

Boundedness of intrinsic square functions on total Fofana-Guliyev spaces

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Abstract. Total Fofana-Guliyev spaces were recently introduced as a variant of Fofana spaces and are closely related to total Morrey spaces. In this paper, we discuss the boundedness of intrinsic square functions and their commutators generated by BMO-functions on these new spaces.

Keywords. Morrey spaces, Fofana spaces, total Fofana-Guliyev spaces, intrinsic square functions, commutator

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1 Introduction and main results

Let d be a fixed positive integer and $\mathbb{R}^d$ be d -dimensional Euclidean space equipped with the Lebesgue measure dx . For any subset E of $\mathbb{R}^d$, we denote by χ_E its characteristic function and by $|E|$ its Lebesgue measure when it is measurable. By $L^0(\mathbb{R}^d)$, we denote the complex vector space of equivalent classes (modulo equality Lebesgue almost everywhere) of Lebesgue measurable complex-valued functions on $\mathbb{R}^d$.

Let $0 < \gamma \leq 1$. we denote by $\mathcal{C}_\gamma$ the family of functions φ defined on $\mathbb{R}^d$ with support in the closed unit ball $\mathbb{B} = \{x \in \mathbb{R}^d : |x| \leq 1\}$ and vanishing integral, i.e, $\int_{\mathbb{R}^d} \varphi(x) dx = 0$, and such that for all $x, x' \in \mathbb{R}^d$, $|\varphi(x) - \varphi(x')| \leq |x - x'|^\gamma$. For every $(x, t) \in \mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0, \infty)$, we write $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$ and $\Gamma(x)$ is the usual "cone of aperture one",

$$\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{d+1} : |x - y| < t\}.$$

Let $f \in L^1_{loc}(\mathbb{R}^d)$. We define the intrinsic square function of f (of order γ) by the formula

$$S_\gamma(f)(x) = \left[\int_{\Gamma(x)} \left(\sup_{\varphi \in \mathcal{C}_\gamma} |f * \varphi_t(y)| \right)^2 \frac{dy dt}{t^{d+1}} \right]^{\frac{1}{2}}, \quad (1.1)$$

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the intrinsic Littlewood-Paley g -function $g_\gamma(f)$ by

$$g_\gamma(f)(x) = \left[\int_0^\infty \left(\sup_{\varphi \in \mathcal{C}_\gamma} |f * \varphi_t(x)| \right)^2 \frac{dt}{t} \right]^{\frac{1}{2}} \quad (1.2)$$

and the intrinsic Littlewood-Paley $g_{\mu,\gamma}^*$ -function $g_{\mu,\gamma}^*(f)$ by

$$g_{\mu,\gamma}^*(f)(x) = \left[\int_{\mathbb{R}^{d+1}_+} \left(\frac{t}{t + |x - y|} \right)^{\mu d} \left(\sup_{\varphi \in \mathcal{C}_\gamma} |f * \varphi_t(y)| \right)^2 \frac{dy dt}{t^{d+1}} \right]^{\frac{1}{2}}. \quad (1.3)$$

Note that, the intrinsic square functions were first introduced by Wilson [24] in order to settle a conjecture proposed by Fefferman and Stein on the boundedness of the Lusin area function S from the weighted Lebesgue space $L^2_{M(v)}(\mathbb{R}^d)$ to $L^2_v(\mathbb{R}^d)$, where $0 \leq v \in L^1_{loc}(\mathbb{R}^d)$ and M denotes the Hardy-Littlewood maximal function. Later, he proved in [25] that these functions are bounded on the weighted Lebesgue spaces $L^q_w(\mathbb{R}^d)$ whenever $q \in (1, \infty)$ and $w \in \mathcal{A}_q(\mathbb{R}^d)$ (the class of Muckenhoupt weights). Therefore, many results on boundedness of intrinsic square functions have been obtained (see, for example, [4, 16, 17, 21–23] and the references therein). In [21], Wang established the boundedness of these operators and their commutators on weighted Morrey spaces $L^{q,\kappa}_w(\mathbb{R}^d)$ ($0 < \kappa < 1$ and $0 < q < 1$). Wang's work was later extended to various Morrey-type spaces (see [16]) and specifically to the weighted Fofana spaces $(L^q_\omega, L^p)^\alpha(\mathbb{R}^d)$ ($1 \leq q \leq \alpha \leq p \leq \infty$) by Feuto in [4]. We recall that Fofana spaces were originally introduced by Fofana [9] in 1988 during the study of Fourier multipliers.

