D.: *A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator*, Chaos: An Interdisciplinary Journal of Nonlinear Science. **29**(8), 083127 (2019).
14. Ebrahimzadeh, A., Jajarmi, A., Baleanu, D.: *Enhancing water pollution management through a comprehensive fractional modeling framework and optimal con trol techniques*, J. Nonlinear Math. Phys. **31**, 48 (2024).
15. Baleanu, D., Jajarmi, A., Defterli, O., Wannan, R., Sajjadi, S.S., Asad, J.H.: *Fractional investigation of time-dependent mass pendulum*, J. low freq. noise vib. act. control. **43**(1), 196-207 (2024).
16. Niyozov, I. E., Juraev, D. A., Efendiev, R. F., Jalalov, M. J.: *The Cauchy problem for the system of elasticity*, J. Contemp. Appl. Math. **14** (2), 92107 (2024).
17. Efendiev, R. F., Orudzhev, H. D., Bahlulzade, S. J.: *Spectral analysis of the discontinuous SturmLiouville operator with almost-periodic potentials*, Adv. Math. Models & Appl. **6** (3), 266277 (2021).
18. Efendiev, R., Gasimov, Y.: *Spectral analysis of the indefinite non-self-adjoint Sturm-Liouville operator*, Partial Differ. Equ. Appl. Math. **11**, 100831 (2024).
19. Khalouta, A., Efendiev, R. F. : *New methodology for a system of nonlinear time-fractional partial differential equations based on the Khalouta-Caputo-Katugampola fractional derivative*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, **45**(4), 105-122 (2025).
20. Bandaliyev, R.A., Alekperova, I.T.: *On some nonlinear fractional integro-differential equation and its connection with the Riemann-Liouville fractional integral in*

Lebesgue spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, **43**(4), 65-71 (2023).

21. Sarikaya, M.Z., Set, E., Yaldiz, H., Başak, N.: *Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model. **57**, 2403–2407 (2013).
22. Öğülmüs, H., Sarikaya, M.Z.: *Hermite-Hadamard-Mercer type inequalities for fractional integrals*, Filomat. **35**, 2425–2436 (2021).
23. Set, E., Akdemir, A.O., Özdemir, M.E.: *Simpson type integral inequalities for convex functions via Riemann–Liouville integrals*, Filomat. **31**, 4415–4420 (2017).
24. Dragomir, S.S.: *Ostrowski type inequalities for Riemann–Liouville fractional integrals of absolutely continuous functions in terms of norms*, RGMIA Res. Rep. Collect. **20**, 49 (2017).
25. Chen, H., Katugampola, U.N.: *Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl. **446**, 1274–1291 (2017).
26. Tariq, M., Ahmad, H., Sahoo, S.K., Kashuri, A., Nofal, T.A., Hsu, C.H.: *Inequalities of Simpson-Mercer-type including Atangana-Baleanu fractional operators and their applications*, AIMS Math. **7**, 15159–15181 (2022).
27. Hanson, M.A.: *On sufficiency of the Kuhn–Tucker conditions*, J. Math. Anal. Appl. **80**, 545–550 (1981).
28. Weir, T., Mond, B.: *Preinvex functions in multiple-objective optimization*, J. Math. Anal. Appl. **136**, 29–38 (1988).
29. Ben-Isreal, A., Mond, B.: *What is invexity?*, J. Aust. Math. Soc. Ser. B. **28**, 1–9 (1986).
30. Barani, A., Ghazanfari, G., Dragomir, S. S.: *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. **2012**, 247, (2012).
31. Du, T.T., Liao, J.G., Li, Y. J.: *Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)–preinvex functions*, J. Nonlinear Sci. Appl. **9**, 3112–3126 (2016).
32. Deng, Y., Kalsoom, H., Wu, S.: *Some new Quantum Hermite–Hadamard-type estimates within a class of generalized (s, m)—preinvex functions*, Symmetry. **11**, 1283 (2019).
33. Farajzadeh, A., Noor, M.A., Noor, K.I.: *Vector nonsmooth variational-like inequalities and optimization problems*, Nonlinear Anal. **71**, 3471–3476 (2009).
34. Noor, M.A.: *Variational–like inequalities*, Optimization. **30**, 323–330 (1994).
35. Du, T.S., Liao, J.G., Chen, L.G., Awan, M.U.: *Properties and Riemann–Liouville fractional Hermite–Hadamard inequalities for the generalized (α, m)-preinvex functions*, J. Inequal. Appl. **2016**, 306 (2016).
36. Mitrinovic, D.S., Pecaric, J.E., Fink, A.M.: *Classical and New Inequalities in Analysis*; Kluwer Academic: Dordrecht, The Netherlands, (1993).
37. Abdeljawad, T., Baleanu, D: *On fractional derivatives with exponential kernel and their discrete versions*, Rep. Math. Phys. **80**, 11–27 (2017).
38. Tariq, M., Ahmad, H., Shaikh, A.G., Sahoo, S.K., Khedher, M., Gia, T.N.: *New fractional integral inequalities for preinvex functions involving Caputo-Fabrizio operator*, AIMS Math. **7**, 3440–3455 (2021).
39. Atangana, A., Baleanu, D.: *New fractional derivatives with non-local and non-singular kernel*, Theory Appl. Heat Transfer Model, Thermal Science. **20**, 763–769 (2016).
40. Abdeljawad, T., Baleanu, D.: *Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel*, J. Nonlinear Sci. Appl. **10**, 1098–1107 (2017).

41. Hadamard, J.: *Étude sur les propriétés des fonctions entiéres en particulier dúne fonction considéréé par Riemann*, J. Math. Pures Appl. **58**, 171-215 (1893).
42. İşcan, I.: *Hermite-Hadamards inequalities for preinvex function via fractional integrals and related fractional inequalities*, Amer. J. Math. Anal. **1**, 33-38 (2013).
43. İşcan, I., Kadakal, M., Kadakal, H.: *On two times differentiable preinvex and prequasiinvex functions*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68**, 950-963 (2019).
44. Fernandez, A., Mohammed, P.: *Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels*, Math. Methods Appl. Sci. **44**, 1-8 (2020).
45. Noor, M.A.: *Hermite-Hadamard integral inequalities for log-preinvex functions*, J. Math. Anal. Approx. Theory. **2**, 126-131 (2007).
46. Pachpatte, B.G.: *On some inequalities for convex functions*, RGMIA Res. Rep. Coll. **6**, 1-8 (2003).
47. Cristescu, G.: *Improved integral inequalities for product of convex functions*, J. Inequal. Pure. Appl. Math. **6**, 1-6 (2005).