Let $1 \leq q, p \leq \infty$ and $r > 0$. For $f \in L^0(\mathbb{R}^d)$ we define

$$r \|f\|_{q,p} := \left\| \left[\int_{\mathbb{R}^d} |f \chi_{B(y,r)}|^q(x) dx \right]^{\frac{1}{q}} \right\|_p,$$

with $\|\cdot\|_p$ denoting the usual norm of the Lebesgue space $L^p(\mathbb{R}^d)$, taken with respect to the variable y and $B(y, r) = \{x \in \mathbb{R}^d : |y - x| < r\}$. We adopt the usual convention $\frac{1}{\infty} = 0$. For $1 \leq q \leq \alpha \leq p \leq \infty$, the Fofana space $(L^q, L^p)^\alpha(\mathbb{R}^d)$ is defined as the set of all functions $f \in L^0(\mathbb{R}^d)$ satisfying $\|f\|_{q,p,\alpha} < \infty$, where

$$\|f\|_{q,p,\alpha} = \sup_{r>0} r^{d(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})} r \|f\|_{q,p}.$$

It is proved in [9] that, for $1 \leq q < \alpha$ fixed and $p \in [\alpha, \infty]$, the spaces $(L^q, L^p)^\alpha(\mathbb{R}^d)$ form a chain of distinct Banach spaces beginning with the Lebesgue space $L^\alpha(\mathbb{R}^d)$ and ending by the classical Morrey space $L^{q,\kappa}(\mathbb{R}^d) = (L^q, L^\infty)^\alpha(\mathbb{R}^d)$, where $\kappa = d(1 - \frac{q}{\alpha})$. For more properties of Fofana spaces we refer to [2, 5–8] and the references therein.

Note that, in recent years, Fofana spaces have enjoyed a lot of interest. Then, inspired by Guliyev's works (see [10–13]), Kpata and Nagacy [15] introduced new spaces of Fofana-type, known as total Fofana-Guliyev spaces, see also [1, 14, 18, 20]. Let $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$. The total Fofana-Guliyev space $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is defined as the set of all elements $f \in L^0(\mathbb{R}^d)$ such that $\|f\|_{(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)} < \infty$, where

$$\|f\|_{(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)} = \sup_{r>0} [r]_1^{d(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda} + \frac{1}{q} + \frac{1}{p})} r \|f\|_{q,p},$$

with $[r]_1 = \min\{1, r\}$, for all $r > 0$. $\left((L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d), \|\cdot\|_{(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\right)$ is a complex Banach space (see [15]). In Section 2, we summarize basic results on total Fofana-Guliyev

spaces. Nevertheless, let us mention that these spaces are generalizations of classical Fofana spaces and are exactly total Fofana spaces defined in [19]. Boundedness properties of classical operators, including the Hardy-Littlewood maximal operator and its commutator, fractional integral, and fractional maximal operators, have been obtained for $(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)$ -spaces (see [15, 19]).

The aim of this paper is to prove that the intrinsic square functions, defined in (1.1), (1.2) and (1.3), and their corresponding commutators are bounded on $(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)$.

Throughout this note, we will use the following abbreviation $A \lesssim B$ for the inequalities $A \leq CB$, where C is a positive constant. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

Our first result reads as follows.

Theorem 1.1 *Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Then*

$$\|S_\gamma f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d).$$

Since $S_\gamma(f)$ and $g_\gamma(f)$ are pointwise comparable (see [24]), Theorem 1.1 obviously implies what follows. We omit the details.

Corollary 1.1 *Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Then*

$$\|g_\gamma(f)\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d).$$

Our next result concerns the intrinsic Littlewood-Paley $g_{\mu, \gamma}^*$ -function.

Theorem 1.2 *Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. If $\mu > \max\{q, 3\}$, then*

$$\|g_{\mu, \gamma}^*(f)\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d).$$

We shall consider the commutators generated by a function $b \in L_{loc}^1(\mathbb{R}^d)$ and intrinsic square functions, which are defined by the following expressions. For all $x \in \mathbb{R}^d$,

$$[b, S_\gamma](f)(x) = \left[\int_{\Gamma(x)} \sup_{\varphi \in \mathcal{C}_\gamma} \left| \int_{\mathbb{R}^d} (b(x) - b(z)) \varphi_t(y - z) f(z) dz \right|^2 \frac{dy dt}{t^{d+1}} \right]^{\frac{1}{2}},$$

$$[b, g_\gamma](f)(x) = \left[\int_0^\infty \sup_{\varphi \in \mathcal{C}_\gamma} \left| \int_{\mathbb{R}^d} (b(x) - b(z)) \varphi_t(y - z) f(z) dz \right|^2 \frac{dt}{t} \right]^{\frac{1}{2}}$$

and

$$[b, g_{\mu, \gamma}^*](f)(x) = \left[\int_{\mathbb{R}_+^{d+1}} \left(\frac{t}{t + |x - y|} \right)^{\mu d} \sup_{\varphi \in \mathcal{C}_\gamma} \left| \int_{\mathbb{R}^d} (b(x) - b(z)) \varphi_t(y - z) f(z) dz \right|^2 \frac{dy dt}{t^{d+1}} \right]^{\frac{1}{2}}.$$

Let $b \in L_{loc}^1(\mathbb{R}^d)$. b belongs to $BMO(\mathbb{R}^d)$ if $\|b\|_{BMO(\mathbb{R}^d)} < \infty$, where

$$\|b\|_{BMO(\mathbb{R}^d)} = \sup_{r>0, x \in \mathbb{R}^d} |B(x, r)|^{-1} \int_{B(x, r)} |b(y) - b_{B(x, r)}| dy,$$

with $b_{B(x, r)} = |B(x, r)|^{-1} \int_{B(x, r)} b(y) dy$.

For these commutators, we have the following boundedness properties.

Theorem 1.3 *Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. If $b \in BMO(\mathbb{R}^d)$, then*

$$\|[b, S_\gamma](f)\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d).$$

As an immediate consequence of Theorem 1.3, we obtain the following corollary. We omit the proof.

Corollary 1.2 *Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. If $b \in BMO(\mathbb{R}^d)$, then*

$$\|[b, g_\gamma](f)\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d).$$

Theorem 1.4 *Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. If $\mu > \max\{q, 3\}$ and $b \in BMO(\mathbb{R}^d)$, then*

$$\|[b, g_{\mu, \gamma}^*](f)\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d).$$

The remainder of the paper is organized as follows. In Section 2, we recall some properties of total Fofana-Guliyev spaces. Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.2, and in the last section we give the proofs of Theorem 1.3 and Theorem 1.4.

2 Preliminary results on Total Fofana-Guliyev spaces

The propositions of this section are proved in [15].

Proposition 2.1 *Let $1 \leq q \leq \alpha$, $\lambda \leq p \leq \infty$. Then*

$$(L^q, L^p)^\alpha(\mathbb{R}^d) \cap (L^q, L^p)^\lambda(\mathbb{R}^d) \hookrightarrow (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)$$

and for $f \in (L^q, L^p)^\alpha(\mathbb{R}^d) \cap (L^q, L^p)^\lambda(\mathbb{R}^d)$,

$$\|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \leq \max\{\|f\|_{q, p, \alpha}, \|f\|_{q, p, \lambda}\}.$$

Proposition 2.2 *Let $1 \leq q \leq \lambda \leq \alpha \leq p \leq \infty$. Then*

$$(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d) = (L^q, L^p)^\alpha(\mathbb{R}^d) \cap (L^q, L^p)^\lambda(\mathbb{R}^d)$$

and for $f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)$,

$$\|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} = \max\{\|f\|_{q, p, \alpha}, \|f\|_{q, p, \lambda}\}.$$

The result of Proposition 2.2 implies that for $1 \leq q \leq \alpha \leq p \leq \infty$, $(L^q, L^p)^{\alpha, \alpha}(\mathbb{R}^d) = (L^q, L^p)^\alpha(\mathbb{R}^d)$. So Total Fofana-Guliyev spaces are generalizations of classical Fofana spaces. We recall that for $\alpha \in \{q, p\}$, $(L^q, L^p)^\alpha(\mathbb{R}^d) = L^\alpha(\mathbb{R}^d)$ (see [9]).

The family of spaces $(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)$ is decreasing with respect to the q power and increasing with respect to the p power. More precisely, we have the following

Proposition 2.3 *Let $1 \leq q_1 \leq q_2 \leq \alpha$, $\lambda \leq p_1 \leq p_2 \leq \infty$. Then:*

$$(1) \quad \|f\|_{(L^{q_1}, L^{p_1})^{\alpha, \lambda}(\mathbb{R}^d)} \leq \|f\|_{(L^{q_2}, L^{p_2})^{\alpha, \lambda}(\mathbb{R}^d)}, \quad f \in L^0(\mathbb{R}^d)$$

and consequently, $(L^{q_2}, L^{p_2})^{\alpha, \lambda}(\mathbb{R}^d) \subset (L^{q_1}, L^{p_1})^{\alpha, \lambda}(\mathbb{R}^d)$;

$$(2) \quad \|f\|_{(L^q, L^{p_2})^{\alpha, \lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q, L^{p_1})^{\alpha, \lambda}(\mathbb{R}^d)}, \quad f \in L^0(\mathbb{R}^d)$$

and consequently, $(L^q, L^{p_1})^{\alpha, \lambda}(\mathbb{R}^d) \subset (L^q, L^{p_2})^{\alpha, \lambda}(\mathbb{R}^d)$.

Total Fofana-Guliyev spaces are subspaces of total Morrey spaces since, for $1 \leq q \leq \alpha$, $\lambda \leq p \leq \infty$, part (2) of Proposition 2.3 asserts that $(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d) \subset (L^q, L^\infty)^{\alpha, \lambda}(\mathbb{R}^d)$. Also, it is easy to see that for $1 \leq q \leq \alpha$, $\lambda < \infty$, the space $(L^q, L^\infty)^{\alpha, \lambda}(\mathbb{R}^d)$ is the total Morrey space $L^{q, d(1-\frac{q}{\alpha}), d(1-\frac{q}{\lambda})}(\mathbb{R}^d)$ defined in [12].

3 Proofs of Theorems 1.1 and 1.2

The following result will be useful in the proof of Theorem 1.1. We use it in the case where $\omega = 1$.

Theorem 3.1 [25] *Let $0 < \gamma \leq 1$, $1 < q < \infty$ and $\omega \in A_q$. Then there exists a constant $C > 0$ independent of f such that*

$$\|S_\gamma f\|_{q,\omega} \leq C \|f\|_{q,\omega}.$$

Proof of Theorem 1.1. Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$. Let $r > 0$ and $y \in \mathbb{R}^d$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{B(y,2r)}$. Since S_γ is a sublinear operator, we have

$$\|S_\gamma(f)\chi_{B(y,r)}\|_q \leq \|S_\gamma(f_1)\chi_{B(y,r)}\|_q + \|S_\gamma(f_2)\chi_{B(y,r)}\|_q. \quad (3.1)$$

For the term in f_1 , Theorem 3.1 imply

$$\|S_\gamma(f_1)\chi_{B(y,r)}\|_q \lesssim \|f\chi_{B(y,2r)}\|_q. \quad (3.2)$$

Next, for the term in f_2 , proceeding as in the proof of Theorem 3.1 in [4], we obtain

$$\|S_\gamma(f_2)\chi_{B(y,r)}\|_q \lesssim \sum_{k=1}^{\infty} \frac{|B(y,r)|^{\frac{1}{q}}}{|B(y,2^{k+1}r)|^{\frac{1}{q}}} \|f\chi_{B(y,2^{k+1}r)}\|_q. \quad (3.3)$$

Therefore the L^p norm of both sides of (3.1) leads to

$$r \|S_\gamma(f)\|_{q,p} \lesssim 2r \|f\|_{q,p} + \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} 2^{k+1}r \|f\|_{q,p}.$$

On the one hand, we have

$$\begin{aligned} 2r \|f\|_{q,p} &= \frac{[2r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/2r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/2r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}} 2r \|f\|_{q,p} \\ &\leq [2r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/2r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \end{aligned}$$

Hence,

$$2r \|f\|_{q,p} \lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \quad (3.4)$$

On the other hand,

$$\begin{aligned} &\sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} 2^{k+1}r \|f\|_{q,p} \\ &= \sum_{i=k}^{\infty} 2^{-\frac{kd}{q}} \frac{[2^{k+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/2^{k+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2^{k+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/2^{k+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}} 2^{k+1}r \|f\|_{q,p} \\ &\leq \|f\|_{(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} [2^{k+1}r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/2^{k+1}r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \\ &\lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \sum_{k=1}^{\infty} (2^k)^{d(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}. \end{aligned}$$

Since $-\frac{1}{\alpha} - \frac{1}{\lambda} + \frac{1}{q} + \frac{2}{p} < 0$ then $\sum_{k=1}^{\infty} (2^k)^{d(-\frac{1}{\alpha} - \frac{1}{\lambda} + \frac{1}{q} + \frac{2}{p})} < \infty$.

Therefore,

$$\begin{aligned} & \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} 2^{k+1} r \|f\|_{q,p} \\ & \lesssim [r]_1^{d(-\frac{1}{\alpha} + \frac{1}{q} + \frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda} - \frac{1}{q} - \frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}. \end{aligned} \quad (3.5)$$

From (3.4) and (3.5), we deduce that

$$r \|S_{\gamma}(f)\|_{q,p} \lesssim [r]_1^{d(-\frac{1}{\alpha} + \frac{1}{q} + \frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda} - \frac{1}{q} - \frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}.$$

It follows that

$$[r]_1^{d(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda} + \frac{1}{q} + \frac{1}{p})} r \|S_{\gamma}(f)\|_{q,p} \lesssim \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}. \quad (3.6)$$

We obtain the desired result by taking the supremum over all $r > 0$ in the left hand side of (3.6).

Remark 3.1 For $\alpha = \lambda$ and $\omega = 1$, Theorem 1.1 and Corollary 1.1 were established in [4, Theorem 3.1] and [4, Corollary 3.5], without the hypothesis $\frac{1}{q} + \frac{2}{p} < \frac{2}{\alpha}$.

For the proof of Theorem 1.2, we need the following varying-aperture versions of S_{γ} . For $0 < \gamma \leq 1$ and $\beta > 0$, we define $S_{\gamma, \beta}$ by

$$S_{\gamma, \beta}(f)(x) = \left[\int_{\Gamma_{\beta}(x)} \left(\sup_{\varphi \in \mathcal{C}_{\gamma}} |f * \varphi_t(y)| \right)^2 \frac{dy dt}{t^{d+1}} \right]^{\frac{1}{2}},$$

where for $x \in \mathbb{R}^d$ and $\Gamma_{\beta}(x) = \{(y, t) \in \mathbb{R}_+^{d+1} : |x - y| < \beta t\}$.

The following lemma will be useful for proving Theorem 1.2. We use it in the case $\omega = 1$.

Lemma 3.1 [21] Let $0 < \gamma \leq 1$, $1 < q < \infty$. Then for all non negative integer j , $S_{\gamma, 2^j}$ is bounded on $L_{\omega}^q(\mathbb{R}^d)$. Moreover

$$\|S_{\gamma, 2^j}(f)\|_{q, \omega} \lesssim (2^{dj} + 2^{\frac{dj}{2}}) \|f\|_{q, \omega}.$$

Proof of Theorem 1.2. Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. We suppose that $\mu > \max\{q, 3\}$ and $f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)$.

For $g_{\mu, \gamma}^*$, by the definition, we know that, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} & [g_{\mu, \gamma}^*(f)(x)]^2 \\ &= \int_0^{\infty} \int_{|x-y| < t} \left(\frac{t}{t + |x-y|} \right)^{\mu d} \left(\sup_{\varphi \in \mathcal{C}_{\gamma}} |f * \varphi_t(y)| \right)^2 \frac{dy dt}{t^{d+1}} \\ &+ \sum_{j=1}^{\infty} \int_0^{\infty} \int_{2^{j-1}t \leq |x-y| < 2^j t} \left(\frac{t}{t + |x-y|} \right)^{\mu d} \left(\sup_{\varphi \in \mathcal{C}_{\gamma}} |f * \varphi_t(y)| \right)^2 \frac{dy dt}{t^{d+1}} \\ &\lesssim [S_{\gamma}(f)(x)]^2 + \sum_{j=1}^{\infty} 2^{-j\mu d} [S_{\gamma, 2^j}(f)(x)]^2. \end{aligned}$$

Thus, for all $x \in \mathbb{R}^d$ it holds true that

$$g_{\mu,\gamma}^*(f)(x) \lesssim S_\gamma(f)(x) + \sum_{j=1}^{\infty} 2^{-j\mu d/2} S_{\gamma,2^j}(f)(x). \quad (3.7)$$

Let $r > 0$. It follows that

$$r \|g_{\mu,\gamma}^*(f)\|_{q,p} \leq r \|S_\gamma(f)\|_{q,p} + \sum_{j=1}^{\infty} 2^{-j\mu d/2} r \|S_{\gamma,2^j}(f)\|_{q,p}. \quad (3.8)$$

The first term on the right can be controlled in this way.

$$\begin{aligned} r \|S_\gamma(f)\|_{q,p} &= \frac{[r]_1^{d(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda} + \frac{1}{q} + \frac{1}{p})}}{[r]_1^{d(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda} + \frac{1}{q} + \frac{1}{p})}} r \|S_\gamma(f)\|_{q,p} \\ &\leq [r]_1^{d(-\frac{1}{\alpha} + \frac{1}{q} + \frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda} - \frac{1}{q} - \frac{1}{p})} \|S_\gamma(f)\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}. \end{aligned}$$

By Theorem 1.1, we have

$$\begin{aligned} r \|S_\gamma(f)\|_{q,p} & \\ &\lesssim [r]_1^{d(-\frac{1}{\alpha} + \frac{1}{q} + \frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda} - \frac{1}{q} - \frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}. \end{aligned} \quad (3.9)$$

Let $j \in \mathbb{N}^*$ and $y \in \mathbb{R}^d$. Let us estimate now

$$r \|S_{\gamma,2^j}(f)\|_{q,p} = \left\| \|S_{\gamma,2^j}(f) \chi_{B(y,r)}\|_q \right\|_p.$$

We proceed as in the proof of Theorem 1.1. So, for $f = f_1 + f_2$ with $f_1 = f \chi_{B(y,2r)}$, we have

$$\begin{aligned} &\|S_{\gamma,2^j}(f) \chi_{B(y,r)}\|_q \\ &\leq \|S_{\gamma,2^j}(f_1) \chi_{B(y,r)}\|_q + \|S_{\gamma,2^j}(f_2) \chi_{B(y,r)}\|_q. \end{aligned} \quad (3.10)$$

By applying Lemma 3.1, we get

$$\|S_{\gamma,2^j}(f_1) \chi_{B(y,r)}\|_q \lesssim (2^{dj} + 2^{\frac{dj}{2}}) \|f \chi_{B(y,2r)}\|_q. \quad (3.11)$$

We now estimate $\|S_{\gamma,2^j}(f_2) \chi_{B(y,r)}\|_q$. By proceeding as in the proof of Theorem 3.2 in [4] and by using Hölder's inequality, we have

$$\begin{aligned} &|S_{\gamma,2^j}(f_2)(x)| \\ &\lesssim 2^{3jd/2} \sum_{k=1}^{\infty} |B(y, 2^{k+1}r)|^{-\frac{1}{q}} \|f \chi_{B(y, 2^{k+1}r)}\|_q \end{aligned} \quad (3.12)$$

for all $x \in B(y, r)$. Thus, taking the $L^q(B(y, r))$ -norm of both sides of the above estimation leads to

$$\|S_{\gamma,2^j}(f_2) \chi_{B(y,r)}\|_q \lesssim 2^{3jd/2} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} \|f \chi_{B(y, 2^{k+1}r)}\|_q. \quad (3.13)$$

Taking estimates (3.11) and (3.13) in (3.10), we get

$$\begin{aligned} \|S_{\gamma,2^j}(f)\chi_{B(y,r)}\|_q &\lesssim (2^{dj} + 2^{\frac{dj}{2}}) \|f\chi_{B(y,2r)}\|_q \\ &\quad + 2^{3jd/2} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} \|f\chi_{B(y,2^{k+1}r)}\|_q, \end{aligned}$$

so that the $L^p(\mathbb{R}^d)$ -norms of both sides leads to

$$r \|S_{\gamma,2^j}(f)\|_{q,p} \lesssim (2^{dj} + 2^{\frac{dj}{2}}) 2r \|f\|_{q,p} + 2^{3jd/2} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} 2^{k+1}r \|f\|_{q,p}.$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} &r \|S_{\gamma,2^j}(f)\|_{q,p} \\ &\lesssim (2^{dj} + 2^{\frac{dj}{2}} + 2^{3jd/2}) [r]_1^{d(-\frac{1}{\alpha} + \frac{1}{q} + \frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda} - \frac{1}{q} - \frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}. \end{aligned} \quad (3.14)$$

Taking estimates (3.9) and (3.14) in (3.8), we have

$$\begin{aligned} &r \|g_{\mu, \gamma}^*(f)\|_{q,p} \\ &\lesssim \left(1 + \sum_{j=1}^{\infty} 2^{-j\mu d/2} (2^{dj} + 2^{\frac{dj}{2}} + 2^{3jd/2}) \right) \\ &\quad \times [r]_1^{d(-\frac{1}{\alpha} + \frac{1}{q} + \frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda} - \frac{1}{q} - \frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}, \end{aligned}$$

where the convergence of the series is due to the fact that $\mu > \max\{q, 3\}$.

It follows that

$$\begin{aligned} &[r]_1^{d(\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda} + \frac{1}{q} + \frac{1}{p})} r \|g_{\mu, \gamma}^*(f)\|_{q,p} \\ &\lesssim \left(1 + \sum_{j=1}^{\infty} 2^{-j\mu d/2} (2^{dj} + 2^{jqd/2} + 2^{3jd/2}) \right) \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}. \end{aligned}$$

We end the proof by taking the supremum over all $r > 0$.

Remark 3.2 Note that Theorem 1.2 in the case $\alpha = \lambda$ was established in [4, Theorem 3.2] for $\omega = 1$. However, the proof given in [4] does not use the hypothesis $\frac{1}{q} + \frac{2}{p} < \frac{2}{\alpha}$.

4 Proofs of Theorem 1.3 and Theorem 1.4.

Let b be a locally integrable function. It is proved in [3] that if $b \in BMO(\mathbb{R}^d)$, then for every $1 < q < \infty$, we have

$$\begin{aligned} &\|b\|_{BMO(\mathbb{R}^d)} \\ &\approx \sup_{r>0, x \in \mathbb{R}^d} \left(|B(x, r)|^{-1} \int_{B(x, r)} |b(y) - b_{B(x, r)}|^q dy \right)^{\frac{1}{q}}. \end{aligned} \quad (4.1)$$

The following lemma, proved in [16], will be very useful for the following theorem proofs. We use it in the case $\omega = 1$.

Lemma 4.1 [21, Theorem 3.1]

Let $0 < \gamma \leq 1$, $1 < q < \infty$. Then the commutators $[b, S_\gamma]$ and $[b, g_{\mu, \gamma}^*]$ are bounded on $L_\omega^q(\mathbb{R}^d)$ whenever $b \in BMO(\mathbb{R}^d)$.

Proof of Theorem 1.3. Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. We suppose that $b \in BMO(\mathbb{R}^d)$ and $f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)$. Let $r > 0$ and $y \in \mathbb{R}^d$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{B(y, 2r)}$. By definition of $[b, S_\gamma](f)$, we have

$$r \|[b, S_\gamma](f)\|_{q, p} \leq r \|[b, S_\gamma](f_1)\|_{q, p} + r \|[b, S_\gamma](f_2)\|_{q, p}. \quad (4.2)$$

For the term in f_1 , we have

$$\|[b, S_\gamma](f_1)\chi_{B(y, r)}\|_q \lesssim \|f\chi_{B(y, 2r)}\|_q \quad (4.3)$$

as an immediate consequence of the boundedness of Lemma 4.1. Then, taking the $L^p(\mathbb{R}^d)$ -norm on both sides of the previous inequality and taking into consideration (3.4), we obtain

$$\begin{aligned} & r \|[b, S_\gamma](f_1)\|_{q, p} \\ & \lesssim [r]_1^{d(-\frac{1}{\alpha} + \frac{1}{q} + \frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda} - \frac{1}{q} - \frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}. \end{aligned} \quad (4.4)$$

Next, we turn to estimate $r \|[b, S_\gamma](f_2)\|_{q, p}$. Let $x \in \mathbb{R}^d$. For $(u, t) \in \Gamma(x)$, we have

$$\begin{aligned} & \sup_{\varphi \in \mathcal{C}_\gamma} \left| \int_{\mathbb{R}^d} (b(x) - b(z)) \varphi_t(u - z) f_2(z) dz \right| \\ & \leq |b(x) - b_{B(y, r)}| \sup_{\varphi \in \mathcal{C}_\gamma} |f_2 * \varphi_t(u)| + \sup_{\varphi \in \mathcal{C}_\gamma} |[(b - b_{B(y, r)})f_2] * \varphi_t(u)| \end{aligned}$$

so that the $L^2(\Gamma(x), \frac{dudt}{t^{d+1}})$ -norm of both sides leads to

$$\begin{aligned} & |[b, S_\gamma](f_2)(x)| \\ & \leq |b(x) - b_{B(y, r)}| S_\gamma(f_2)(x) \\ & + \left[\int_{\Gamma(x)} \left(\sup_{\varphi \in \mathcal{C}_\gamma} |[(b - b_{B(y, r)})f_2] * \varphi_t(u)| \right)^2 \frac{dudt}{t^{d+1}} \right]^{\frac{1}{2}} \\ & = I_1(x) + I_2(x). \end{aligned} \quad (4.5)$$

We take $x \in B(y, r)$. Following the ideas of the proof of Theorem 3.1 in [4] and by using Hölder's inequality, we have

$$|S_\gamma(f_2)(x)| \lesssim \sum_{k=1}^{\infty} |B(y, 2^{k+1}r)|^{-\frac{1}{q}} \left\| f\chi_{B(y, 2^{k+1}r)} \right\|_q. \quad (4.6)$$

Taking the $L^q(B(y, r))$ -norm of I_1 and taking into account (4.1), we obtain

$$\begin{aligned} & \| |b - b_{B(y, r)}| S_\gamma(f_2) \chi_{B(y, r)} \|_q \\ & \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} \left\| f\chi_{B(y, 2^{k+1}r)} \right\|_q. \end{aligned} \quad (4.7)$$

On the other hand, by proceeding as in the proof of Theorem 3.3 in [4], we have

$$\begin{aligned}
 I_2(x) &\lesssim \sum_{k=1}^{\infty} |B(y, 2^{k+1}r)|^{-1} \int_{B(y, 2^{k+1}r) \setminus B(y, 2^k r)} |b(z) - b_{B(y, r)}| |f(z)| dz \\
 &\lesssim \sum_{k=1}^{\infty} |B(y, 2^{k+1}r)|^{-1} \int_{B(y, 2^{k+1}r)} |b(z) - b_{B(y, 2^{k+1}r)}| |f(z)| dz \\
 &\quad + \sum_{k=1}^{\infty} \frac{|b_{B(y, 2^{k+1}r)} - b_{B(y, r)}|}{|B(y, 2^{k+1}r)|} \int_{B(y, 2^{k+1}r)} |f(z)| dz \\
 &:= J_1(x) + J_2(x)
 \end{aligned}$$

for all $x \in B(y, r)$. Applying Hölder's inequality and using (4.1), we have

$$J_1(x) \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} |B(y, 2^{k+1}r)|^{-\frac{1}{q}} \left\| f \chi_{B(y, 2^{k+1}r)} \right\|_q$$

for all $x \in B(y, r)$, and the $L^q(B(y, r))$ -norm of both sides leads to

$$\|J_1 \chi_{B(y, r)}\|_q \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} \left\| f \chi_{B(y, 2^{k+1}r)} \right\|_q. \quad (4.8)$$

For J_2 , since $b \in BMO(\mathbb{R}^d)$, we have

$$|b_{B(y, 2^{k+1}r)} - b_{B(y, r)}| \lesssim (k+1) \|b\|_{BMO(\mathbb{R}^d)}.$$

It follows that

$$\begin{aligned}
 J_2(x) &\lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+1)}{|B(y, 2^{k+1}r)|} \int_{B(y, 2^{k+1}r) \setminus B(y, 2^k r)} |f(z)| dz \\
 &\lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+1)}{|B(y, 2^{k+1}r)|^{\frac{1}{q}}} \left\| f \chi_{B(y, 2^{k+1}r)} \right\|_q
 \end{aligned}$$

according to Hölder's inequality.

Hence,

$$\|J_2 \chi_{B(y, r)}\|_q \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+1)}{2^{\frac{kd}{q}}} \left\| f \chi_{B(y, 2^{k+1}r)} \right\|_q. \quad (4.9)$$

Taking the $L^q(B(y, r))$ -norm on both sides of (4.5) and taking into account inequalities (4.7), (4.8) and (4.9), we obtain

$$\|S_\gamma(f_2) \chi_{B(y, r)}\|_q \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} \left\| f \chi_{B(y, 2^{k+1}r)} \right\|_q.$$

Therefore, the $L^p(\mathbb{R}^d)$ -norm on both sides of the previous inequality, gives

$$\begin{aligned}
 &{}_r \|[b, S_\gamma](f_2)\|_{q,p} \\
 &\lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} 2^{k+1}r \|f\|_{q,p}.
 \end{aligned} \quad (4.10)$$

We have,

$$\begin{aligned}
& \sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} 2^{k+1} r \|f\|_{q,p} \\
&= \sum_{i=k}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} \frac{[2^{k+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/2^{k+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2^{k+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/2^{k+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}} 2^{k+1} r \|f\|_{q,p} \\
&\leq \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} [2^{k+1}r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/2^{k+1}r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \\
&\lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{k+3}{2^{kd(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q}-\frac{2}{p})}}.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} \frac{k+3}{2^{kd(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q}-\frac{2}{p})}}$ converges then inequality (4.10) becomes

$$\begin{aligned}
& r \|[b, S_{\gamma}](f_2)\|_{q,p} \\
&\lesssim \|b\|_{BMO(\mathbb{R}^d)} [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}.
\end{aligned} \tag{4.11}$$

Taking estimates (4.4) and (4.11) in (4.2), we obtain

$$\begin{aligned}
r \|[b, S_{\gamma}](f)\|_{q,p} &\lesssim (1 + \|b\|_{BMO(\mathbb{R}^d)}) \\
&\quad \times [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& [r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} r \|[b, S_{\gamma}](f)\|_{q,p} \\
&\lesssim (1 + \|b\|_{BMO(\mathbb{R}^d)}) \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}.
\end{aligned} \tag{4.12}$$

We obtain the desired result by taking the supremum over all $r > 0$ in the left hand side of (4.12).

Proof of Theorem 1.4. [Proof of Theorem 1.4] Let $0 < \gamma \leq 1$, $1 < q \leq \alpha$, $\lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. We suppose that $\mu > \max\{q, 3\}$, $b \in BMO(\mathbb{R}^d)$ and $f \in (L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)$.

Let $r > 0$ and $y \in \mathbb{R}^d$. We write $f = f_1 + f_2$ with $f_1 = f \chi_{B(y, 2r)}$.

By definition of $[b, g_{\mu, \gamma}^*](f)$ and norm $r \|\cdot\|_{q,p}$, we have

$$\begin{aligned}
& r \|[b, g_{\mu, \gamma}^*](f)\|_{q,p} \\
&\lesssim r \|[b, g_{\mu, \gamma}^*](f_1)\|_{q,p} + r \|[b, g_{\mu, \gamma}^*](f_2)\|_{q,p}.
\end{aligned} \tag{4.13}$$

By proceeding as in the proof of Theorem 1.3, we get the following estimation for the term in f_1 ,

$$\begin{aligned}
& r \|[b, g_{\mu, \gamma}^*](f_1)\|_{q,p} \\
&\lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}.
\end{aligned} \tag{4.14}$$

Next, we turn to estimate $_r \| [b, g_{\mu, \gamma}^*](f_2) \|_{q, p}$. For $x \in B(y, r)$, we have

$$\begin{aligned} & |[b, g_{\mu, \gamma}^*](f_2)(x)| \\ & \leq |b(x) - b_{B(y, r)}| g_{\mu, \gamma}^*(f_2)(x) \\ & + \left[\int_{\mathbb{R}_+^{d+1}} \left(\frac{t}{t + |x - y|} \right)^{\mu d} \left(\sup_{\varphi \in \mathcal{C}_\gamma} |[(b - b_{B(y, r)})f_2] * \varphi_t(u)| \right)^2 \frac{dudt}{t^{d+1}} \right]^{\frac{1}{2}} \\ & := A(x) + B(x). \end{aligned} \quad (4.15)$$

For any $x \in B(y, r)$, by using inequalities (3.7), (3.12), (4.6) and the fact that $\mu > 3$, we conclude that

$$g_{\mu, \gamma}^*(f_2)(x) \lesssim \sum_{k=1}^{\infty} |B(y, 2^{k+1}r)|^{-\frac{1}{q}} \|f \chi_{B(y, 2^{k+1}r)}\|_q,$$

which further implies that

$$\|A \chi_{B(y, r)}\|_q \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} \|f \chi_{B(y, 2^{k+1}r)}\|_q. \quad (4.16)$$

On the other hand, by proceeding as in the proof of [16, Theorem 3.10], we have

$$B(x) \lesssim \sum_{j=0}^{\infty} 2^{-j\mu d/2} I_j(x) \quad (4.17)$$

for all $x \in B(y, r)$, where

$$\begin{aligned} I_j(x) & \lesssim 2^{3jd/2} \sum_{k=1}^{\infty} |B(y, 2^{k+1}r)|^{-1} \int_{B(y, 2^{k+1}r)} |b(z) - b_{B(y, 2^{k+1}r)}| |f(z)| dz \\ & + 2^{3jd/2} \sum_{k=1}^{\infty} \frac{|b_{B(y, 2^{k+1}r)} - b_{B(y, r)}|}{|B(y, 2^{k+1}r)|} \int_{B(y, 2^{k+1}r)} |f(z)| dz \\ & \lesssim 2^{3jd/2} (J_1(x) + J_2(x)). \end{aligned}$$

It follows that

$$\|B \chi_{B(y, r)}\|_q \lesssim \sum_{j=0}^{\infty} 2^{(-\mu+3)jd/2} (\|J_1 \chi_{B(y, r)}\|_q + \|J_2 \chi_{B(y, r)}\|_q).$$

Taking estimates (4.8) and (4.9) in the previous inequality and taking into account $\mu > 3$, we obtain

$$\|B \chi_{B(y, r)}\|_q \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+2)}{2^{\frac{kd}{q}}} \|f \chi_{B(y, 2^{k+1}r)}\|_q. \quad (4.18)$$

Next, taking the $L^q(B(y, r))$ -norm on both sides of (4.15) and taking into account inequalities (4.16) and (4.18), we obtain

$$\|[b, g_{\mu, \gamma}^*](f_2) \chi_{B(y, r)}\|_q \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} \|f \chi_{B(y, 2^{k+1}r)}\|_q.$$

Therefore, the $L^p(\mathbb{R}^d)$ -norm on both sides of the previous inequality, gives

$$r \left\| [b, g_{\mu, \gamma}^*](f_2) \right\|_{q,p} \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{k d}{q}}} 2^{k+1} r \|f\|_{q,p}.$$

Using the same arguments as in proof in Theorem 1.3, we obtain

$$\begin{aligned} r \left\| [b, g_{\mu, \gamma}^*](f_2) \right\|_{q,p} \\ \lesssim \|b\|_{BMO(\mathbb{R}^d)} [r]_1^{d(-\frac{1}{\alpha} + \frac{1}{q} + \frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda} - \frac{1}{q} - \frac{1}{p})} \|f\|_{(L^q, L^p)^{\alpha, \lambda}(\mathbb{R}^d)}, \end{aligned} \quad (4.19)$$

which, combined with (4.14), completes the proof of Theorem 1.4.

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